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DEPARTMENT OF MATHEMATICS

ENGINEERING MATHEMATICS -I

SUBJECT CODE: MA8151

(Regulation 2017)

UNIT – III

INTEGRAL CALCULUS

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3.1Introduction

One of the important step followed by any mathematics student after learning Differential Calculus is to think about the reverse process. Integration can be considered as the reverse process of Differentiation. The word Integration means the Summations.

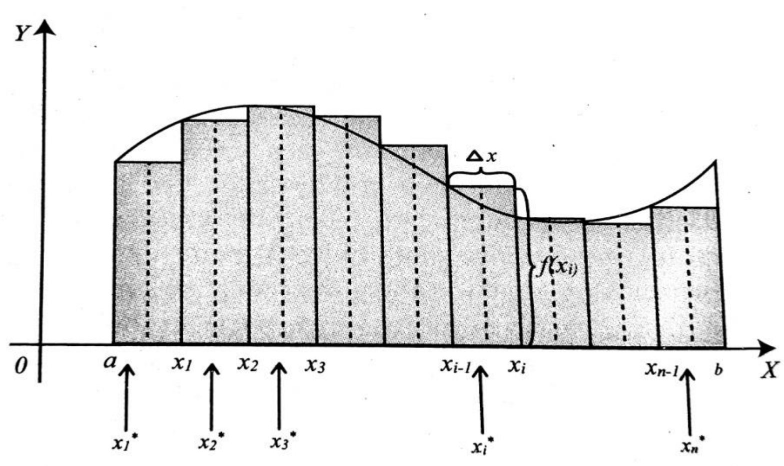
In other words, the process of finding $f(x)$ from $g(x)$given that $\frac{d}{dx}\{f(x)\} = g(x)$ is integration.

We say that $f(x)$ is the integral of $g(x)$ and write symbolically that$\int g(x)dx = f(x)$. The symbol $\int$ is the symbol of integration introduced by Leibnitz and it is an elongated S and was chosen because an integral is a limit of sums, $g(x)$ is called the integrand and dx indicates the variable x with respect to which integration is performed.

The Area Problem

In this section we try to find the area of irregular regions. We know that for regular region, the area is defined by using formulae. For example area of a triangle is half the base times the height and area of the circle π time's radius square. However it is not so easy to find the area of a irregular regions or region with curved sides. Let us see the precise idea for giving an exact definition of area.

Let S be the region that lies under the curve $y = f(x)$from $a\ to\ b$. This means that S, illustrated in below figure is bounded by the graph of a continuous function f (where $f(x) > 0$) , the vertical lines $x = a$ and $x = b$, and the $x-$ axis.



The given interval $[a, b]$ has width $b - a$ so that can be subdivided into n strips as

$$\Delta x = \frac{b-a}{n}$$

Hence the interval is subdivided into n subinterval as

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$$

The left end points are

$$x_0, x_1, x_2, x_{3,\dots,}x_{n-1}$$

The right end points are

$$x_1, x_2, x_3, x_{4,\dots,}x_n$$

The sum of the areas of the lower approximate rectangles is

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x$$

$$= \sum_{i=0}^{n} f(x_i)\Delta x$$

The sum of the areas of the upper approximate rectangles is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

Definition:

The area A of the region S that lies under the gragh of the continuous function$f$ is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}(f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Or

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}(f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

If x_i^* be any point is subinterval $[x_{i-1}, x_i]$then $x_1{}^*, x_2{}^*, x_3{}^*, \ \dots, x_n{}^*$ are the sample points. Then a more general expression for the area S is

$$A = \lim_{n\to\infty}(f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

3.1(a) Definite Integral

The integral which has definite value is called Definite Integral. In other words, when $\int g(x)dx = f(x) + C$, then $[f(b) - f(a)]$ is called the Definite Integral of $g(x)$ between the limits (or end values) a and b and denoted by the symbol $\int_a^b g(x)dx$,a is called the lower limit and b is called the upper limit and is denoted by $[f(x)]_a^b$

Thus $\int_a^b g(x)dx = [f(x)]_a^b = [f(b) - f(a)]$

Theorem 1: If f is continuous on $[a, b]$, (or) if f has only a finite number of discontinuous, then f is integrable on $[a, b]$

$$\text{i.e., } \int_a^b f(x)dx \text{ exists.}$$

Theorem 2: If f is integrable on $[a, b]$ then $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

Example : 3.1

Evaluate $\int_0^3 (x^2 - 2x)\, dx$ by using Riemann sum by taking right end points as the sample points.

Solution:

Take n subintervals, we have $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$$x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{3}{n}, x_3 = \frac{3}{n}, \ldots, x_i = \frac{3i}{n}$$

Since we are using right end points.

$$\therefore \int_0^3 (x^2 - 2x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{9}{n^2} i^2 - \frac{6}{n} i\right]$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right] - \lim_{n\to\infty} \frac{18}{n^2} \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{27}{6n^3} n^3 \left[1 + \frac{1}{n}\right]\left[2 + \frac{1}{n}\right] - \lim_{n\to\infty} \frac{9}{n^2} n^2 \left[1 + \frac{1}{n}\right]$$

$$= \left(\frac{27}{6}\right)(1)(2) - 9 = 9 - 9 = 0$$

Example: 3.2

Evaluate the Riemann sum for f(x) = $x^3 - 6x$, taking the sample points to be right end points and a = 0, b = 3 and n= 6

Solution:

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

The right end points are 0.5, 1, 1.5, 2, 2.5 and 3

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = \sum_{i=1}^{6} f(x_i)\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{i=1}^{6} f(x_i)$$

$$= \frac{1}{2}[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)]$$

$$= \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9] = -3.9375$$

Example: 3.3

Use the definition of area to find an expression for the area under the curve of $\boldsymbol{f(x) = e^{-x}}$ between $\boldsymbol{x = 0}$, $\boldsymbol{x = 2}$. Do not evaluate the limit.

Solution:

Given that $f(x) = e^{-x}, a = 0, b = 2$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right)$$

Area under the curve $f(x) = e^{-x}$ between $x = 0$ and $x = 2$ is given by

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(e^{-2i/n}\right)\left(\frac{2}{n}\right)$$

The Mid Point

The Riemann sum which is the approximation to a given integral using the midpoint is given by

$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n} f(\overline{x_i})\Delta x$$

$$= \Delta x\left[f(\overline{x_1}) + \cdots + f(\overline{x_n})\right]$$

Where $\Delta x = \frac{b-a}{n}$ and $(\overline{x_i}) = \frac{1}{2}[x_{i-1} + x_i]$

= midpoint of $[x_{i-1},\ x_i]$

Example : 3.4

Using midpoint rule find the approximate area for $f(x) = x^2$ between $x = 0$, $x = 1$ and $n = 4$

Solution:

Given that $f(x) = x^2$, $a = 0$ and $b = 1$ and $n = 4$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$$

Hence the interval is subdivided into four equal parts as $[0, 0.25][\ 0.25, 0.5]$, $[0.5, 0.75]$, $[0.75,\ 1]$

The mid points of the subintervals are 0.125, 0.375, 0.625, 0.875

$$\int_0^1 x^2 dx \simeq \sum_{i=1}^{4} f(\overline{x_i})\Delta x$$

$$= 0.25\ [f(0.125) + f(0.375) + f(0.625) + f(0.875)] = 0.3281$$

Example : 3.5

Using the midpoint rule with $n = 5$ find the approximate value of $\int_1^2 \frac{1}{x} dx$

Solution:

Given that $f(x) = \frac{1}{x}$, $a = 1$ and $b = 2$ and $n = 5$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

Hence the interval is subdivided into five equal parts as $[1, 1.2][\ 1.2, 1.4]$, $[1.4, 1.6]$, $[1.6,\ 1.8][1.8,\ 2]$,

The mid points of the subintervals are 1.1, 1.3, 1.5, 1.7, 1.9

$$\int_1^2 \frac{1}{x} dx \simeq \sum_{i=1}^{5} f(\overline{x_i})\Delta x$$

$$= 0.2\ [f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= 0.2\left[\left(\frac{1}{1.1}\right) + \left(\frac{1}{1.3}\right) + \left(\frac{1}{1.5}\right) + \left(\frac{1}{1.7}\right) + \left(\frac{1}{1.9}\right)\right] = 0.691908$$

The Fundamental theorem of Calculus

Part 1: If f is continuous on $[a, b]$ then the function g is defined by

$$g(x) = \int_a^x f(t)dt\,; \;\; a \le x \;\le b$$

is continuous on $[a, b]$ and differentiable on $(a\,, b), and \;\; g'(x) = f(x)$

The Fundamental theorem of Calculus

Part 2: If f is continuous on $[a, b]$ then $\int_a^b f(x)dx = F(b) - \,F(a)$

Where F is any anti derivative of f, that is, a function such that $F' = f$

Problems based on Fundamental theorem of calculus I

Example : 3.6

Find the derivative of the following

(i) $\boldsymbol{g(\,x\,) = \int_0^x (t^2 + \;1)\,dt}$

Solution:

Given $g(\,x\,) = \int_0^x (t^2 + \;1)\,dt$

$\therefore\;\; g'(x\,) = (x^2 + \;1)$ $\qquad$ ($\because\; f(t) = t^2 + \;1$ is continuous by FTC1)

(ii) $\boldsymbol{h(x) = \int_1^{e^x} \log t\; dt}$

Solution:

Given $h(\,x) = \int_1^{e^x} \log t\; dt$

Put $u = \;e^x \quad \Rightarrow \;\; du = \;e^x\,dx \;\Rightarrow\; \frac{du}{dx} = \;e^x$

$$\frac{dh}{dx} \;=\; \frac{dh}{du}\,\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_1^u \log t\; dt\right]e^x = \;\log u\,(e^x\,) \;= \log(e^x)\,e^x \;= xe^x$$

(iii) $\boldsymbol{f(x) = \;\int_0^{\tan x} \sqrt{t + \;\sqrt{t}}\;dt}$

Solution:

Given $f(x) = \;\int_0^{\tan x} \sqrt{t + \;\sqrt{t}}\;dt$

Put $u = \;tanx \quad \Rightarrow\;\; du = \;sec^2x\,dx \;\Rightarrow\; \frac{du}{dx} = \;sec^2x$

$$\frac{df}{dx} \;=\; \frac{df}{du}\,\frac{du}{dx}$$

$$= \frac{d}{du}\left[\int_0^u \sqrt{t + \sqrt{t}}\, dt\right] sec^2 x = \sqrt{u + \sqrt{u}}\, sec^2 x = \sqrt{tan\, x + \sqrt{tan\, x}}\, sec^2 x$$

Problems based on Fundamental theorem of calculus II

Example : 3.7

Evaluate $\int_1^3 e^x dx$ by fundamental theorem of calculus

Solution:

The function $f(x) = e^x$ is continuous everywhere.

By fundamental theorem of calculus part II, Anti derivative $F(x) = e^x$

$$\int_1^3 e^x dx = F(3) - F(1) = e^3 - e$$

Example : 3.8

Evaluate $\int_3^6 \frac{1}{x} dx$ by fundamental theorem of calculus

Solution:

The function $f(x) = \frac{1}{x}$ is continuous in $3 \leq x \leq 6$.

By fundamental theorem of calculus part II, Anti derivative $F(x) = \log x$

$$\int_3^6 \frac{1}{x} dx = [logx]_3^6 = log6 - log3$$

$$= log\left(\frac{6}{3}\right) = log2$$

Example: 3.9

Find the derivative of the following

(i) $\int_{-1}^2 (x^3 - 2x) dx$

Solution:

Given $f(x) = x^3 - 2x$ is continuous in $-1 \leq x \leq 2$

By FTC 2, Anti derivative $F(x) = \frac{x^4}{4} - \frac{2x^2}{2} = \frac{x^4}{4} - x^2$

$\int_{-1}^2 (x^3 - 2x)dx = F(b) - F(a) = F(2) - F(-1)$

$$= \left[\frac{2^4}{4} - 2^2\right] - \left[\frac{(-1)^4}{4} - (-1)^2\right] = \frac{3}{4}$$

(ii) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx$

Solution:

Given $f(x) = \frac{8}{1+x^2}$ is continuous in the given interval.

By FTC 2, Anti derivative $F(x) = 8tan^{-1} x$

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\, dx = F(b) - F(a) = F(\sqrt{3}) - F\left(\frac{1}{\sqrt{3}}\right)$$

$$= 8\, tan^{-1}(\sqrt{3}) - 8\, tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 8\left(\frac{\pi}{3}\right) - 8\left(\frac{\pi}{6}\right) = \frac{4}{3}\pi$$

(iii) $\int_1^9 \frac{x-1}{\sqrt{x}}\, dx$

Solution:

Given $f(x) = \frac{x-1}{\sqrt{x}} = \sqrt{x} - \frac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2}$ is continuous in the given interval.

By FTC 2, Anti derivative $F(x) = \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} = \frac{2}{3}x^{3/2} - 2\, x^{1/2}$

$$\int_1^9 \frac{x-1}{\sqrt{x}}\, dx = F(b) - F(a) = F(9) - F(1)$$

$$= \left[\frac{2}{3}(9)^{3/2} - 2\,(9)^{1/2}\right] - \left[\frac{2}{3} - 2\right]$$

$$= (18-6) - \left(-\frac{4}{3}\right) = 12 + \frac{4}{3} = \frac{40}{3}$$

Example: 3.10

What is wrong with the calculation $\int_0^\pi sec^2 x\; dx = 0$

Solution:

Given $f(x) = sec^2 x = \frac{1}{cos^2 x}\;\; 0 \le x \le \pi$

The fundamental theorem of calculus applies to continuous function.

Here, $f(x) = sec^2 x = \frac{1}{cos^2 x}$ is not continuous at $x = \frac{\pi}{2}$.

Since $f\left(\frac{\pi}{2}\right) = \frac{1}{cos^2 \frac{\pi}{2}} = \frac{1}{0} = \infty$

At $x = \frac{\pi}{2}$ the function $f(x) = sec^2 x$ is discontinuous.

So $\int_0^\pi sec^2 x\; dx$ does not exist.

Example: 3.11

What is wrong with the calculation $\int_{-1}^{3} \frac{dx}{x^2} = -\frac{4}{3}$

Solution:

The fundamental theorem of calculus applies to continuous function.

Here, $f(x) = \frac{1}{x^2}$ is not continuous at $[-1, 3]$.

That is f(x) is discontinuous at $x = 0$. So $\int_{-1}^{3} \frac{dx}{x^2}$ does not exist.

Example: 3.12

What is wrong with the calculation $\int_{\pi/3}^{\pi} \sec\theta \tan\theta \, d\theta = -3$

Solution:

Given $\int_{\pi/3}^{\pi} sec\ \theta\ tan\ \theta\ d\theta$

$$\int_{\pi/3}^{\pi} sec\ \theta\ tan\ \theta\ d\theta = [sec\ \theta]_{\pi/3}^{\pi} = -3$$

The fundamental theorem of calculus applies to continuous function.

Here, $f(\theta) = \sec\theta \tan\theta$ is not continuous on the interval $\left[\frac{\pi}{3}, \pi\right]$, since $\tan\frac{\pi}{2} = \infty$

Exercise 3. 1(a)

1. Find the approximate area for $f(x) = x^2$ between $x = 0$ and $x = 1$ for $n = 4$.

   Ans: $L_4 = 0.21875$, $R_4 = 0.46875$

2. Find the approximate area for $f(x) = e^{-x}$ between $x = 0$ and $x = 2$ for $n = 4$.

   Ans: $L_4 = 1.09877$, $R_4 = 0.66644$

3. Find the approximate area for $f(x) = sin\,x$ between $x = 0$ and $x = \pi$ for $n = 6$.

   Ans: $L_6 = (2 + \sqrt{3})\frac{\pi}{6}$, $R_6 = (2 + \sqrt{3})\frac{\pi}{6}$

4. Evaluate $\int_0^1 x^3 - 3x^2\, dx$ by using Riemann sum by taking right end points as the sample points.

   Ans: $-\frac{3}{4}$

5. Using mid pot rule find the approximate area for $f(x) = x^3$ between $x = 0$ and $x = 1$ and $n = 6$.

   Ans: 0.24653

6. Using fundamental theorem of calculus find the derivative of the following

   $(i)\ g(x) = \int_1^x \frac{dt}{t^3 + 1}$ **Ans:** $g'(x) = \frac{1}{x^3 + 1}$

   $(ii) y(x) = \int_0^{x^2} \cos t\, dt$ **Ans:** $y'(x) = 2x\cos x^2$

   $(iii)\ h(x) = \int_0^{\sin x} \frac{1}{\sqrt{1 - t^2}}\, dt, \quad x < \frac{\pi}{2}$ **Ans:** 1

7. What is wrong with the calculations

   $(i) \int_{-2}^{1} \frac{dx}{x^4} = -\frac{3}{8}$ **Ans:** The function $f(x) = \frac{1}{x^4}$ is not continuous at x = 0.

   $(ii) \int_{-1}^{2} \frac{4}{x^3}\, dx = \frac{3}{2}$ **Ans:** The function $f(x) = \frac{1}{x^3}$ is not continuous at x = 0.

3.1(b) Indefinite Integral

$\int g(x)dx = f(x) + C$ where C is the arbitrary constant of integration. By taking different values C we get any number of solution. Therefore $f(x) + C$ is called the indefinite integral of $g(x)$.

For convenience, we normally omit C when we evaluate an indefinite integral.

As the fundamental theorem of calculus establish a connection between anti derivative and integrals. Thus $\int g(x)dx = f(x)\ means\ f'(x) = g(x)$.

Formulae

1. $\int k\ dx = kx + C$
2. $\int e^x\ dx = e^x + C$
3. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C\ (n \neq 1)$
4. $\int \frac{dx}{x} = \log x + C$
5. $\int a^x\ dx = a^x \log a + C$
6. $\int sinx\ dx = -\cos x + C$
7. $\int cosx\ dx = \sin x + C$
8. $\int sec^2\ x\ dx = tanx + C$
9. $\int cosec^2\ x\ dx = -cotx + C$
10. $\int sec\ x\ tan\ x\ dx = \sec x + C$
11. $\int cosec\ x\ cot\ x\ dx = -\operatorname{cosec} x + C$
12. $\int \tan x\ dx = \log \sec x + C$
13. $\int \cot x\ dx = \log \sin x + C$
14. $\int \sec x\ dx = \log(\sec x + \tan x) + C$
15. $\int cosec\ x\ dx = \log(\ cosec\ x - \cot x\ ) + C$
16. $\int \frac{dx}{1+x^2} = tan^{-1}\ x + C$
17. $\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1}\ x + C$
18. $\int sinhx\ dx = \cos hx + C$
19. $\int coshx\ dx = \sin \text{h}\, x + C$

Problems based on Indefinite Integrals

Example: 3.13

Evaluate $\int \frac{x^3 + 2x + 1}{x^4}\, dx$

Solution:

Given $\int \frac{x^3 + 2x+1}{x^4}\, dx$

$$= \int \left(\frac{1}{x} + \frac{2}{x^3} + \frac{1}{x^4}\right) dx \quad = \int \left(\frac{1}{x} + 2x^{-3} + x^{-4}\right) dx$$

$$= logx + 2\frac{x^{-2}}{(-2)} + \frac{x^{-3}}{(-3)} + C$$

$$= logx - \frac{1}{x^2} - \frac{1}{3x^3} + C$$

Example: 3.14

Evaluate $\int \frac{\boldsymbol{x^3 - 2\sqrt{x}}}{\boldsymbol{x}}\, \boldsymbol{dx}$

Solution:

Given $\int \frac{x^3 - 2\sqrt{x}}{x}\, dx$

$$= \int \left(x^2 - \frac{2}{\sqrt{x}}\right) dx \quad = \int \left(x^2 - 2x^{-1/2}\right) dx$$

$$= \frac{x^3}{3} - 2\frac{x^{1/2}}{1/2} + C \ = \frac{1}{3}\, x^3 - 4\sqrt{x} + C$$

Example: 3.15

Evaluate $\int \left(\boldsymbol{x^{2/5} - x^{-3/5}}\right)^{\boldsymbol{2}}\, \boldsymbol{dx}$

Solution:

Given $\int \left(x^{2/5} - x^{-3/5}\right)^2\, dx$

$$= \int \left[\left(x^{2/5}\right)^2 + \left(x^{-3/5}\right)^2 - 2\left(x^{2/5}\right)\left(x^{-3/5}\right) dx\right]$$

$$= \int \left[x^{4/5} + x^{-6/5} - 2\left(x^{-1/5}\right)\, dx\right]$$

$$= \frac{x^{\frac{4}{5}+1}}{\left(\frac{4}{5}+1\right)} + \frac{x^{\frac{-6}{5}+1}}{\left(-\frac{6}{5}+1\right)} - \frac{x^{\frac{-1}{5}+1}}{\left(-\frac{1}{5}+1\right)} + C$$

$$= \frac{5}{9}\, x^{9/5} - 5x^{-1/5} - \frac{5}{2}\, x^{4/5} + C$$

Example: 3.16

Evaluate $\int \boldsymbol{x^2\, (1 - x)^2\, dx}$

Solution:

Given $\int x^2\, (1 - x)^2\, dx$

$$= \int x^2\, (1 + x^2 - 2x)\, dx$$

$$= \int (x^2 + x^4 - 2x^3)\, dx$$

$$= \frac{x^3}{3} + \frac{x^5}{5} - 2\frac{x^4}{4} + C$$

Example: 3.17

Evaluate $\int \frac{1}{1+\sin x}\, dx$

Solution:

Given $\int \frac{1}{1+\sin x}\, dx$

$$\int \frac{1}{1+\sin x}\, dx = \int \frac{1}{1+\sin x} \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\, dx = \int \frac{1-\sin x}{\cos^2 x}\, dx$$

$$= \int [\sec^2 x - \sec x \tan x]\, dx \qquad \left[\because \frac{1}{\cos x} = \sec x\,;\ \frac{\sin x}{\cos x} = \tan x\right]$$

$$= \tan x - \sec x + C$$

Example: 3.18

Evaluate $\int \sqrt{1 + \sin 2x}\, dx$

Solution:

Given $\int \sqrt{1 + \sin 2x}\, dx$

$$= \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}\, dx \qquad [\because \sin^2 x + \cos^2 x = 1]$$

$$= \int \sqrt{(\sin x + \cos x)^2\, dx} \qquad [\sin 2x = 2 \sin x \cos x]$$

$$= \int (\sin x + \cos x)\, dx \qquad [(a+b)^2 = a^2 + b^2 + 2ab]$$

$$= (-\cos x + \sin x) + C$$

Example: 3.19

Evaluate $\int \frac{\sin^2 x}{1+\cos x}\, dx$

Solution:

Given $\int \frac{\sin^2 x}{1+\cos x}\, dx = \int \frac{1-\cos^2 x}{1+\cos x}\, dx \qquad [\because \sin^2 x = 1 - \cos^2 x]$

$$= \int \frac{(1-\cos x)\,(1+\cos x)}{(1+\cos x)}\, dx \qquad [\because a^2 - b^2 = (a-b)\,(a+b)]$$

$$= \int (1 - \cos x)\, dx$$

$$= x - \sin x + C$$

Example: 3.20

Evaluate $\int \frac{1}{\sin^2 x \cos^2 x}\, dx$

Solution:

Given $\int \frac{1}{\sin^2 x \cos^2 x}\, dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx \qquad [\because \sin^2 x + \cos^2 x = 1]$

$= \int \left[\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right] dx$

$= \int (\sec^2 x + \text{cosec}^2 x)\, dx$

$= (\tan x - \cot x) + C$

Exercise 3. 1(b)

Evaluate:

1. $\int (x+4)(2x+1)\, dx$ **Ans:** $\frac{2}{3} x^3 + \frac{9}{2} x^2 + 4x + C$

2. $\int \left(\sqrt{x^3} - \sqrt[3]{x^2}\right) dx$ **Ans:** $\frac{1}{3} x^{1/3} - \frac{1}{3} x^{-1/3} + C$

3. $\int \frac{1}{1-\cos x}\, dx$ **Ans:** $-\cos x - \text{cosec}\, x + C$

4. $\int \frac{\cos^2 x}{1-\sin x}\, dx$ **Ans:** $x - \cos x + C$

5. $\int \frac{1}{\sin x \cos^2 x}\, dx$ **Ans:** $\sec x - \log(\text{cosec}\, x + \cot x) + C$

3.1(c) Properties of Definite Integral

We assume that f and g are continuous functions.

1. $\int_a^b C\, f(x)dx = C \int_a^b f(x)dx$, Where C is any constant.
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a < c < b$
4. $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
6. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
7. $\int_a^a f(x)dx = 0$
8. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \quad iff \quad f(2a-x) = f(x)$

9. $\int_0^{2a} f(x)dx = 0 \ \ iff \ \ f(2a-x) = -f(x)$

10. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx\,, & if\ f(x)\ is\ even \\ 0\,, & if\ f(x) is\ odd \end{cases}$

11. If $f(x) \geq 0\ for\ a \leq x \leq b,\ then\ \int_a^b f(x)\,dx \ \geq 0$

12. If $f(x) \geq g(x)\ for\ a \leq x \leq b,\ then\ \int_a^b f(x)\,dx \ \geq \int_a^b g(x)\,dx$

13. If $m \leq f(x) \leq M\ for\ a \leq x \leq b,\ then\ m\,(b-a) \leq \int_a^b f(x)dx \leq M(\,b-a)$

Problems based on Properties of Definite Integral

Example: 3.21

Evaluate $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ **(or)** $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \cdots (1)$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f\,(a-x)dx\right]$$

$$\text{I} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \cdots (2)$$

$$(1) + (2) \Rightarrow 2\,I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right] dx$$

$$= \int_0^{\pi/2} 1 \cdot dx$$

$$= [x]_0^{\pi/2}$$

$$= \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$\therefore I = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

Example: 3.22

Evaluate $\int_0^{\pi/2} \log(\tan x)\,dx$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \log(\tan x)\,dx \cdots (1)$$

$$= \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \log(\cot x)\, dx \quad \cdots (2)$$

$$(1) + (2) \Rightarrow 2I = \int_0^{\pi/2} \log(\tan x)\, dx + \int_0^{\pi/2} \log(\cot x)\, dx$$

$$= \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\, dx$$

$$= \int_0^{\pi/2} \log(\tan x \cdot \cot x)\, dx \qquad [\because \log a + log b = log ab]$$

$$= \int_0^{\pi/2} log\, 1 \cdot dx$$

$$= \int_0^{\pi/2} 0 \cdot dx \quad [\because \log 1 = 0]$$

$$\therefore I = 0$$

Example: 3.23

Evaluate $\int_{-\pi/2}^{\pi/2} sin^{199} x\, dx$

Solution:

Given $\int_{-\pi/2}^{\pi/2} sin^{199} x\, dx$

Let $f(x) = sin^{199} x$

$$\Rightarrow f(-x) = sin^{199}(-x) = [sin(-x)]^{199}$$

$$= [-sin\, x]^{199} = -sin^{199} x = -f(x)$$

$\therefore f(x)$ is an odd function.

Hence $\int_{-\pi/2}^{\pi/2} sin^{199} x\, dx = 0 \qquad [\because \int_{-a}^{a} f(x)dx = 0,\ where\ f(x)$ is odd function.]

Example: 3.24

Evaluate $\int_{-2}^{2} |x+1|\, dx$

Solution:

Given $\int_{-2}^{2} |x+1|\, dx$

$$|x+1| = \begin{cases} -(x+1); & if\ -2 < x < -1 \\ (x+1); & if\ -1 < x < 2 \end{cases}$$

$$\int_{-2}^{2} |x+1|\, dx = \int_{-2}^{-1} |x+1|\, dx + \int_{-1}^{2} |x+1|\, dx$$

$$= \int_{-2}^{-1} -(x+1)\, dx + \int_{-1}^{2} (x+1)\, dx$$

$$= -\left[\frac{x^2}{2} + x\right]_{-2}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{2}$$

$$= -\left[\left(\frac{1}{2} - 1\right) - \left(\frac{4}{2} - 2\right)\right] + \left[\left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= -\left[-\frac{1}{2} - 0\right] + \left[4 - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 4 + \frac{1}{2} = 4 + 1 = 5$$

Example: 3.25

Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx$

Solution:

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\, dx$

$Put\ x = \tan\theta$ $\qquad\qquad dx = sec^2\theta\, d\theta$

$when\ x = 0,\ \tan\theta = 0 \Rightarrow \theta = 0\ and\ when\ x = 1, \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore I = \int_0^{\pi/4} \frac{\log(1+tan\theta)}{(1+tan^2\theta)}\, sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+tan\theta)}{sec^2\theta}\, sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+tan\theta)\, d\theta$$

$$I = \int_0^{\pi/4} \log(1+tanx)\, dx \cdots (1)$$

$$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/4} \log\left[1 + \frac{1-\tan x}{1+\tan x}\right] dx \qquad \left[\because \tan(45^o - A) = \frac{1-tanA}{1+tanA}\right]$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x + 1-\tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right] dx$$

$$I = \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\, dx \cdots (2) \quad \left[\because log\left(\frac{a}{b}\right) = loga - logb\right]$$

$$(1) + (2) \Rightarrow 2I = \int_0^{\pi/4} \log(1+tanx)\, dx + \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\, dx$$

$$= \int_0^{\pi/4} \log 2\, dx$$

$$= \log 2 \int_0^{\pi/4} dx$$

$$= log2[x]_0^{\pi/4}$$

$$= log2\left[\frac{\pi}{4} - 0\right]$$

$$\therefore I = \frac{\pi}{8} log2$$

Exercise 3.1(c)

Evaluate:

1. $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$ **Ans:** $\boldsymbol{\frac{\pi}{4}}$

2. $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$ **Ans: 0**

3. $\int_{-1}^1 \log\left(\frac{4-x}{4+x}\right) dx$ **Ans: 0**

4. $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}\, dx$ **Ans:** $\boldsymbol{\frac{a}{2}}$

5. $\int_{-1}^1 (2 - |x|)\, dx$ **Ans: 3**

3.2 Method of Substitution

The method of substitution is one of the powerful rules when the given integrand is product of two functions.

Let us see the suitable substitution to convert the given integral into a standard form.

The integrand of the form

(i) $\int F\big(f(x)\big) f'(x) dx$ (ii) $\int \big(f(x)\big)^n f'(x) dx$

(iii) $\int \frac{f'(x)}{(f(x))^n} dx$ (iv) $\int \frac{f'(x)}{F(f(x))} dx$

(v) $\int \frac{e^{f(x)}}{f'(x)} dx$ (vi) $\int e^{f(x)} f'(x) dx$

Substitute $u = f(x) \therefore du = f'(x)$ and then proceed.

Algebraic functions:

Example:3.26

(i) Evaluate $\boldsymbol{\int \sqrt{2x+1}\, dx}$**.**

Solution:

Put $u = 2x + 1 \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{2}$

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u} \frac{du}{2} = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{2}{2\times 3}(u)^{3/2} + C = \frac{(2x+1)^{3/2}}{3} + C$$

(ii)Evaluate $\boldsymbol{\int \frac{1}{(ax+b)^4} dx}$.

Solution:

Put u $= ax + b \quad \Rightarrow \quad du = adx \quad \Rightarrow \quad dx = \frac{du}{a}$

$$\int \frac{1}{(ax+b)^4}\,dx = \int \frac{1}{u^4}\frac{du}{a}$$

$$= \frac{1}{a}\int u^{-4}du$$

$$= \frac{1}{a}\left[\frac{u^{-3}}{-3}\right] + C$$

$$= \frac{-1}{3a}\left[\frac{1}{u^3}\right] + C \quad = \frac{-1}{3a}\left[\frac{1}{(ax+b)^3}\right] + C$$

(iii) Evaluate $\int \boldsymbol{x^5\sqrt{x^2+1}dx}$.

Solution:

Put $u = x^2 + 1 \ \Rightarrow x^2 = u - 1; \quad du = 2xdx \Rightarrow xdx = \frac{du}{2}$

$$\int x^5\sqrt{x^2+1}dx = \int x^4\sqrt{x^2+1}\,x\,dx$$

$$= \int \sqrt{u}\,(u-1)^2\,\frac{du}{2}$$

$$= \frac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\,du$$

$$= \frac{1}{2}\int\left(u^{5/2} - 2\,u^{3/2} + u^{1/2}\right)du$$

$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - \frac{2u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C$$

$$= \left(\frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} - \frac{u^{3/2}}{3}\right) + C$$

$$= \left(\frac{(x^2+1)^{7/2}}{7} - \frac{2(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3}\right) + C$$

(iv)Evaluate $\int \frac{\boldsymbol{x^2}}{\sqrt{\boldsymbol{x+5}}}\,\boldsymbol{dx}$

Solution:

Given $\int \frac{x^2}{\sqrt{x+5}}\,dx$

$$Put\ u = \sqrt{x+5} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{x+5}}dx$$

$$\Rightarrow \ 2du = \frac{1}{\sqrt{x+5}}dx$$

$$u^2 = x + 5 \quad \Rightarrow x = u^2 - 5 \quad \Rightarrow \quad x^2 = (u^2-5)^2 = u^4 - 10u^2 + 25$$

$$\int \frac{x^2}{\sqrt{x+5}}\,dx = \int (u^4 - 10u^2 + 25)\,2\,du = 2\int(u^4 - 10u^2 + 25)du$$

$$= 2\left[\frac{u^5}{5} - 10\frac{u^3}{3} + 25u\right] + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50(x+5)^{1/2} + C$$

(v)Evaluate $\int \frac{\mathbf{1}}{\sqrt{x}(\mathbf{1}+\sqrt{x})^{\mathbf{2}}}\ \boldsymbol{dx}$

Solution:

Given $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\ dx$

$Put\ u = 1+\sqrt{x} \quad \Rightarrow\ du = \frac{1}{2\sqrt{x}}dx \quad \Rightarrow\ 2du = \frac{1}{\sqrt{x}}dx$

$$= \int \frac{1}{u^2}\ 2\ du \ = \ 2\int u^{-2}\ du \ = 2\left(\frac{u^{-1}}{-1}\right) + C$$

$$= -\frac{2}{u} + C$$

$$= -\frac{2}{1+\sqrt{x}} + C$$

(vi)Evaluate $\int \frac{(\mathbf{1}+\sqrt{x})^{\mathbf{1/3}}}{\sqrt{x}}\ \boldsymbol{dx}$

Solution:

Given $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\ dx$

$Put \quad u = 1+\sqrt{x} \quad \Rightarrow\ du = \frac{1}{2\sqrt{x}}dx \quad \Rightarrow\ 2du = \frac{1}{\sqrt{x}}dx$

$$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\ dx \ = \ \int u^{1/3}\ 2\ du \ = \ 2\int u^{1/3}\ du = 2\ \frac{u^{4/3}}{(4/3)} + C$$

$$= \frac{3}{2}(u)^{4/3} + C$$

$$= \frac{3}{2}(1+\sqrt{x})^{4/3} + C$$

(vii)Evaluate $\int (\boldsymbol{x}+\mathbf{1})\sqrt{\mathbf{2}\boldsymbol{x}+\boldsymbol{x}^{\mathbf{2}}}\ \boldsymbol{dx}$

Solution:

Given $\int (x+1)\sqrt{2x+x^2}dx$

$Put\ u = 2x + u^2 \quad \Rightarrow du = (2+2x)dx \quad \Rightarrow du = 2(1+x)dx$

$$\Rightarrow \frac{du}{2} = (1+x)dx$$

$$= \int \sqrt{u}\frac{du}{2} \ = \ \frac{1}{2}\int \frac{u^{3/2}}{(3/2)} + C$$

$$= \frac{1}{3}u^{3/2} + \mathrm{C}$$

$$= \frac{1}{3}(2x+x^2)^{3/2} + C$$

(viii)Evaluate $\int \frac{\boldsymbol{x}}{\sqrt{\mathbf{1-4}\boldsymbol{x}^{\mathbf{2}}}}\,\boldsymbol{dx}$

Solution:

Given $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Put $u = 1 - 4x^2 \quad \Rightarrow\ du = -8x\,dx \quad \Rightarrow\ xdx = -\frac{du}{8}$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{u}}\left(-\frac{du}{8}\right) = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}\left(\frac{u^{1/2}}{1/2}\right) + C$$

$$= -\frac{2\sqrt{u}}{8} + C$$

$$= -\frac{\sqrt{1-4x^2}}{4} + C$$

(ix)Evaluate $\int \frac{(\mathbf{4}\boldsymbol{x}+\mathbf{3})}{(\mathbf{2}\boldsymbol{x}^{\mathbf{2}}+\mathbf{3}\boldsymbol{x}+\mathbf{5})}\,\boldsymbol{dx}$

Solution:

Given $\int \frac{(4x+3)}{(2x^2+3x+5)}\,dx$

Put $u = 2x^2 + 3x + 5 \quad \Rightarrow du = 4xdx + 3dx$

$\Rightarrow du = (4x+3)dx$

$$\int \frac{(4x+3)}{(2x^2+3x+5)}\,dx = \int \frac{du}{u} = logu + C$$

$$= \log(2x^2 + 3x + 5) + C$$

Logarithmic functions:

Example :3.27

(i) Evaluate $\int \frac{\boldsymbol{logx}}{\boldsymbol{x}}\,\boldsymbol{dx}$

Solution:

Given $\int \frac{logx}{x}\,dx$

Put $u = logx \ \Rightarrow\ du = \frac{1}{x}\,dx$

$$\int \frac{logx}{x}\,dx = \int udu = \frac{u^2}{2} + C \quad = \frac{(logx)^2}{2} + C$$

(ii)Evaluate: $\int \frac{(\boldsymbol{logx})^{\mathbf{2}}}{\boldsymbol{x}}\,\boldsymbol{dx}$

Solution:

Given $\int \frac{(logx)^2}{x}\ dx$

Put $u = logx \quad \Rightarrow du = \frac{1}{x}\ dx$

$$\int \frac{(logx)^2}{x}\ dx\ =\ \int u^2 du = \frac{u^3}{3} + C\ \ = \frac{(logx)^3}{3} + C$$

(iii)Evaluate $\int \frac{\boldsymbol{sin(2+logx)}}{\boldsymbol{x}}\ \boldsymbol{dx}$

Solution:

Given $\int \frac{sin(2+logx)}{x}\ dx$

Put $u = 2 + logx \quad \Rightarrow du = \frac{1}{x}\ dx$

$$\int \frac{sin(2+logx)}{x}\ dx\ =\ \int sin\ u\ du = -cos\ u + C$$

$$= -\cos(2 + logx) + C$$

(iv)Evaluate $\int \frac{\boldsymbol{dx}}{\boldsymbol{x\sqrt{logx}}}$

Solution:

Given $\int \frac{dx}{x\sqrt{logx}}$

Put $u\ =\ logx \quad \Rightarrow du = \frac{1}{x}\ dx$

$$\int \frac{dx}{x\sqrt{logx}} =\ \int \frac{du}{\sqrt{u}} = 2\sqrt{u}\ +\ C \quad = 2\sqrt{logx} +\ C$$

(v)Evaluate $\int \boldsymbol{secx}\ \mathbf{log}(\boldsymbol{secx} + \boldsymbol{tanx})\ \boldsymbol{dx}$

Solution:

Given $\int secx\ \log(secx + tanx)\ dx$

Put $u = \log(secx + tanx) \quad \Rightarrow\ du = \frac{1}{(secx+tanx)}(secxtanx + sec^2x)dx$

$$\Rightarrow du = \frac{sec(tanx+secx)}{(secx+tanx)}dx$$

$$= \int udu = \frac{u^2}{2} +\ C$$

$$= \frac{1}{2}[log(secx + tanx)]^2 + C$$

Exponential functions

Example:3.28

(i)Evaluate$\int \boldsymbol{e^{cosx}sinx\ dx}$

Solution:

Given $\int e^{cosx} sinx\, dx$

$$Put\ u = e^{cosx} \quad \Rightarrow \quad du = e^{cosx}(-sinx)dx$$

$$\int e^{cosx} sinx\, dx = \int (-du) \quad = -\int du \quad = -u + C \quad = -e^{cosx} + C$$

(ii)Evaluate $\int e^{x^3} x^2 dx$

Solution:

Given $\int e^{x^3} x^2 dx$

$$Put\ \ u = e^{x^3} \quad \Rightarrow \quad du = e^{x^3} 3x^2 dx \quad \Rightarrow \quad \frac{du}{3} = e^{x^3} x^2 dx$$

$$\int e^{x^3} x^2 dx = \int \frac{du}{3} \quad = \frac{1}{3}\int du \quad = \frac{1}{3}u + C \quad = \frac{1}{3}e^{x^3} + C$$

(iii)Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution:

Given $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$Put\ u = e^{\sqrt{x}} \quad \Rightarrow \quad du = e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \quad 2du = \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int 2du \quad = 2\int du \quad = 2u + C \quad = 2e^{\sqrt{x}} + C$$

(iv)Evaluate $\int \frac{e^{tan^{-1}x}}{1+x^2}\, dx$

Solution:

Given $\int \frac{e^{tan^{-1}x}}{1+x^2}\, dx$

Put $u = tan^{-1}x$ $\qquad du = \frac{1}{1+x^2}\, dx$

$$\int \frac{e^{tan^{-1}x}}{1+x^2}\, dx = \int e^u du = e^u + C$$

$$= e^{tan^{-1}x} + C$$

(v)Evaluate $\int \frac{1}{e^x + e^{-x}}\, dx$

Solution:

Given $\int \frac{1}{e^x + \frac{1}{e^x}}\, dx \quad = \quad \int \frac{e^x dx}{e^{2x}+1}$

Put $e^x = u \quad \Rightarrow \quad e^x dx = du$

$$= \int \frac{du}{u^2+1}$$

$$= tan^{-1}u + C \quad = tan^{-1}e^{x} + C$$

Trigonometric functions

Example: 3.29

(i)Evaluate $\int \boldsymbol{cos^3\theta sin\theta\, d\theta}$

Solution:

Given $\int cos^3\theta sin\theta\, d\theta$

$$Put\ u = cos\theta \quad \Rightarrow \quad du = -sin\theta d\theta$$

$$\int cos^3\theta sin\theta\, d\theta = \int u^3(-du) \quad = -\int u^3 du \quad = -\frac{u^4}{4} + C \quad = -\frac{cos^4\theta}{4} + C$$

(ii)Evaluate $\int \boldsymbol{xsin(x^2)\, dx}$

Solution:

Given $\int xsin(x^2)\, dx$

$$Put\ u = x^2 \quad \Rightarrow \quad du = 2xdx \quad \Rightarrow \quad \frac{du}{2} = xdx$$

$$\int xsin(x^2)\, dx = \int \sin(u)\frac{du}{2} \quad = \frac{1}{2}\int sinudu \quad = \frac{1}{2}(-cosu) + C \quad = -\frac{1}{2}cos(x^2) + C$$

(iii)Evaluate $\int_{0}^{\pi/2} \boldsymbol{cosxsin(sinx)dx}$

Solution:

Given $\int_{0}^{\pi/2} cosxsin(sinx)dx$

$$Put\ u = sinx \qquad \Rightarrow du = cosxdx$$

$$when\ x = 0 \ \Rightarrow u = 0$$

$$x = \frac{\pi}{2} \Rightarrow u = 1$$

$$\int_{0}^{\pi/2} cosxsin(sinx)dx = \int_{0}^{1} sinudu = [-cosu]_{0}^{1} = (-cos1) - 1 = 1 - cos1$$

(iv)Evaluate $\int \boldsymbol{x^3 \cos(x^4 + 2)dx}$

Solution:

Given $\int x^3 \cos(x^4 + 2)dx$

Put $u = x^4 + 2 \quad \Rightarrow du = 4x^3dx \quad \Rightarrow x^3\, dx = \frac{du}{4}$

$$\int x^3 \cos(x^4 + 2)dx \ = \int \cos u\ \frac{du}{4}$$

$$= \frac{\sin u}{4} + C$$

$$= \frac{\sin(x^4+2)}{4} + C$$

(v)Evaluate $\int \frac{1}{(1+x^2)tan^{-1}x} dx$

Solution:

Given $\int \frac{1}{(1+x^2)tan^{-1}x} dx$

$$Put\ u = tan^{-1}x \quad \Rightarrow \quad du = \frac{1}{1+x^2} dx$$

$$\int \frac{1}{(1+x^2)tan^{-1}x} dx = \int \frac{1}{u} du = logu + C = \log(tan^{-1}x) + C$$

(vi)Evaluate $\int e^{tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right] dx$

Solution:

Given $\int e^{tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right] dx$

$$Put\ u = tan^{-1}x \quad \Rightarrow \quad du = \frac{1}{1+x^2} dx$$

$$tanu = x \quad \Rightarrow 1 + x + x^2 = 1 + tanu + tan^2u = tanu + sec^2u$$

$$\int e^{tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right] dx = \int e^u(tanu + sec^2u)du \qquad [\because\ 1 + tan^2x = sec^2x]$$

$$Put\ \ t = e^u tanu \qquad \Rightarrow dt = (e^u sec^2u + tanue^u)du$$

$$\int e^u(tanu + sec^2u)du = \int dt$$

$$= t + C$$

$$= [e^u tanu] + C = xe^{tan^{-1}x} + C$$

(vii)Evaluate $\int \frac{sec^2x}{5+4tanx} dx$

Solution:

Given $\int \frac{sec^2x}{5+4tanx} dx$

$$Put\ u = 5 + 4tanx \quad \Rightarrow du = 4sec^2xdx \quad \Rightarrow \frac{du}{4} = sec^2xdx$$

$$\int \frac{sec^2x}{5+4tanx} dx = \int \frac{1}{u}\frac{du}{4} = \frac{1}{4}logu + C = \frac{1}{4}log(5 + 4tanx) + C$$

(viii)Evaluate $\int \frac{\sin 2x}{1+\cos^2 x} dx$

Solution:

Given $\int \frac{sin\ 2x}{1+ cos^2 x} dx$

$Put\ u = 1 + cos^2 x \Rightarrow du = 2\cos x(-\sin x)dx = -sin\ 2xdx\ [\because 2sinx \cos x = \sin 2x]$

$$\int \frac{sin\ 2x}{1+ cos^2 x} dx = \int \frac{1}{u}(-du)$$

$$= -\int \frac{1}{u} du = -\log u + C$$

$$= -\log(1 + cos^2 x) + C$$

Exercise 3. 2

Evaluate:

1. $\int x^2 \sqrt{x^3+1}\, dx$ **Ans:** $\frac{2}{9}(x^3 + 1)^{3/2} + C$

2. $\int \frac{1}{x^2}\sqrt{2 - \frac{1}{x}}\, dx$ **Ans:** $\frac{2}{3}\left(2 - \frac{1}{x}\right)^{3/2} + C$

3. $\int \frac{1}{(3x-4)^{3/2}}\, dx$ **Ans:** $-\frac{2}{3}\frac{1}{\sqrt{3x-4}} + C$

4. $\int \frac{1}{x\log x}\, dx$ **Ans:** $\log(\log x) + C$

5. $\int \frac{sec^2(\log x)}{x}\, dx$ **Ans:** $\tan(\log x) + C$

6. $\int_1^e \frac{log\ x}{x}\, dx$ **Ans:** $\frac{1}{2}$

7. $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$ **Ans:** $e - \sqrt{e}$

8. $\int_0^1 \frac{e^x + 1}{e^x + x}\, dx$ **Ans:** $log\ (e+1)$

9. $\int e^x\sqrt{1+ e^x}\, dx$ **Ans:** $\frac{2}{3}(1+e^x)^{3/2} + C$

10. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$ **Ans:** $-2\, cos\sqrt{x} + C$

11. $\int (1+\sin x)^4 \cos x\, dx$ **Ans:** $\frac{1}{5}(1+sin\ x)^5 + C$

12. $\int \frac{dx}{sin^{-1}x\sqrt{1-x^2}}$ **Ans:** $log\ (sin^{-1}x) + C$

3.3Techniques of integration

3.3(a) Integration by parts

If the integrand is either a product or quotient of polynomial and a transcendental function such as trigonometric, exponential or logarithmic function then have to develop different methods to evaluate them.

The most powerful rule to integrate the product of two differentiable function is integration by parts which corresponds to the product rule for differentiation.

The product rule states that if f and g are differentiable functions then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Now integrating both side of the equation, we have

$$\int \frac{d}{dx}[f(x)g(x)] = \int \Big(f(x)g'(x) + g(x)f'(x)\Big)dx$$

$f(x)\, g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx$

$\int f(x)g'(x)dx = f(x)\, g(x) - \int g(x)f'(x)dx$

The above formula is called the integration by parts.

Let if we take $u = f(x)$ and $v = g(x)$

Then the above formula becomes $\int udv = uv - \int vdu$

To choose u, we should follow the following order

I – Inverse function

L – Logarithmic function

A – Algebraic function

T – Trigonometric function

E – Exponential function

Note:

The generalized integration by parts formula is known as Bernoulli's formula

Bernoulli's formula states that

$$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Where $v_1, v_2, v_3, \ldots$ are functions obtained by integrating v successively with respect to x and $u', u'', \ldots$ are functions obtained by differentiating u successively with respect to x.

Problems based on Integration by parts

Example: 3.30

Evaluate $\int xe^{-x}dx$

Solution:

Let $u = x$ $\qquad dv = e^{-x}dx$

$du = dx$ $\qquad v = -e^{-x}$

$$\int udv = uv - \int vdu$$

$$\int xe^{-x}dx \ = x(-e^{-x}) - \ \int -e^{-x}\, dx$$

$$= \ -xe^{-x} + \ \int e^{-x}\, dx$$

$$= -xe^{-x} + (-e^{-x}) + C \ = -(xe^{-x} + e^{-x} + C)$$

$$= \ -e^{-x}(x+1) + C$$

Example: 3.31

Evaluate $\int \boldsymbol{xsinxdx}$

Solution:

Let $u = x$ $\qquad dv = sinxdx$

$du = dx$ $\qquad v = \int sinxdx = \ -cosx$

$$\int udv = uv - \int vdu$$

$$\int xsinxdx = (x)(-cosx) - \int(-\cos x)dx$$

$$= -xcosx + \int cosxdx = \ -xcosx + sinx + c$$

Example :3.32

Evaluate $\int \boldsymbol{x^4logxdx}$

Solution:

Let $u = logx$ $\qquad dv = \ x^4dx$

$du \ = \ \frac{1}{x}dxv = \ \int x^4dx = \ \frac{x^5}{5}$

$$\int udv = uv - \int vdu$$

$$\int x^4logxdx = (logx)\left(\frac{x^5}{5}\right) - \int \frac{x^5}{5}\frac{1}{x}dx$$

$$= \ \frac{x^5}{5}logx - \frac{1}{5}\int \ x^4dx$$

$$= \ \frac{x^5}{5}logx - \frac{1}{5}\frac{x^5}{5} + \ C \ = \ \frac{x^5}{5}logx - \frac{x^5}{25} + \ C$$

Example :3.33

Evaluate $\int(\log \boldsymbol{x})^2$ dx

Solution:

Let $u = (\log x)^2$ $\qquad dv = dx$

$du = 2\ logx\left(\frac{1}{x}\right)dx$ $\qquad v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int (\log x)^2 dx = (\log x)^2 x - \int \left(x\, 2\, logx \left(\frac{1}{x}\right) dx \right)$$

$$= x(\log x)^2 - 2 \int logxdx \ldots (1)$$

Take $\int logxdx$

Let $u = logx$ $\quad\quad dv = dx$

$du = \frac{1}{x}dx$ $\quad\quad v = \int dx = x$

$$\int udv = uv - \int vdu$$

$$\int logxdx = (logx)(x) - \int x\frac{1}{x}dx = xlogx - \int dx = x\, logx - x$$

$$(1) \Rightarrow \int (\log x)^2 dx = x(\log x)^2 - 2\,[xlogx - x] + C$$

Example :3.34

Evaluate $\int \boldsymbol{xsec^2 2xdx}$

Solution:

Let $u = x$ $\quad\quad dv = sec^2 2xdx$

$du = dxv = \int sec^2 2xdx = \frac{\tan 2x}{2}$

$$\int udv = uv - \int vdu$$

$$\int xsec^2 2xdx = (x)\left(\frac{\tan 2x}{2}\right) - \int \frac{\tan 2x}{2}dx$$

$$= \frac{1}{2}x\tan 2x - \frac{1}{2}\int \tan 2xdx$$

$$= \frac{1}{2}x\tan 2x - \frac{1}{2}\left[\frac{\log(\sec 2x)}{2}\right] + C$$

$$= \frac{1}{2}x\tan 2x - \frac{1}{4}\log(\sec 2x) + C$$

Example :3.35

Evaluate $\int \boldsymbol{xsin^2 xdx}$

Solution:

Let $u = x$ $\quad\quad dv = sin^2 xdx$

$du = dx$ $\quad\quad v = \frac{1}{2}\int (1 - cos2x)dx = \frac{1}{2}\ (x - \frac{sin2x}{2})$

$$\int udv = uv - \int vdu$$

$$\int xsin^2 xdx = \frac{x}{2}\left(x - \frac{sin2x}{2}\right) - \frac{1}{2}\int \left(x - \frac{sin2x}{2}\right) dx$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{1}{2}\int \left(x - \frac{sin2x}{2}\right) dx$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{1}{2}\left(\frac{x^2}{2} + \frac{cos2x}{4}\right) + C$$

$$= \frac{x^2}{2} - \frac{xsin2x}{4} - \frac{x^2}{4} - \frac{cos2x}{8} + C$$

$$= \frac{x^2}{4} - \frac{xsin2x}{4} - \frac{cos2x}{8} + C$$

Example :3.36

Evaluate $\int xe^{x/2}\, dx$

Solution:

Let $u = x$ $\qquad dv = e^{x/2}\, dx$

$du = dx$ $\qquad v = \int e^{x/2}\, dx = \frac{e^{x/2}}{1/2} = 2e^{x/2}$

$$\int udv = uv - \int vdu$$

$$\int xe^{x/2}\, dx = (x)\left(2e^{x/2}\right) - 2\int 2e^{x/2}\, dx$$

$$= 2\, xe^{x/2} - 2\int e^{x/2}\, dx$$

$$= 2\, xe^{x/2} - 2\frac{e^{x/2}}{1/2} + C$$

$$= 2\, xe^{x/2} - 4e^{x/2} + C$$

$$= 2(x-2)e^{x/2} + C$$

Example :3.37

Evaluate $\int \frac{x}{1+cosx}\, dx$

Solution:

$$\int \frac{x}{1+cosx} dx = \int \frac{x}{2cos^2\frac{x}{2}} dx \qquad \left[\because 1 + cosx = 2cos^2\frac{x}{2}\right]$$

$$= \frac{1}{2}\int xsec^2\frac{x}{2} dx \cdots (1)$$

Let $u = x$ $\qquad dv = sec^2\frac{x}{2} dx$

$du = dx$ $\qquad v = \int sec^2\frac{x}{2} dx = \frac{\tan\left(\frac{x}{2}\right)}{\frac{1}{2}} = 2\tan\frac{x}{2}$

$$\int udv = uv - \int vdu$$

$$(1) \Rightarrow \int \frac{x}{1+cosx} dx = \frac{1}{2}\left[x\left(2\tan\frac{x}{2}\right) - \int 2\tan\frac{x}{2} dx\right]$$

$$= x\tan\frac{x}{2} - \frac{log\left[\sec\left(\frac{x}{2}\right)\right]}{\frac{1}{2}} + C$$

$$= x\tan\frac{x}{2} - 2\,log\left[\sec\left(\frac{x}{2}\right)\right] + C$$

Example :3.38

Evaluate $\int \frac{x}{1+sinx}dx$

Solution:

$$\int \frac{x}{1+sinx}dx = \int \frac{x\,(1-sinx)}{(1+sinx)(1-sinx)}dx$$

$$= \int \frac{x\,(1-sinx)}{1-sin^2x}dx = \int \frac{x\,(1-sinx)}{cos^2x}dx$$

$$= \int (xsec^2x - xsecxtanx)\,dx$$

$$= \int xsec^2xdx - \int xsecxtanxdx \;\ldots(1)$$

Take $\int xsec^2xdx$

Let $u = x$ $\qquad dv = sec^2xdx$

$du = dx$ $\qquad v = \int sec^2xdx = tanx$

$$\int udv = uv - \int vdu$$

$$\int xsec^2xdx = (x)(tanx) - \int tanxdx$$

$$= xtanx - \log(secx) \ldots(2)$$

Take $\int xsecxtanxdx$

Let $u = x$ $\qquad dv = secxtanxdx$

$du = dx$ $\qquad v = \int secxtanxdx = secx$

$$\int udv = uv - \int vdu \quad = \int xsecxtanxdx = (x)(secx) - \int secxdx$$

$$= xsecx - \log(secx + \tan x) \quad \ldots(3)$$

(1) $\Rightarrow \int \frac{x}{1+sinx}dx = xtanx - \log(secx) - xsecx + \log(secx + tanx) + C \;[\because by(2)and(3)\,]$

Example :3.39

Evaluate $\int \frac{x}{secx+1}dx$

Solution:

$$\int \frac{x}{secx+1}dx = \int \frac{x}{\left(\frac{1}{cosx}+1\right)}dx = \int \frac{xcosx}{1+cosx}dx = \int \left[x - \frac{x}{1+\cos x}\right]dx$$

$$= \int xdx - \int \frac{x}{1+\cos x}dx$$

$$= \frac{x^2}{2} - xtan\frac{x}{2} + 2\log\left[\sec\left(\frac{x}{2}\right)\right] + C \qquad \text{(by problem 3.37)}$$

Example :3.40

Evaluate $\int (x^2 + 2x)\, cosxdx$

Solution:

Let $u = x^2 + 2x$, $u' = 2x + 2,$ $u'' = 2,$ $u''' = 0$

$dv = cosxdx,$ $v = sinx,$ $v_1 = -cosx,$ $v_2 = -sinx$

$$\int udv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int (x^2 + 2x)\, cosxdx = (x^2 + 2x)sinx - (2x + 2)(-cosx) + (2)(-sinx) + C$$

$$= (x^2 + 2x - 2)sinx + (2x + 2)(cosx) + C$$

Example :3.41

Evaluate $\int (x^2 e^{2x})\, dx$

Solution:

Let $u = x^2$, $u' = 2x,$ $u'' = 2,$

$dv = e^{2x}dx,$ $v = \frac{e^{2x}}{2},$ $v_1 = \frac{e^{2x}}{4}$, $v_2 = \frac{e^{2x}}{8}$

$$\int udv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int (x^2 e^{2x})\, dx = (x^2)\frac{e^{2x}}{2} - (2x)\frac{e^{2x}}{4} + (2)\frac{e^{2x}}{8} + C$$

$$= (x^2)\frac{e^{2x}}{2} - (x)\frac{e^{2x}}{2} + \frac{e^{2x}}{4} + C$$

Example :3.42

Evaluate $\int e^x cosxdx$

Solution:

Let $u = e^x$ $dv = cosxdx$

$du = e^x dx$ $v = \int cosxdx = sinx$

$$\int udv = uv - \int vdu$$

$$I = \int e^x cosxdx = e^x sinx - \int sinxe^x dx \quad \dots (1)$$

Take $\int e^x sinxdx$

Let $u = e^x$ $dv = sin\, xdx$

$du = e^x dx$ $v = \int sinxdx = -cosx$

$\int udv = uv - \int vdu$

$$\int e^x sinx dx = (e^x)(-cosx) - \int(-cosx)(e^x)\,dx$$

$$= -e^x cosx + \int e^x cosx dx = -e^x cosx + I$$

$$(1) \Rightarrow I = e^x sinx - [-e^x cosx + I] + C$$

$$I = e^x sinx + e^x cosx - I + C$$

$$2I = e^x sinx + e^x cosx + C$$

$$I = \frac{1}{2}[e^x sinx + e^x cosx] + C$$

$$\therefore \int e^x cosx dx = \frac{e^x}{2}[sinx + cosx] + C$$

Example :3.43

Evaluate $\int e^{2x} sinx dx$

Solution:

$$I = \int e^{2x} sinx dx \quad \dots (1)$$

Let $u = sinx$ $\qquad dv = e^{2x}dx$

$du = cosx\,dx$ $\qquad v = \frac{e^{2x}}{2}$

$$\int udv = uv - \int vdu$$

$$I = sinx\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,cosx\,dx \quad = \frac{e^{2x}}{2}sinx - \frac{1}{2}I_1 \dots (2)$$

Take $I_1 = \int e^{2x} \cos x\, dx$

Let $u = cosx$ $\qquad dv = e^{2x}dx$

$du = -sinxdx$ $\qquad v = \frac{e^{2x}}{2}$

$$I_1 = cosx\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}(-sinx)dx$$

$$= \frac{e^{2x}}{2}\,cosx + \frac{1}{2}\int e^{2x} sinxdx$$

$$= \frac{e^{2x}}{2}\,cosx + \frac{1}{2}I$$

$$(2) \Rightarrow I = \frac{e^{2x}}{2}\,sinx - \frac{1}{2}\left[\frac{e^{2x}}{2}\,cosx + \frac{1}{2}I\right]$$

$$I = \frac{e^{2x}}{2}sinx - \frac{e^{2x}}{4}\,cosx - \frac{1}{4}I$$

$$I + \frac{1}{4}I = \frac{e^{2x}}{2}sinx - \frac{e^{2x}}{4}\,cosx$$

$$\frac{5}{4}\,I = \frac{e^{2x}}{4}(2\,sinx - cosx)$$

$\therefore \; I = \frac{e^{2x}}{5}(2\,sinx - cosx) + \; C$

Example :3.44

Evaluate $\int tan^{-1}xdx.\; Also find \int_0^1 tan^{-1}xdx$

Solution:

Let $u = tan^{-1}x$ $\qquad dv = dx$

$du = \frac{1}{1+x^2}dx v = \int dx = x$

$\int udv = uv - \int vdu$

$\int tan^{-1}xdx = xtan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx$

$= xtan^{-1}x - \int\left(\frac{x}{1+x^2}\right)dx$ ... (1)

Take $\int\left(\frac{x}{1+x^2}\right)dx$

Put $t = \; 1 + x^2$, $\; dt = 2xdx$

$\int\left(\frac{x}{1+x^2}\right)dx \quad = \; \int\frac{1}{t}\frac{1}{2}dt = \; \frac{1}{2}\int\frac{1}{t}dt \; = \; \frac{1}{2}logt = \; \frac{1}{2}\log(\,1 + x^2)$

(1) ⇒ $\int tan^{-1}xdx = \; xtan^{-1}x - \frac{1}{2}\log(1 + x^2) + C$ ... (2)

To find $\int_0^1 tan^{-1}x$

(2) ⇒ $\int_0^1 tan^{-1}x = [xtan^{-1}x]_0^1 - \left[\frac{1}{2}\log(1 + x^2)\right]_0^1$

$= \; tan^{-1}1 - 0 - \left[\frac{1}{2}log2 - \frac{1}{2}log1\right]$

$= \; \frac{\pi}{4} - \frac{1}{2}log2 \qquad [\because \log 1 = 0]$

Example :3.45

Evaluate $\int sin^{-1}x$ dx

Solution:

Let $u = sin^{-1}x$ $\qquad dv = \; dx$

$du = \frac{1}{\sqrt{1-x^2}}dx v = \int dx = x$

$\int udv = uv - \int vdu$

$\int sin^{-1}x$dx $\quad = \quad xsin^{-1}x - \int x\frac{1}{\sqrt{1-x^2}}dx$

$= \quad xsin^{-1}x - \int\frac{x}{\sqrt{1-x^2}}dx$ $\qquad$ … (1)

Take $\int \frac{x}{\sqrt{1-x^2}} dx$

Put $t = 1 - x^2$, $dt = -2xdx$, $\frac{-dt}{2} = xdx$

$$= -\frac{1}{2} \int \sqrt{\frac{1}{\sqrt{t}}} dt = \frac{-1}{2}\left(2\sqrt{t}\right) = -\sqrt{t} = -\sqrt{1 - x^2}$$

(1) ⇒ $\int sin^{-1}x$ dx = x $sin^{-1}x + \sqrt{1 - x^2} + C$

Exercise 3.3(a)

Evaluate:

1. $\int x \cos 5x\, dx$ **Ans:** $\frac{1}{5} x \sin 5x + \frac{1}{25} \cos 5x + C$

2. $\int x^2 \log x\, dx$ **Ans:** $\frac{x^3}{3} \log x - \frac{1}{9} x^3 + C$

3. $\int t^2 \cos t\ dt$ **Ans:** $t^2\, sint + 2t \cos t - 2 sint + C$

4. $\int_0^{1/2} x \cos \pi\, x\, dx$ **Ans:** $\frac{\pi - 2}{2\pi^2}$

5. $\int x^3\, e^{2x}\, dx$ **Ans:** $\frac{e^{2x}}{8}\ [4x^3 - 6x^2 + 6x - 3] + C$

6. $\int x^4\, e^x\, dx$ **Ans:** $e^x[x^4 - 4x^3 + 12x^2 - 24x + 24] + C$

7. $\int e^{2t} \sin 3t\ dt$ **Ans:** $\frac{e^{2t}}{13}\ [2 \sin 3t - 3 \cos 3t] + C$

8. $\int e^{2\theta} \cos 3\theta\, dx\theta$ **Ans:** $\frac{e^{2\theta}}{13}\ [3 \sin 3\theta + 2 \cos 3\theta] + C$

9. $\int sec^{-1}x\ dx$ **Ans:** $x\, sec^{-1} x - \log\left(x + \sqrt{x^2 + 1}\right) + C$

10. $\int_0^{1/2} cos^{-1}x\, dx$ **Ans:** $\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$

11. $\int x^2\, sin^{-1}x\, dx$ **Ans:** $\frac{x^3}{3}\, sin^{-1}x + \frac{1}{9}\sqrt{1 - x^2}\,(2 + x^2) + C$

12. $\int x\, tan^{-1}x\, dx$ **Ans:** $\frac{1}{2}\ [(x^2 + 1)\, tan^{-1}x - x] + C$

3.3(b) Reduction Formula

(I) Find the reduction formula for $\int sin^n x\, dx\, ; n \geq 2$ is an integer

Solution:

Consider $I_n = \int sin^n x\, dx = \int sin^{n-1}x\, sinx\, dx$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = sin^{n-1}x$ $\qquad dv = sinx\, dx;$

$$du = (n-1)sin^{n-2}x\ cosx\ dx \qquad v = \int sinx\ dx = -cosx$$

$$I_n = -cosx\ sin^{n-1}x - \int(-cosx)(n-1)sin^{n-2}\ x\ cosx\ dx$$

$$= -cosx\ sin^{n-1}x + (n-1)\int cos^2\ xsin^{n-2}x\ \ dx$$

$$= -cosx\ sin^{n-1}x + (n-1)\int(1-sin^2x)\ sin^{n-2}x\ \ dx$$

$$= -cosx\ sin^{n-1}x + (n-1)\int\ sin^{n-2}x\ \ dx - (n-1)\int sin^n x\ dx$$

$$= -cosx\ sin^{n-1}x + (n-1)\int\ sin^{n-2}x\ \ dx - (n-1)I_n$$

$$I_n + (n-1)I_n = -cosx\ sin^{n-1}x + (n-1)\int\ sin^{n-2}x\ \ dx$$

$$nI_n = -cosx\ sin^{n-1}x + (n-1)\int\ sin^{n-2}x\ \ dx$$

$$I_n = -\frac{cosx\ sin^{n-1}x}{n} + \frac{(n-1)}{n}\int\ sin^{n-2}x\ \ dx$$

The ultimate integral is I_0 or I_1

n even: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd: $I_1 = \int sinx\ dx = -cosx + C$ [Put n = 1 in (1)]

(II) Find the reduction formula for $\int cos^n x\ dx; n \geq 2$ is an integer

Solution:

Consider $I_n = \int cos^n x\ dx = \int cos^{n-1}x\ cosx\ dx$

We know by the method of integration by part

$$\int u\ dv = uv - \int v\ du$$

Let $u = cos^{n-1}x$ $\qquad dv = cosx\ dx$

$$du = (n-1)cos^{n-2}x\ (-sinx)dx \qquad v = \int cosx\ dx = sinx$$

$$I_n = sinx\ cos^{n-1}x - \int(sinx)[-(n-1)cos^{n-2}\ x\ sinx]\ dx$$

$$= sinx\ cos^{n-1}x + (n-1)\int sin^2\ xcos^{n-2}x\ \ dx$$

$$= sinx\ cos^{n-1}x + (n-1)\int(1-cos^2x)\ cos^{n-2}x\ \ dx$$

$$= sinx\ cos^{n-1}x + (n-1)\int\ cos^{n-2}x\ \ dx - (n-1)\int cos^n x\ dx$$

$$= sinx\ cos^{n-1}x + (n-1)\int\ cos^{n-2}x\ \ dx - (n-1)I_n$$

$$I_n + (n-1)I_n = sinx\ cos^{n-1}x + (n-1)\int\ cos^{n-2}x\ \ dx$$

$$nI_n = sinx\ cos^{n-1}x + (n-1)\int\ cos^{n-2}x\ \ dx$$

$$I_n = \frac{sinx\ cos^{n-1}x}{n} + \frac{(n-1)}{n}\int\ cos^{n-2}x\ \ dx$$

The ultimate integral is I_0 or I_1

n even: $I_0 = \int dx = x + C$ [Put n = 0 in (1)]

n odd: $I_1 = \int cosx\ dx = sinx + C$ [Put n = 1 in (1)]

(III)Find the reduction formula for $\int_0^{\pi/2} sin^n x\ dx$

Solution:

Consider $I_n = \int_0^{\pi/2} sin^n x\ dx$

We know that $\int sin^n x\ dx = -\frac{cosx\ sin^{n-1}x}{n} + \frac{n-1}{n}\int sin^{n-2}x\ dx$

$$\int_0^{\pi/2} sin^n x\ dx = \left[-\frac{cosx\ sin^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} sin^{n-2} x\ dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} sin^{n-2} x\ dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} sin^{n-4} x\ dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} sin^{n-6} x\ dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots I$$

If n is even then,

$$I = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

If n is odd then,

$$I = \int_0^{\pi/2} sinx\ dx = (-cosx)_0^{\pi/2} = -cos\frac{\pi}{2} + cos0 = 0 + 1 = 1$$

Thus,

$$\int_0^{\pi/2} sin^n x\ dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & if\ \ n\ is\ even \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}\cdot 1, & if\ n\ is\ odd \end{cases}$$

(IV)Find the reduction formula for $\int_0^{\pi/2} cos^n x\ dx$

Solution:

Consider $I_n = \int_0^{\pi/2} cos^n x\ dx$

We know that $\int cos^n x\ dx = \frac{sinx\ cos^{n-1}x}{n} + \frac{(n-1)}{n}\int cos^{n-2}x\ dx$

$$\int_0^{\pi/2} cos^n x\ dx = \left[\frac{sinx\ cos^{n-1}x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} cos^{n-2} x\ dx$$

$$= \ 0 + \frac{n-1}{n}\int_0^{\pi/2} cos^{n-2}\, x\ d$$

$$= \ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} cos^{n-4}\, x\ dx$$

$$= \ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} cos^{n-6}\, x\ dx$$

$$= \ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\ \cdots I$$

If n is even then,

$$I \ = \ \int_0^{\pi/2} dx \ = \ (x)_0^{\pi/2} \ = \frac{\pi}{2} - \ 0 \ = \ \frac{\pi}{2}$$

If n is odd then,

$$I \ = \ \int_0^{\pi/2} cosx\ dx \ = \ (sinx)_0^{\pi/2} \ = sin\ \frac{\pi}{2} - sin0 \ = \ 1 - 0 = \ 1$$

Thus,

$$\int_0^{\pi/2} cos^n x\ dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\ \cdots\frac{1}{2}\cdot\frac{\pi}{2}, & if \ \ n\ is\ \ even \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\ \cdots\frac{2}{3}\cdot 1, & if\ n\ is\ odd \end{cases}$$

(V) Find the reduction formula for $\int sec^n\, xdx,\ n \geq 2$ is an integer.

Solution:

Consider $I_n = \int sec^n\, xdx = \int sec^{n-2}\, x sec^2 xdx$ ... (1)

We know by the method of integration by part

$$\int u\ dv = uv - \int v\ du$$

Let $u = \ sec^{n-2}x$ $\qquad dv = sec^2 x\ dx$

$du = (n-2)cos^{n-3}x\ (secxtanx)dx \qquad v = \ \int sec^2 x\ dx = \ tanx$

$$I_n = \ sec^{n-2}x\ tanx - \ \int (tanx)[(n-2)sec^{n-3}\ x\ secx\ tanx]\ dx$$

$$= sec^{n-2}x\ tanx - (n-2)\ \int tan^2\ x\ sec^{n-2}x\ \ dx$$

$$= sec^{n-2}x\ tanx - (n-2)\ \int (sec^2 x - 1)\ sec^{n-2}x\ \ dx$$

$$= sec^{n-2}x\ tanx - (n-2)\int \ sec^n x\ \ dx + (n-2)\int sec^{n-2}x\ dx$$

$$= sec^{n-2}x\ tanx - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = \ sec^{n-2}x\ tanx + (n-2)I_{n-2}$$

$$(n-1)I_n \ = \ sec^{n-2}x\ tanx + (n-2)I_{n-2}$$

$$I_n \ = \ \frac{1}{n-1}sec^{n-2}x\ tanx + \frac{n-2}{n-1}I_{n-2}$$

The ultimate integral is $I_0\ or\ I_1$

$n\ even\ \ :\ \ I_0\ =\ \int dx\ = x + C$ [Put n = 0 in (1)]

$n\ odd\ \ :\ \ I_1\ =\ \int secx\ dx\ = \log(secx + tanx) + C$ [Put n = 1 in (1)]

(VI) Find the reduction formula for $\int cosec^n\, xdx,\ n \geq 2$ is an integer.

Solution:

Consider $I_n = \int cosec^n\, xdx = \int cosec^{n-2}\, x\, cosec^2 xdx\ \ldots(1)$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = cosec^{n-2}x$ $\qquad dv = cosec^2 x\, dx$

$du = (n-2)sec^{n-3}x\,(-cosecx\, cotx)dx$ $\qquad v = \int cosec^2 x\, dx = -cotx$

$$I_n = cosec^{n-2}x\,(-cotx) - \int(-cotx)[(n-2)cosec^{n-3}\, x\,(-cosecx\, cotx)]\, dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)\int cot^2\, x\, cosec^{n-2}x\ dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)\int(cosec^2 x - 1)\, cosec^{n-2}x\ dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)\int\ cosec^n x\ dx + (n-2)\int cosec^{n-2}x\, dx$$

$$= -cosec^{n-2}x\, cotx - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = -cosec^{n-2}x\, cotx + (n-2)I_{n-2}$$

$$(n-1)I_n = -cosec^{n-2}x\, cotx + (n-2)I_{n-2}$$

$$I_n = -\frac{1}{n-1}cosec^{n-2}x\, cotx + \frac{n-2}{n-1}I_{n-2}$$

The ultimate integral is $I_0\ or\ I_1$

$n\ even\ \ :\ \ I_0\ =\ \int dx\ = x + C$ [Put n = 0 in (1)]

$n\ odd\ \ :\ \ I_1\ =\ \int cosecx\ dx\ = \log(cosecx - cotx) + C$ [Put n = 1 in (1)]

(VII) Find the reduction formula for $\int cot^n\, xdx,\ n \neq 1$

Solution:

Consider $I_n = \int cot^n\, xdx = \int cot^{n-2}x\, cot^2 x\, dx\ \ldots(1)$

$$= \int cot^{n-2}x\,(cosec^2 x - 1)\, dx$$

$$= -\int cot^{n-2}x\,(cosec^2 x)dx - \int cot^{n-2}x\, dx$$

$$= -\int\ cot^{n-2}x\ \ d(cotx) - I_{n-2}$$

$$= -\frac{1}{n-1}cot^{n-1}x - I_{n-2}$$

The ultimate integral is $I_0\ or\ I_1$

$n\ even \quad : \quad I_0 = \int dx = x + C$ [Put n = 0 in (1)]

$n\ odd \quad : \quad I_1 = \int cotx\, dx = \log(sinx) + C$ [Put n = 1 in (1)]

(VIII)Find the reduction formula for $\int tan^n xdx,\ n \neq 1$

Solution:

Consider $I_n = \int tan^n xdx$ ... (1)

$$= \int tan^{n-2}x\ tan^2x\ dx$$

$$= \int tan^{n-2}x\ (sec^2x - 1)\ dx$$

$$= \int tan^{n-2}x\ sec^2xdx - \int tan^{n-2}x\ dx$$

$$= \int\ tan^{n-2}x\ \ d(tanx) - I_{n-2}$$

$$= \frac{1}{n-1} tan^{n-1}x - I_{n-2}$$

The ultimate integral is $I_0\ or\ I_1$

$n\ even \quad : \quad I_0 = \int dx = x + C$ [Put n = 0 in (1)]

$n\ odd \quad : \quad I_1 = \int tanx\, dx = \log(secx) + C$ [Put n = 1 in (1)]

Problems on Reduction formula

Example: 3.46

i) **Evaluate** $\int \boldsymbol{sin^7x\ dx}$

Solution:

Given $\int sin^7x\ dx$

We know that $I_n = -\frac{cosx\ sin^{n-1}x}{n} + \frac{n-1}{n}\int\ sin^{n-2}x\ \ dx \cdots$ (1)

Put n = 7 in equation (1)

$$\int sin^7x\ dx = -\frac{cosx\ sin^{7-1}x}{7} + \frac{7-1}{7}\int\ sin^{7-2}x\ \ dx$$

$$\int sin^7x\ dx = -\frac{cosx\ sin^6x}{7} + \frac{6}{7}\int\ sin^5x\ \ dx\ \cdots (2)$$

Put n = 5 in equation (1)

$$\int sin^5x\ dx = -\frac{cosx\ sin^4x}{5} + \frac{4}{5}\int\ sin^3x\ \ dx\ \cdots (3)$$

Put n = 3 in equation (1)

$$\int sin^5x\ dx = -\frac{cosx\ sin^2x}{3} + \frac{2}{3}\int sinx\ \ dx$$

$$= -\frac{cosx\ sin^2x}{3} + \frac{2}{3}(-cosx)$$

$$\therefore (3) \quad gives \int sin^5x\, dx = -\frac{cosx\, sin^4x}{5} + \frac{4}{5}\left[\frac{-sin^2x\, cosx}{3} - \frac{2}{3}cosx\right]$$

$$= -\frac{cosx\, sin^4x}{5} - \frac{4}{15}sin^2x\, cosx - \frac{8}{15}\, cosx$$

and (2) gives

$$\int sin^7x\, dx = -\frac{cosx\, sin^6x}{7} + \frac{6}{7}\left[\frac{-sin^4x\, cosx}{5} - \frac{4}{15}sin^2\, x\, cosx - \frac{8}{15}\, cosx\right]$$

$$= -\frac{1}{7}cosx\, sin^6x - \frac{6}{35}sin^4x\, cosx - \frac{8}{35} - sin^2x\, cosx - \frac{16}{35}cosx$$

(ii) Evaluate $\int cos^4x\, dx$

Solution:

Given $\int cos^4x\, dx$

We know that $I_n = \frac{sin\, x\, cos^{n-1}x}{n} + \frac{(n-1)}{n}\int cos^{n-2}x\ dx \cdots (1)$

Put n = 4 in equation (1)

$$\int cos^4x\, dx = \frac{sin\, x\, cos^3x}{4} + \frac{3}{4}\int cos^2x\ dx$$

$$= \frac{sin\, x\, cos^3x}{4} + \frac{3}{4}\int\left(\frac{1 + cos\, 2x}{2}\right)dx$$

$$= \frac{sin\, x\, cos^3x}{4} + \frac{3}{8}\left(x + \frac{\sin 2x}{2}\right)$$

(iii)Evaluate$\int_0^{\pi/2} sin^7\, x\, dx$

Solution:

Given $\int_0^{\pi/2} sin^7\, x\, dx$

We know that $\int_0^{\pi/2} sin^n\, x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}.1,\ when\, n\, is\, odd \cdots (1)$

Put n = 7 in equation (1)

$$\int_0^{\pi/2} sin^7\, x\, dx = \left(\frac{7-1}{7}\right)\left(\frac{7-3}{7-2}\right)\left(\frac{7-5}{7-4}\right)(1)$$

$$= \left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)(1)$$

(iv)Evaluate$\int_0^{\pi/2} cos^{10}\, x\, dx$

Solution:

Given $\int_0^{\pi/2} cos^{10}\, x\, dx$

We know that $\int_0^{\pi/2} cos^n\, x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{1}{2}\cdot\frac{\pi}{2},\ when\, n\, is\, even \cdots (1)$

Put n = 10 in equation (1)

$$\int_0^{\pi/2} cos^n\, x\, dx = \left(\frac{10-1}{10}\right)\left(\frac{10-3}{10-2}\right)\left(\frac{10-5}{10-4}\right)\left(\frac{10-7}{10-6}\right)\left(\frac{10-9}{10}\right)\left(\frac{\pi}{2}\right)$$

$$= \left(\frac{9}{10}\right)\left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) \quad = \frac{63}{512}\pi$$

(v) Evaluate $\int_0^{\pi} sin^2 x\, dx$

Solution:

Given $\int_0^{\pi} sin^2 x\, dx$

$$\int_0^{\pi} sin^2 x\, dx = \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^{\pi}(1 - cos\, 2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{0}\right)\right]$$

$$= \frac{1}{2}(\pi - 0 - 0 + 0) = \frac{\pi}{2}$$

(vi) Evaluate $\int_0^{\pi/2} sin^{2n+1}\, x\, dx$

Solution:

Given $\int_0^{\pi/2} sin^{2n+1}\, x\, dx$

We know that $\int_0^{\pi/2} sin^n\, x\, dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6} \cdots \frac{2}{3}\, .1,\ when\, n\, is\, odd \ \cdots (1)$

Put n = 2 n + 1 in equation (1)

$$\int_0^{\pi/2} sin^n\, x\, dx = \frac{(2n+1)-1}{2n+1}\cdot\frac{(2n+1)-3}{(2n+1)-2}\cdot\frac{(2n+1)-5}{(2n+1)-4} \cdots 1$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3} \cdots \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

(vii) Evaluate $\int tan^2\, x\, dx$

Solution:

Given $\int tan^2\, x\, dx$

$$\int tan^2\, x\, dx = \int (sec^2 x - 1)\, dx$$

$$= \int sec^2 x\, dx - \int dx$$

$$= tanx - x + C$$

(viii)Evaluate $\int tan^3\, x\, dx$

Solution:

Given $\int tan^3\, x\, dx$

$$\int tan^3 x\, dx = \int tan^2 x\ \ tanx\, dx$$
$$= \int (sec^2 x - 1)\, tanx\, dx$$
$$= \int sec^2 x\, tanx\, dx - \int tanx\, dx$$
$$= \int tanx\, d(tanx) - \int tanx\, dx$$
$$= \frac{tan^2 x}{2} - \log secx + C$$

(ix) Evaluate $\int sec^3 x\, dx$

Solution:

Let $I = \int sec^3 x\, dx$

$$\int sec^3 x\, dx = \int sec^2 x\ \ sec\, x\, dx$$

Let $u = \sec x$ $\qquad\qquad dv = sec^2 x\, dx$

$du = secx\, tanx\, dx \qquad v = \int sec^2 x\, dx = tanx$

$$\therefore I = \int sec^3 x\, dx = secx\, tanx - \int tanx\, secx\, tanx\, dx$$
$$= secx\, tanx - \int tan^2 x\, secx\ \ dx$$
$$= secx\, tanx - \int (sec^2 x - 1) secx\ \ dx$$
$$= secx\, tanx + \int secx\ \ dx - \int sec^3 x\, dx$$
$$= secx\, tanx + \log(secx + tanx) - I + C$$
$$2I = secx\, tanx + \log(secx + tanx\,) + C$$
$$I = \frac{1}{2}[secx\, tanx + \log(secx + tanx\,) + C]$$

(x) Evaluate $\int_0^\pi cos^4(2x)\, dx$

Solution:

Given $\int_0^\pi cos^4(2x)\, dx$

$$= \int_0^\pi \left[\frac{1+\cos 4x}{2}\right]^2 dx$$
$$= \frac{1}{4}\int_0^\pi [1 + 2\cos 4x + cos^2 4x]\ dx$$
$$= \frac{1}{4}\int_0^\pi \left[1 + 2\cos 4x + \frac{1+\cos 8x}{2}\right] dx$$
$$= \frac{1}{8}\int_0^\pi [2 + 4\cos 4x + 1 + \cos 8x]\ dx$$
$$= \frac{1}{8}\left[3x + 4\frac{\sin 4x}{4} + \frac{\sin 8x}{8}\right]_0^\pi$$
$$= \frac{1}{8}[(3\pi + 0 + 0) - (0 + 0 + 0)] = \frac{3}{8}\pi$$

(xi) Evaluate $\int_{\pi/6}^{\pi/3} cosec^3\, x\, dx$

Solution:

Given $\int_{\pi/6}^{\pi/3} cosec^3\, x\, dx$

Let I = $\int_{\pi/6}^{\pi/3} cosec^3\, x\, dx = \int_{\pi/6}^{\pi/3} cosec^2\, x\;\; cosec\, x\, dx$

We know by the method of integration by part

$$\int u\, dv = uv - \int v\, du$$

Let $u = cosecx$ $\qquad dv = cosec^2\, x\, dx$

$du = -cosecx\; cotx\; dx \qquad v = \int cosec^2 x\, dx = -cotx$

$$= [(cosecx)(-cotx)]_{\pi/6}^{\pi/3} - \int_{\pi/6}^{\pi/3}(-cotx)(-cosecx)\cot x\, dx$$

$$= \left(\frac{-2}{\sqrt{3}}\frac{1}{\sqrt{3}}\right) - \left(-2\sqrt{3}\right) - \int_{\pi/6}^{\pi/3} cosec\, x\, cot^2\, x\, dx$$

$$= -\frac{2}{3} + 2\sqrt{3} - \int_{\pi/6}^{\pi/3} cosec\, x\,(cosec^2\, x - 1)\, dx$$

$$= -\frac{2}{3} + 2\sqrt{3} - \int_{\pi/6}^{\pi/3} cosec^3\, x\, dx + \int_{\pi/6}^{\pi/3} cosecx\, dx$$

$$= -\frac{2}{3} + 2\sqrt{3} - I + [log(cosec\, x - \cot x)]_{\pi/6}^{\pi/3}$$

$$2\, I = -\frac{2}{3} + 2\sqrt{3} + log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) - \log\left(2 - \sqrt{3}\right)$$

$$= -\frac{2}{3} + 2\sqrt{3} + log\left(\frac{1}{\sqrt{3}}\right) - \log\left(2 - \sqrt{3}\right)$$

$$= -\frac{2}{3} + 2\sqrt{3} + log 1 - \log\sqrt{3} - \log\left(2 - \sqrt{3}\right)$$

$$= -\frac{2}{3} + 2\sqrt{3} - \left[\log\sqrt{3} - \log\left(2 - \sqrt{3}\right)\right] \qquad [\because \log 1 = 0\,]$$

$$= -\frac{2}{3} + 2\sqrt{3} - \left[\log\sqrt{3}\,(2 - \sqrt{3})\right]$$

$$= -\frac{2}{3} + 2\sqrt{3} - \left[\log\left(2\sqrt{3} - 3\right)\right]$$

$$2I = -\frac{1}{3} + \sqrt{3} - \frac{1}{2}\left[\log\left(2\sqrt{3} - 3\right)\right]$$

$$\therefore I = -\frac{1}{3} + \sqrt{3} - \frac{1}{2}\left[\log\sqrt{3}\left(2 - \sqrt{3}\right)\right]$$

(xii)Evaluate $\int_{\pi/6}^{\pi/2} cot^2\, x\, dx$

Solution:

Given $\int_{\pi/6}^{\pi/2} cot^2\, x\, dx$

$$\int_{\pi/6}^{\pi/2} cot^2\, x\, dx = \int_{\pi/6}^{\pi/2} (cosec^2\, x - 1)\, dx$$

$$= \int_{\pi/6}^{\pi/2} cosec^2\, x\, dx - \int_{\pi/6}^{\pi/2} dx$$

$$= [-\cot x]_{\pi/6}^{\pi/2} - [x]_{\pi/6}^{\pi/2}$$

$$= (-0) - (-\sqrt{3}) - \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{1}{3}\pi$$

Exercise3. 3(b)

Evaluate:

1. $\int sin^4 x\, dx$ **Ans:** $-\frac{1}{4}\cos x\, sin^3\, x - \frac{3}{16}\sin 2x + \frac{3}{8}\, x + C$

2. $\int_0^{\pi/2} sin^{2n}\, x\, dx$ **Ans:** $\frac{2n-1}{2} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

3. $\int_0^{\pi/2} sin^8\, x\, dx$ **Ans:** $\left(\frac{7}{8}\right)\left(\frac{5}{6}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$

4. $\int cos^3 x\, dx$ **Ans:** $\frac{1}{3}\, cos^2 x \sin x + \frac{2}{3}\sin x + C$

5. $\int_0^{\pi/2} cos^9\, x\, dx$ **Ans:** $\left(\frac{8}{9}\right)\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)$

6. $\int cot^5 x\, dx$ **Ans:** $-\frac{cot^4\, x}{4} + \frac{cot^2\, x}{x} + \log(\sin x) + C$

7. $\int cosec^6 x\, dx$ **Ans:** $-\frac{cosec^4\, x \cot x}{5} + \frac{4}{5}\left[-\frac{cosec^2\, x \cot x}{3} + \frac{2}{3}\,(-\cot x)\right] + C$

8. Find the reduction formula for $\int x^n \sin mx\, dx$ (n being a positive).

3.3(c) TRIGONOMETRIC INTEGRALS

(I) Products of powers of sines and cosines

Evaluating $\int sin^m x cos^n x dx$

Case (i) If n is odd $(n = 2k + 1)$, then

$$\int sin^m x cos^{2k+1} x dx = \int sin^m\, x (cos^2 x)^k cos x dx$$

$$= \int sin^m x (1 - sin^2 x)^k cos x dx$$

Here, substitute $u = sinx$

Case (ii) If m is odd $(m = 2k + 1)$, then

$$\int sin^{2k+1} x cos^n x dx = \int (sin^2\, x)^k cos^n x sin x dx$$

$$= \int (1 - cos^2 x)^k cos^n x sin x dx$$

Here, substitute $u = cosx$

Note: If both m and n are odd apply case (i) or case (ii)

Case(iii) If both m and n are even, use half- angle identities

$$sin^2x = \frac{1}{2}(1 - cos2x), \qquad cos^2x = \frac{1}{2}(1 + cos2x)$$

$$\int_0^{\pi/2} sin^m x cos^n x dx = \frac{m-1}{m+n}\frac{m-3}{m+n-2}\ldots\frac{2}{3+n}\frac{1}{1+n}$$

(if m is odd, n may be even or odd)

$$= \frac{m-1}{m+n}\frac{m-3}{m+n-2}\ldots\frac{1}{2+n}\frac{n-1}{n}\frac{n-3}{n-2}\ldots\frac{2}{3}.1$$

(if m is even, n is odd)

$$= \frac{m-1}{m+n}\frac{m-3}{m+n-2}\ldots\frac{1}{2+n}\frac{n-1}{n}\frac{n-3}{n-2}\ldots\frac{1}{2}\frac{\pi}{2}$$

(if m is even, n is even)

(II) Products of powers of $secx$ and $tanx$

Evaluating $\int tan^m x sec^n x dx$

Case (i) If m is odd $(m = 2k + 1)$, then

$$\int tan^{2k+1} x sec^n x dx = \int (tan^2 x)^k sec^{n-1} x secx tanx dx$$

$$= \int (sec^2 x - 1)^k sec^{n-1} x secx tanx dx$$

Here, substitute $u = secx$

Case (ii) If n is even $(n = 2k)$, then

$$\int tan^m x sec^{2k} x dx = \int tan^m x (sec^2 x)^{k-1} sec^2 x dx$$

$$= \int tan^m x (1 + tan^2 x)^{k-1} sec^2 x dx$$

Here, substitute $u = tanx$

(III) Products of sines and cosines of multiples of x

Evaluating $\int sinmx\, sinnx dx, \quad \int sinmx\, cosnx dx \;\; and \int cosmx\, cosnx\, dx$

Use the following identities

$$sinmxsinnx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$sinmxcosnx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$cosmxcosnx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

Problems based on $\int sin^m x cos^n x d x$

Example: 3.47

(i)Evaluate $\int sin^6 x cos^3 x dx$

Solution:

Given $\int sin^6 x cos^3 x dx$ Here m = 6, n = 3 (odd)

$$= \int sin^6 x cos^2 x cos x dx$$

$$= \int sin^6 x(1 - sin^2 x) cos x dx \ldots.(1)$$

$$Put \quad u = sinx; \quad du = cosxdx$$

$$(1) \Rightarrow \int u^6(1 - u^2)\, du = \int (u^6 - u^8) du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C$$

$$= \frac{sin^7 x}{7} - \frac{sin^9 x}{9} + C$$

(ii) Evaluate $\int sin^2(\pi x) cos^5(\pi x) dx$

Solution:

Given $\int sin^2(\pi x) cos^5(\pi x) dx$ (Here m = 2, n = 5 (odd))

$$= \int sin^2(\pi x) cos^4(\pi x) \cos(\pi x)\, dx$$

$$= \int sin^2(\pi x)[1 - sin^2(\pi x)]^2 \cos(\pi x)\, dx \ldots (1)$$

$$Put \ u = sin\pi x; \quad du = \pi cos\pi x dx$$

$$(1) \Rightarrow \int u^2(1 - u^2)^2 \frac{du}{\pi} = \frac{1}{\pi}\int u^2(1 - 2u^2 + u^4) du$$

$$= \frac{1}{\pi}\int (u^2 - 2u^4 + u^6) du$$

$$= \frac{1}{\pi}\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$$

$$= \frac{1}{3\pi} sin^3(\pi x) - \frac{2}{5\pi} sin^5(\pi x) + \frac{1}{7\pi} sin^7(\pi x) + C$$

Problems on (m is odd and n is even)

Example: 3.48

Evaluate $\int sin^5 x cos^2 x dx$

Solution:

Given $\int sin^5 x cos^2 x dx$ (Here m = 5 (odd), n = 2)

$$= \int sin^4 x cos^2 x sin x dx$$

$$= \int (1 - cos^2 x)^2 cos^2 x sin x dx \ldots.(1)$$

$$Put \ u = cosx; \quad du = -sinxdx$$

$$(1) \Rightarrow \int (1 - u^2)^2 u^2(-du) = -\int (1 - 2u^2 + u^4) u^2 du$$

$$= -\int (u^2 - 2u^4 + u^6) du$$

$$= -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$$

$$= -\frac{1}{3}cos^3(x) + \frac{2}{5}cos^5(x) - \frac{1}{7}cos^7(x) + C$$

Problems on (m and n are odd)

Example: 3.49

Evaluate $\int \boldsymbol{cos^2 xsin2xdx}$

Solution:

Given $\int cos^2 xsin2xdx$

$$= \int 2sinxcosxcos^2xdx$$

$= \int 2sinxcos^3xdx$ (Here, m = 1, n = 3)

$$= 2 \int sinxcos^3xdx \ldots (1)$$

$Put \quad u = cosx; \qquad du = -sinxdx$

$$(1) \Rightarrow 2\int u^3(-du) = -2\int u^3du$$

$$= -2\frac{u^4}{4} + C = -\frac{1}{2}cos^4x + c$$

Problems on (m and n are even)

Example:3.50

Evaluate $\int \boldsymbol{sin^2 xcos^4xdx}$

Solution:

Given $\int sin^2 xcos^4xdx$ (Here, m = 2, n = 4)

$$= \int \left(\frac{1-cos2x}{2}\right)\left(\frac{1+cos2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int(1 - cos2x)(1 + 2cos2x + cos^22x)dx$$

$$= \frac{1}{8}\int(1 + cos2x - cos^22x - cos^32x)dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}sin2x - \int(cos^22x + cos^32x)dx\right] \ldots (1)$$

$$\int cos^22xdx = \int \frac{1+cos4x}{2}dx = \frac{1}{2}\left(x + \frac{1}{4}sin4x\right)$$

$$\int cos^32xdx = \int cos^22xcos2xdx = \int(1 - sin^22x)\, cos2xdx$$

$Put \quad u = sin2x; \qquad du = 2cos2xdx$

$$\therefore \int cos^32xdx = \int(1 - u^2)\frac{du}{2} = \frac{1}{2}\left[u - \frac{u^3}{3}\right] = \frac{1}{2}\left[sin2x - \frac{1}{3}sin^32x\right]$$

$$(1) \Rightarrow \frac{1}{8}\left[x + \frac{1}{2}sin2x - \frac{1}{2}x - \frac{1}{8}sin4x - \frac{1}{2}sin2x + \frac{1}{6}sin^32x\right] + C$$

$$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}sin4x + \frac{1}{6}sin^3 2x\right] + C$$

$$= \frac{1}{16}\left[x - \frac{1}{4}sin4x + \frac{1}{3}sin^3 2x\right] + C$$

Problems based on $\int tan^m x sec^n x d\, x$

Example: 3.51

(i)Evaluate $\int tan\, x sec^3 x dx$

Solution:

Given $\int tan\, x sec^3 x dx$ (Here m =1 (odd))

$$= \int sec^2 x (secx\; tanx) dx$$

$$Put\; u = secx; \qquad du = secx\; tanxdx$$

$$= \int u^2\, du \quad = \frac{u^3}{3} + C \;=\; \frac{sec^3 x}{3} + C$$

(ii)Evaluate $\int_0^{\pi/3} tan^5 x sec^4 x dx$

Solution:

Given $\int_0^{\pi/3} tan^5 x sec^4 x dx$ (Here m = 5 (odd))

$$= \int_0^{\pi/3} tan^4 x\; sec^3 x\; (secx\; tanx) dx$$

$$= \int_0^{\pi/3} (sec^2 x - 1)^2 sec^3 x (secx\; tanx)\; dx \quad \ldots (1)$$

$Put\; u = secx$ when $x = 0 \Rightarrow u = 1$

$du = secx\; tanx\; dx \quad x = \frac{\pi}{3} \Rightarrow u = 2$

$$\therefore (1) \Rightarrow \int_1^2 (u^2 - 1)^2 u^3\, du \;=\; \int_1^2 (u^4 - 2u^2 + 1) u^3\, du$$

$$= \int_1^2 (u^3 - 2u^5 + u^7)\, du$$

$$= \left[\frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8}\right]_1^2$$

$$= \left(4 - \frac{64}{3} + 32\right) - \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) \;=\; \frac{117}{8}$$

Problems based on n is even

Example: 3.52

(i)Evaluate $\int tan^2 x sec^4 x d\, x$

Solution:

Given $\int tan^2 x sec^4 x d\, x$

$$= \int tan^2x sec^2x \; sec^2x \; d\,x$$

$$= \int tan^2x(1 + \; tan^2x)sec^2x \; dx \dots (1)$$

$$Put \; u = tanx \;\; ; \qquad du = sec^2x \;\; dx$$

$$(1) \Rightarrow \int u^2(1 + u^2) \, du \;\; = \;\int (u^2 + u^4) du$$

$$= \left[\frac{u^3}{3} + \frac{u^5}{5}\right] + C$$

$$= \frac{1}{3} tan^3(x) + \frac{1}{5} tan^5(x) + C$$

(ii)Evaluate $\int tanx sec^2x d\,x$

Solution:

Given $\int tanx sec^2x d\,x$

$$Put \; u = tanx \;\; ; \quad du = sec^2x \;\; dx$$

$$= \;\int u du$$

$$= \left[\frac{u^2}{2}\right] + C = \frac{1}{2} tan^2(x) + C$$

Problems on m is even (or) n is odd

Example: 3.53

(i) Evaluate $\int sec^3x \; dx$

Solution:

Given I = $\int sec^3x \; dx \;\; = \int sec^2x \; secx \; dx$

$$Put \; u = secx \qquad\qquad dv = \; sec^2x \; dx$$

$$du = secx \; tanx \; dx \qquad v = \; \int sec^2x \; dx = \; tanx$$

$$\int u \; dv = uv - \int v \; du$$

$$I \quad = (secx)tanx - \; \int tanx \; (\; secx \; tanx \;)dx$$

$$= (secx)tanx - \; \int tan^2x \; secx \; dx$$

$$= (secx)tanx - \; \int (sec^2x - 1) \; secx \; dx$$

$$= (secx)tanx - \; \int sec^3x \; dx + \int secx \; dx$$

$$= secx \; tanx - I + \log(\; secx + tanx \;)$$

$$2I \quad = secx \; tanx + \log(\; secx + tanx)$$

$$I \quad = \frac{1}{2} secx \; tanx + \frac{1}{2} \log(\; secx + tanx) + C$$

(ii)Evaluate $\int tan^2x \; secx \; dx$

Solution:

Given $\int tan^2x\ secx\ dx \quad = \int (sec^2x - 1)secx\ dx$

$$= \int sec^3x\ dx - \int secx\ dx$$

$$= \frac{1}{2}secx\ tanx + \frac{1}{2}\log(\ secx + tanx) - \log(\ secx + tanx) + C$$

Using example (3.53(i))

$$= \frac{1}{2}secx\ tanx - \frac{1}{2}\log(\ secx + tanx) + C$$

Problems based on $\int_0^{\pi/2} sin^m x\ cos^n x\ dx$

Example:3.54

(i) Evaluate $\int_0^{\pi/2} sin^7 x\ cos^5 x\ dx$

Solution:

Given $\int_0^{\pi/2} sin^7x\ cos^5x\ dx$ (Here m = 7 , n = 5)

$\int_0^{\pi/2} sin^m x cos^n x dx = \frac{m-1}{m+n}\frac{m-3}{m+n-2} \cdots \frac{2}{3+n}\frac{1}{1+n}$ (m is odd, n even or odd)

$$= \frac{7-1}{7+5}\frac{7-3}{7+5-2} \cdots \frac{2}{3+5}\frac{1}{1+5}$$

$$= \left(\frac{6}{12}\right)\left(\frac{4}{10}\right)\left(\frac{2}{8}\right)\left(\frac{1}{6}\right) \quad = \frac{1}{120}$$

(ii)Evaluate $\int_0^{\pi/2} sin^7 x\ dx$

Solution:

Given $\int_0^{\pi/2} sin^7x\ dx$ (Here m = 7 (odd) , n = 0)

$\int_0^{\pi/2} sin^m x cos^n x dx = \frac{m-1}{m+n}\frac{m-3}{m+n-2} \cdots \frac{2}{3+n}\frac{1}{1+n}$ (m is odd, n even or odd)

$$= \frac{7-1}{7+0}\frac{7-3}{7+0-2} \cdots \frac{2}{3+0}\frac{1}{1+0} = \left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)(1) \quad = \frac{16}{35}$$

Problems based on $\int sinmx\ sinnx\ dx\ (or) \int sinmx\ cosnx\ dx\ (or) \int cosmx\ cosnx\ dx$

Example: 3.55

i)Evaluate $\int sin\ 4x\ cos\ 5x\ dx$

Solution:

Given $\int sin\ 4x\ cos\ 5x\ dx$

We know that, $sinAxcosBx = \frac{1}{2}[\sin(A-B)\,x + \sin(A+B)\,x]$

$$= \frac{1}{2}\int [\sin(-x) + sin9x]dx$$

$$= \frac{1}{2}\int(-sinx + \ sin9x)dx$$

$$= \frac{1}{2}\left[cosx - \frac{1}{9}cos9x\right] + C$$

ii)Evaluate $\int cos\ 3x\ cos\ 4x\ dx$

Solution:

Given $\int cos\ 3x\ cos\ 4x\ dx$

We know that $cosAxcosBx = \ \frac{1}{2}[\cos(A-B)\,x + \cos(A+B)\,x]$

$$= \frac{1}{2}\int(cosx + \ cos7x)dx$$

$$= \frac{1}{2}\left[sinx + \frac{1}{7}sin7x\right] + C$$

$$= \frac{1}{2}sinx + \ \frac{1}{14}sin7x + C$$

iii) Evaluate $\int sin\ 5x\ sinx\ dx$

Solution:

Given $\int sin\ 5x\ sinx\ dx$

We know that $sinAxsinBx = \ \frac{1}{2}[\cos(A-B)\,x - \cos(A+B)\,x]$

$$= \frac{1}{2}\int(cos4x - \ cos6x)dx$$

$$= \frac{1}{2}\left[\frac{1}{4}sin4x - \frac{1}{6}sin6x\right] + C$$

$$= \frac{1}{2}sin4x - \ \frac{1}{12}sin6x + C$$

Problems based on Eliminating square roots

Example: 3.56 i) Evaluate $\int_0^{\pi/4}\sqrt{1 + cos4x}\ dx$

Solution:

Given $\int_0^{\pi/4}\sqrt{1 + cos4x}\ dx$

$$= \int_0^{\pi/4}\sqrt{2cos^2 2x}\ dx$$

$$= \sqrt{2}\int_0^{\pi/4}cos2x\ dx$$

$$= \sqrt{2}\left[\frac{sin2x}{2}\right]_0^{\pi/4}$$

$$= \sqrt{2}\left[\frac{1}{2} - 0\right] \quad = \frac{\sqrt{2}}{2} \quad = \frac{1}{\sqrt{2}}$$

ii)Evaluate $\int_0^{\pi/4}\sqrt{1 - cos4x}\ dx$

Solution:

Given $\int_0^{\pi/4} \sqrt{1 - cos4x}\ dx$

$$= \int_0^{\pi/4} \sqrt{2sin^2 2x}\ dx$$

$$= \sqrt{2} \int_0^{\pi/4} sin2x\ dx = \sqrt{2} \left[\frac{-cos2x}{2}\right]_0^{\pi/4}$$

$$= -\frac{\sqrt{2}}{2} [cos2x]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}} [0 - 1] \quad = \frac{1}{\sqrt{2}}$$

Exercise 3.3(c)

Evaluate:

1. $\int sin^4 2x\ \cos 2x\ dx$ **Ans:** $\frac{1}{10}\ sin^5 (2x) + C$
2. $\int sin^3 x\ \cos^2 x\ dx$ **Ans:** $\frac{1}{5}\ cos^5 x - \frac{1}{3}\ cos^3 x + C$
3. $\int sin^3 x\ \cos^3 x\ dx$ **Ans:** $\frac{1}{4}\ sin^4 x - \frac{1}{6}\ sin^6 x + C$
4. $\int cos^2 x\ \tan^3 x\ dx$ **Ans:** $\log(\sec x) + \frac{1}{2}\ cos^2 x + C$
5. $\int tan^5 x\ \sec^7 x\ dx$ **Ans:** $\frac{sec^{11}x}{11} + \frac{sec^7 x}{7} - 2\frac{sec^9 x}{9} + C$
6. $\int tan^4 x\ dx$ **Ans:** $\frac{1}{3}\ tan^3 x - tanx + x + C$
7. $\int_0^{\pi/2} x\sqrt{1 - \cos 2x}\ dx$ **Ans:** $\sqrt{2}$
8. $\int_0^{\pi/6} \sqrt{1 + \cos 2x}\ dx$ **Ans:** $\frac{1}{\sqrt{2}}$
9. $\int_0^{\pi/4} cos^4 x\ dx$ **Ans:** $\frac{8+3\pi}{2}$
10. $\int_{\pi/4}^{\pi/2} cosec^4\, \theta\ d\theta$ **Ans:** $\frac{4}{3}$

3.3(d) Trigonometric Substitutions

Expression	Substitution	Identity Used
$\sqrt{a^2 - x^2}$	x = asinθ	$cos^2\theta = 1 - sin^2\theta$
$\sqrt{a^2 + x^2}$	x = atanθ	$sec^2\theta = 1 + tan^2\theta$
$\sqrt{x^2 - a^2}$	x = asecθ	$tan^2\theta = sec^2\theta - 1$

Problem based on Trigonometric Substitution

Example: 3.57

Evaluate $\int \frac{x^2}{\sqrt{9-x^2}}dx$

Solution:

Put $x = 3sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}; \qquad dx = 3cos\theta d\theta$

$$= \int \frac{9sin^2\theta}{\sqrt{9-9sin^2\theta}} 3cos\theta d\theta$$

$$= \int \frac{9sin^2\theta}{3cos\theta} 3cos\theta d\theta \qquad \left[\because \sqrt{1-sin^2\theta} = \sqrt{cos^2\theta} = cos\theta\right]$$

$$= 9\int sin^2\theta d\theta$$

$$= 9\int \frac{1-cos2\theta}{2} d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{sin2\theta}{2}\right] + C$$

$$= \frac{9}{2}\theta - \frac{9}{4}sin2\theta + C$$

$$= \frac{9}{2}\theta - \frac{9}{4}(2sin\theta cos\theta) + C$$

$$= \frac{9}{2}\theta - \frac{9}{2}sin\theta cos\theta + C$$

$$= \frac{9}{2}sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

Example: 3.58

Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}dx$ **by using trigonometric substitution.**

Solution:

Put $x = asin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\ ; \qquad dx = acos\theta d\theta$

$$= \int \frac{1}{\sqrt{a^2-a^2sin^2\theta}} acos\theta d\theta$$

$$= \int \frac{1}{acos\theta} acos\theta d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= sin^{-1}\left(\frac{x}{a}\right) + C$$

Example: 5.59

Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}dx$

Solution:

$$\text{Let } 3-2x-x^2 = -(x^2+2x)+3$$

$$= -[(x+1)^2-1]+3$$

$$= -(x+1)^2+4$$

$$\text{Put} \quad u = x+1 \qquad \Rightarrow x = u-1 \qquad \Rightarrow du = dx$$

$$\Rightarrow \int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{u-1}{\sqrt{4-u^2}}du$$

$$Put \quad u = 2sin\theta \quad ; \qquad du = 2cos\theta d\theta$$

$$= \int \frac{2sin\theta-1}{\sqrt{4-4sin^2\theta}} 2cos\theta d\theta$$

$$= \int \frac{2sin\theta-1}{2cos\theta} 2cos\theta d\theta$$

$$= \int (2sin\theta-1)d\theta$$

$$= 2(-cos\theta)-\theta+C$$

$$= -2cos\theta-\theta+C$$

$$= -2\frac{\sqrt{4-u^2}}{2} - sin^{-1}\left(\frac{u}{2}\right)+C$$

$$= -\sqrt{4-(x+1)^2} - sin^{-1}\left(\frac{x+1}{2}\right)+C$$

$$= -\sqrt{3-2x-x^2} - sin^{-1}\left(\frac{x+1}{2}\right)+C$$

Example:3.60

Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$

Solution:

$$\text{Put } x = 2tan\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}; \qquad dx = 2sec^2\theta d\theta$$

$$= \int \frac{1}{\sqrt{4+4tan^2\theta}} 2sec^2\theta d\theta$$

$$= \int \frac{1}{2sec\theta} 2sec^2\theta d\theta \qquad [\because 1+tan^2\theta = sec^2\theta]$$

$$= \int sec\theta d\theta$$

$$= \log(sec\theta+tan\theta)+C$$

$$= log\left[\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right]+C$$

Example: 3.61

Evaluate $\int \frac{x^2}{(a^2-x^2)^{3/2}} dx$

Solution:

Given $\int \frac{x^2}{(a^2-x^2)^{3/2}} dx$

$Put\ \ x = asin\theta \Rightarrow \theta = sin^{-1}\left(\frac{x}{a}\right) \quad \Rightarrow dx = acos\theta d\theta$

$$\int \frac{x^2}{(a^2-x^2)^{3/2}} dx = \int \frac{a^2 sin^2\theta}{(a^2-a^2 sin^2\theta)^{3/2}} acos\theta d\theta$$

$$= \int \frac{a^2 sin^2\theta}{(a^2 cos^2\theta)^{3/2}} acos\theta d\theta$$

$$= \int \frac{a^2 sin^2\theta}{(a^3 cos^3\theta)} acos\theta d\theta$$

$$= \int \frac{sin^2\theta}{cos^2\theta} d\theta$$

$$= \int \frac{1-cos^2\theta}{cos^2\theta} d\theta = \int (sec^2\theta - 1) d\theta$$

$$= \int sec^2\theta\, d\theta - \int d\theta$$

$$= tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{a^2-x^2}} - sin^{-1}\left(\frac{x}{a}\right)$$

Example: 3.62

Evaluate $\int \frac{1}{x^2\sqrt{x^2+2^2}} dx$

Solution:

Put $x = 2tan\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2};\qquad dx = 2sec^2\theta d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+2^2}} dx = \int \frac{1}{4tan^2\theta\sqrt{4tan^2\theta+4}} 2sec^2\theta d\theta$$

$$= \int \frac{1}{4tan^2\theta 2sec\theta} 2sec^2\theta d\theta$$

$$= \int \frac{1}{4}\frac{1}{tan^2\theta} sec\theta d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{sin^2\theta} d\theta$$

$$= \frac{1}{4}\int cosec\theta cot\theta d\theta$$

$$= -\frac{1}{4} cosec\theta + C$$

$$= -\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + c$$

$$= -\frac{\sqrt{x^2+4}}{4x} + C$$

Example: 3.63

Evaluate $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx$

Solution:

Given $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx$

$$= \int_0^{3\sqrt{3}/2} \frac{x^3}{8\left(x^2+\frac{9}{4}\right)^{3/2}} dx$$

Put $x = \frac{3}{2}tan\theta$; $\quad x = \ 0 \ \Rightarrow \theta = 0$

$dx = \frac{3}{2}sec^2\theta d\theta$; $\quad x = \frac{3\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$= \int_0^{\pi/3} \frac{1}{8} \frac{\frac{27}{8}tan^3\theta}{\left(\frac{9}{4}tan^2\theta+\frac{9}{4}\right)^{3/2}} \frac{3}{2} sec^2 d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{8}\right)\left(\frac{27}{8}\right)\left(\frac{3}{2}\right) \frac{tan^3\theta sec^2\theta}{\left(\frac{9}{4}\right)^{3/2}(1+tan^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{8}\right)\left(\frac{27}{8}\right)\left(\frac{3}{2}\right)\left(\frac{4}{9}\right)^{3/2} \frac{tan^3\theta sec^2\theta}{(sec^2\theta)^{3/2}} d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{8}\right)\left(\frac{27}{8}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^3 \frac{tan^3\theta}{sec\theta} d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{8}\right)\left(\frac{3}{2}\right) \frac{tan^3\theta}{sec\theta} d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{tan^2\theta}{sec\theta} tan\theta d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \left(\frac{sec^2\theta-1}{sec\theta}\right) tan\theta d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} (sec\theta - cos\theta)\, tan\theta d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} (sec\theta tan\theta - sin\theta)\, d\theta$$

$$= \frac{3}{16}\left[(sec\theta + cos\theta)\right]_0^{\pi/3}$$

$$= \frac{3}{16}\left[\left(2+\frac{1}{2}\right) - (1+1)\right] = \frac{3}{32}$$

Example:3.64

Evaluate $\int \frac{1}{\sqrt{25x^2-4}} dx, \quad x > \frac{2}{5}$

Solution:

Given $\int \frac{1}{\sqrt{25x^2-4}}dx, \quad x > \frac{2}{5}$

$$= \int \frac{1}{5\sqrt{x^2-\frac{4}{25}}}dx$$

Put $x = 5sec\theta \Rightarrow \quad dx = 5sec\theta tan\theta d\theta$

$$= \int \frac{1}{2(tan\theta)} 5sec\theta tan\theta d\theta$$

$$= \frac{5}{2}\int sec\theta d\theta = \frac{5}{2}\log(sec\theta + tan\theta) + C$$

$$= \frac{5}{2}log\left[\frac{5x}{2} + \frac{5}{2}\sqrt{x^2 - 4/25}\right] + C$$

Example:3.65

Evaluate $\int \frac{1}{\sqrt{x^2-a^2}}dx, \quad a > 0$

Solution:

Given $\int \frac{1}{\sqrt{x^2-a^2}}dx$

Put $x = asec\theta \quad \Rightarrow \quad dx = asec\theta tan\theta d\theta$

$\Rightarrow \sqrt{x^2 - a^2} = \sqrt{a^2sec^2\theta - a^2} = a\sqrt{sec^2\theta - 1}$

$$= a\sqrt{tan^2\theta}$$

$$= atan\theta$$

$$= \int \frac{1}{atan\theta} asec\theta tan\theta d\theta$$

$$= \int sec\theta d\theta$$

$$= \log(sec\theta + tan\theta) + C_1$$

$$= \log\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + C_1$$

$$= log\left(x + \sqrt{x^2 - a^2}\right) - loga + C_1$$

$$= log\left(x + \sqrt{x^2 - a^2}\right) + CwhereC = C_1 - loga$$

Example: 3.66

Evaluate $\int \frac{1}{x^2\sqrt{x^2-1}}dx, \quad x > 1$

Solution:

Given $\int \frac{1}{x^2\sqrt{x^2-1}}dx, \quad x > 1$

Put $x = asec\theta; \qquad dx = asec\theta tan\theta d\theta$

$$= \int \frac{1}{sec^2\theta tan\theta} sec\theta tan\theta d\theta$$
$$= \int cos\theta d\theta$$
$$= sin\theta + C$$
$$= \frac{\sqrt{x^2-1}}{x} + C$$

Exercise3. 3(d)

Evaluate:

1. $\int \frac{\sqrt{9-x^2}}{x^2}\, dx$ **Ans:** $-\frac{\sqrt{9-x^2}}{x} - sin^{-1}\left(\frac{x}{3}\right) + C$
2. $\int \frac{x}{\sqrt{a^2-x^2}}\, dx$ **Ans:** $-\sqrt{a^2-x^2} + C$
3. $\int \sqrt{4-x^2}\, dx$ **Ans:** $2\, sin^{-1} \frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$
4. $\int \frac{1}{x^2\sqrt{x^2+4}}\, dx$ **Ans:** $-\frac{\sqrt{x^2+4}}{4x} + C$
5. $\int \frac{1}{x\sqrt{5-x^2}}\, dx$ **Ans:** $-\frac{1}{\sqrt{5}}\log\left[\frac{\sqrt{5}}{x} + \frac{1}{x}\sqrt{5-x^2}\right]$
6. $\int \frac{t^5}{\sqrt{t^2+2}}\, dt$ **Ans:** $4\sqrt{2}\left[\frac{1}{5}\left(\frac{\sqrt{2+t^2}}{2}\right)^5 + \frac{\sqrt{2+t^2}}{2} + \frac{2}{3}\left(\frac{\sqrt{2+t^2}}{2}\right)^3\right] + C$
7. $\int \frac{\sqrt{1+x^2}}{x}\, dx$ **Ans:** $\log\left(\sqrt{1+x^2} - 1\right) - \log x + \sqrt{1+x^2} + C$
8. $\int \frac{x}{\sqrt{x^2+x+1}}\, dx$ **Ans:** $\sqrt{x^2+x+1} - \frac{1}{2}\log\left(\sqrt{x^2+x+1} + x + \frac{1}{2}\right) + C$
9. $\int \frac{1}{\sqrt{t^2+16}}\, dt$ **Ans:** $log\left(\sqrt{t^2+16} + t\right) + C$
10. $\int \frac{x^2}{\sqrt{2x-x^2}}\, dx$ **Ans:** $\frac{3}{2}\, sin^{-1}(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + C$

3.4 Integration of Rational functions by Partial fraction

Let $f(x) = \frac{P(x)}{Q\,(x)}$ be any rational function where $P\ and\ Q$ are polynomials.

If $\deg P < \deg Q$, then f is proper

If $\deg P \geq \deg Q$, then f is improper then to make them proper divide $P(x)\ by\ Q(x)$ by long division until a remainder $R(x)$ is obtained such that $\deg P < \deg Q$

Hence $\frac{P(x)}{Q\,(x)} = S(x) + \frac{R(x)}{Q\,(x)}$ (or) = Quotient + $\frac{Remainder}{Divisor}$

Where $S\ and\ R$ are also polynomials.

Case (i):

The denominator is a product of distinct linear factors

Example:

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

Case (ii):

The denominator is a product of distinct linear factors, some of which are repeated.

Example:

$$\frac{1}{(x+a)(x+b)^2} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2}$$

Case (iii):

The denominator contains irreducible quadratic factors, none of which is repeated.

Example:

$$\frac{1}{(x^2+a)(x^2+b)} = \frac{Ax+B}{(x^2+a)} + \frac{Cx+D}{(x^2+b)}$$

Problems based on integration of rational functions by partial fractions:

Example:3.67

Evaluate $\int \frac{(x^2+1)}{(x^2-1)(2x+1)}\, dx$

Solution:

$$\frac{(x^2+1)}{(x^2-1)(2x+1)} = \frac{(x^2+1)}{(x-1)(x+1)(2x+1)}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+1)}$$

$$(x^2+1) = A(x+1)(2x+1) + B\,(x-1)(2x+1) + C\,(x-1)(x+1)$$

Put $x = 1$, we get

$2 = A\,(2)\,(3)$

$A = \frac{1}{3}$

Put $x = -1$, we get

$2 = B\,(-2)\,(-1)$

$B = 1$

Put $x = 0$, we get

$1 = A - B - C$

$1 = \frac{1}{3} - 1 - C$

$C = -2 + \frac{1}{3} = \frac{-5}{3}$

$$\Rightarrow \frac{(x^2+1)}{(x^2-1)(2x+1)} = \frac{1}{3}\frac{1}{x-1} + \frac{1}{x+1} - \frac{5}{3}\frac{1}{2x+1}$$

$$\int \frac{(x^2+1)}{(x^2-1)(2x+1)}\, dx = \frac{1}{3}\int \frac{1}{x-1}\, dx + \int \frac{1}{x+1}\, dx - \frac{5}{3}\int \frac{1}{2x+1}\, dx$$

$$= \frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{3}\frac{\log(2x+1)}{2} + C$$

$$= \frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{6}\log(2x+1) + C$$

Example:3.68

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Solution:

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x\,(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx\,(x+2) + Cx\,(2x-1)$$

Put $x = 0$, we get

$-1 = A\,2$

$A = \frac{1}{2}$

Put $x = \frac{1}{2}$, we get

$\frac{1}{4} + 1 - 1 = B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right)$

$\frac{1}{4} = \frac{5B}{4}$

$B = \frac{1}{5}$

Put $x = -2$, we get

$4 - 4 - 1 = C(2)(-5)$

$-1 = 10\,C$

$C = \frac{-1}{10}$

$$\Rightarrow \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\left(\frac{1}{x}\right) + \frac{1}{5}\left(\frac{1}{2x-1}\right) - \frac{1}{10}\left(\frac{1}{x+2}\right)$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\int\left(\frac{1}{x}\right)dx + \frac{1}{5}\int\left(\frac{1}{2x-1}\right)dx - \frac{1}{10}\int\left(\frac{1}{x+2}\right)dx$$

$$= \frac{1}{2}\,logx + \frac{1}{5}\frac{\log(2x-1)}{2} - \frac{1}{10}\log(x+2) + C$$

$$= \frac{1}{2}\,logx + \frac{1}{10}\log\left(\frac{2x-1}{x+2}\right) + C$$

Example:3.69

Evaluate $\int \frac{sec^2x}{tan^2x+3tanx+2}\,dx$

Solution:

Put $u = tanx$; $du = sec^2x\,dx$

$$\therefore I = \int \frac{1}{u^2+3u+2}\,du$$

$$= \int \frac{1}{(u+1)(u+2)}\,du$$

$$= \int \left[\frac{1}{u+1} - \frac{1}{u+2}\right] du$$

$$= \log(u+1) - \log(u+2) + C$$

$$= \log(tanx+1) - \log(tanx+2) + C$$

$$= \log\left(\frac{1+tanx}{2+tanx}\right) + C$$

Example:3.70

Evaluate $\int \frac{x^2}{(x-1)^3(x-2)}\,dx$

Solution:

$$\frac{x^2}{(x-1)^3(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$x^2 = A(x-1)^3$ +B$(x-1)^2(x-2) +\ C\ (x-1)(x-2) +\ D\ (x-2)$

Put $x = 2$,

We get $4\ =\ A$

Equating the coeffs of x^3

On both sides

$0\ =\ A + B$

$B\ =\ -4$

Put x = 1, we get

$1\ = D(-1)$

$D\ =\ -1$

Put x=0, we get

$0\ =\ -A - 2B + 2C - 2D$

$2C\ =\ A + 2B + 2D$

$=\ 4 - 8 - 2$

$C = -3$

$$\Rightarrow \frac{x^2}{(x-1)^3(x-2)} = \frac{4}{x-2} - \frac{4}{x-1} - \frac{3}{(x-1)^2} - \frac{1}{(x-1)^3}$$

$$I = \int \frac{x^2}{(x-1)^3(x-2)}\ dx$$

$$= 4\ \int \frac{1}{x-2}\ dx - 4\ \int \frac{1}{x-1}\ dx - 3\ \int \frac{1}{(x-1)^2}\ dx - \int \frac{1}{(x-1)^3}\ dx$$

$$= 4\log(x-2) - 4\log(x-1) +\ 3\ \left(\frac{1}{x-1}\right) + \frac{1}{2(x-1)^2} + C$$

$$= 4\ log\left(\frac{x-2}{x-1}\right) + \frac{3}{x-1} + \frac{1}{2(x-1)^2} + C$$

Example:3.71

Evaluate $\int \frac{1}{x^2(x-1)}\ dx$

Solution:

Let $I = \int \frac{1}{x^2(x-1)}\ dx$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} \quad \text{... (1)}$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

Put $x = 0$,

We get $1 = -B$

$B = -1$

Put $x\ =\ 1$, we get

$1\ = C$

Equating the Coefficients of x^2on both side

$0\ =\ A + C \Rightarrow A = -C$

$A =\ -1$

(1) $\Rightarrow \frac{1}{x^2(x-1)} = \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{(x-1)}$

$$I = \int \frac{1}{x^2(x-1)}\ dx = -\int \frac{1}{x}\ dx - \int \frac{1}{x^2}\ dx + \int \frac{1}{(x-1)} dx$$

$$= -logx + \frac{1}{x} + \log(x-1) + C\ = log\left(\frac{x-1}{x}\right) + \frac{1}{x} + C$$

Example:3.72

Evaluate $\int \frac{10}{(x-1)(x^2+9)}dx$

Solution:

Let $I = \frac{10}{(x-1)(x^2+9)}dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \qquad \dots(1)$$

$$10 = A\,(x^2+9) + (Bx+C)(x-1)$$

Put $x = 1$, We get

$10 = 10\,A$

A = 1

Equating the Coefficients of x^2

We get

$0 = A + B \Rightarrow B = -A$

$B = -1$

Equating the Coefficients of x,

$0 = -B + C \Rightarrow -B = -C$

$C = -1$

$$(1) \Rightarrow \frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{-x-1}{x^2+9} = \frac{1}{x-1} - \left(\frac{x+1}{x^2+9}\right)$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+9}\,dx - \int \frac{1}{x^2+9}dx$$

$$= \log(x-1) - \frac{1}{2}\,log(x^2+9) - \frac{1}{3}tan^{-1}\left(\frac{x}{3}\right) + C$$

Example:3.73

Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Solution:

Let $I = \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-0x^3-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ \hline & \quad x^3-x^2+3x+1 \\ & \quad x^3-x^2-x+1 \\ \hline & \qquad\qquad 4x \end{array}$$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x+1}{x^3-x^2-x+1}$$

$$= x+1+\frac{4x+1}{(x-1)^2(x+1)}$$

$$[x^3-x^2-x+1 = (x-1)^2(x+1)]$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$\Rightarrow 4x = A(x+1)(x+1) + B(x+1) + C(x+1)^2$$

Put $x = 1$, We get

$$4 = 2B$$

$$B = 2$$

Put $x = -1$, We get

$$-4 = 4C$$

$$C = -1$$

Equating the Coefficient of x^2 on, both sides , we get

$$0 = A + C \Rightarrow A = -C$$

$$A = 1$$

$$\Rightarrow \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{(x+1)}$$

$$I = \int (x+1)dx + \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx - \int \frac{1}{(x+1)}\,dx$$

$$= \frac{x^2}{2} + x + \log(x-1) - \frac{2}{x-1} - \log(x+1) + C$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \log\left(\frac{x-1}{x+1}\right) + C$$

Example:3.74

Evaluate $\int \frac{x^{24}}{x^{10}+1}\,dx$

Solution:

Given $\int \frac{x^{24}}{x^{10}+1}\,dx$

$$\begin{array}{r|l} & x^{14} - x^4 \\ \hline x^{10}+1 & x^{24} \\ & x^{24} + x^{14} \\ \hline & -x^{14} \\ & -x^{14} - x^4 \\ \hline & x^4 \end{array}$$

$$\frac{x^{24}}{x^{10}+1} = x^{14} - x^4 + \frac{x^4}{x^{10}+1}$$

$$\int \frac{x^{24}}{x^{10}+1}\,dx = \int x^{14} - x^4 + \frac{x^4}{x^{10}+1}\,dx$$

$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \frac{1}{5}\int \frac{1}{y^2+1}\,dy$$

$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \frac{1}{5}tan^{-1}(y) + C$$

$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \frac{1}{5}tan^{-1}(x^5) + C$$

Put $y = x^5$

$$dy = 5x^4dx$$

$$x^4dx = \frac{1}{5}dy$$

Integration of Rational Functions

Type II

$$\int \frac{1}{ax^2+bx+c}\, dx$$

There are two methods to evaluate this integral.

(i) By using Partial fraction

(ii) Express the integral in one of the following forms.

(iii) $\frac{1}{a^2+x^2}, \frac{1}{a^2-x^2}, \frac{1}{x^2-a^2}$ and then use the following result.

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} tan^{-1}(x/a) + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + C$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + C$$

Note :

If the denominator of the integrand is not factorizable, only we can use the second method.

Example:3.75

(i) Evaluate $\int \frac{1}{3+2x+x^2}$ **dx**

Solution:

$$(x^2 + 2x + 3 = (x+1)^2 + 3 - 1) = (x+1)^2 + 2 = (x+1)^2 + \left(\sqrt{2}\right)^2)$$

$$\int \frac{1}{3+2x+x^2}\, dx = \int \frac{dx}{(x+1)^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}} tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

(ii)Evaluate $\int \frac{dx}{1+x-x^2}$

Solution:

$$1 + x - x^2 = -(x^2 - x - 1)$$

$$= -\left[\left(x - \frac{1}{2}\right)^2 - 1 - \frac{1}{4}\right]$$

$$= -\left[\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}\right]$$

$$= -\left[\left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2\right] = \left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2$$

$$= \int \frac{dx}{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)} \log\left[\frac{\left(\frac{\sqrt{5}}{2}\right)+\left(x-\frac{1}{2}\right)}{\left(\frac{\sqrt{5}}{2}\right)-\left(x-\frac{1}{2}\right)}\right] + C$$

$$= \frac{1}{\sqrt{5}} \log\left[\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right] + C$$

Type III

$$\int \frac{Px+q}{ax^2+bx+c}\,dx$$

$$Nr = A\frac{d}{dx}(Dr) + B$$

$$\int \frac{Px+q}{ax^2+bx+c}\,dx = A\int\left[\frac{\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}\right]dx$$

$$= A\log(ax^2+bx+c) + B\int\frac{1}{ax^2+bx+c}\,dx \qquad \text{(Proceed as type II)}$$

Example:3.76

(i)Evaluate $\int \frac{2x+3}{x^2+2x+5}\,dx$

Solution:

Let $2x+3 = A\frac{d}{dx}(x^2+2x+5) + B$

$2x+3 = A\,(2x+2) + B$ ... (1)

Equating the coefficients of x we get

$2 = 2A$

$A = 1$

Put $x = 0$ we get

$3 = 2A + B \Rightarrow 3 = 2 + B$

$B = 1$

$\therefore (1) \Rightarrow 2x+3 = (2x+2) + 1$

$$= \int \frac{(2x+2)+1}{x^2+2x+5}\,dx$$

$$= \int \frac{(2x+2)+1}{x^2+2x+5}\,dx + \int\frac{1}{x^2+2x+5}\,dx$$

$$= \log(x^2+2x+5) + \int\frac{1}{(x+1)^2+5-1}$$

$$= \log(x^2+2x+5) + \int\frac{dx}{(x+1)^2+2^2}$$

$$= \log(x^2+2x+5) + \frac{1}{2}tan^{-1}\left(\frac{x+1}{2}\right) + C$$

(ii)Evaluate $\int \frac{x^2+x+1}{x^2-x+1}\,dx$

Solution:

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2-x+1 & x^2+x+1 \\ & x^2-x+1 \\ \hline & \quad 2x \end{array}$$

$$\frac{x^2+x+1}{x^2-x+1} = 1 + \frac{2x}{x^2-x+1}$$

$$= \int 1 + \frac{2x}{x^2-x+1}\, dx$$

$$= x + \int \frac{2x}{x^2-x+1}\, dx$$

$$= x + I \quad \ldots(1)$$

Where $I = \int \frac{2x}{x^2-x+1}\, dx$

$$2x = A\, \frac{d}{dx}(x^2 - x + 1) + B$$

$$2x = A\ (2x - 1) + B$$

Equating the coefficients of x we get,

$2 = 2A$

$A = 1$

Put $x = 0$ we get

$0 = A + B \Rightarrow 0 = -1 + B$

$B = 1$

$$\therefore \int \frac{2x}{x^2-x+1}\, dx = \int \frac{(2x-1)+1}{x^2-x+1}\, dx$$

$$= \int \frac{2x-1}{x^2-x+1}\, dx + \int \frac{1}{x^2-x+1}\, dx$$

$$= log(x^2 - x + 1) + \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + 1 - \frac{1}{4}}\, dx$$

$$= log(x^2 - x + 1) + \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\, dx$$

$$= log(x^2 - x + 1) + \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$= log(x^2 - x + 1) + \frac{1}{\frac{\sqrt{3}}{2}} tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$(1) \Rightarrow \int \frac{x^2+x+1}{x^2-x+1}\, dx = x + log(x^2 - x + 1) + \frac{2}{\sqrt{3}} tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

(iii)Evaluate $\int \frac{3x+5}{x^2+4x+7}\, dx$

Solution:

Let $3x + 5 = A\frac{d}{dx}(x^2 + 4x + 7) + B$

$$= A\,(2x + 4) + B$$

$$3x + 5 = 2Ax + (4A + B)$$

Equating the coefficients of x and constant terms on both sides

$$3 = 2A \ \Rightarrow A = \frac{3}{2}\ and\ \ 5 = 4A + B \ \ \Rightarrow B = -1$$

$$\int \frac{3x+5}{x^2+4x+7} = \frac{3}{2}\int \frac{\frac{d}{dx}(x^2+4x+7)dx}{x^2+4x+7} - \int \frac{1}{x^2+4x+7}\, dx$$

$$= \frac{3}{2}\log(x^2+4x+7) - \int \frac{1}{(x+2)^2+3}\, dx$$

$$= \frac{3}{2}\log(x^2+4x+7) - \frac{1}{\sqrt{3}} tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + C$$

Exercise3. 4

Evaluate:

1. $\int \frac{2x-3}{x^2+2x-3}\, dx$ **Ans:** $-\frac{1}{4}\, log\left(\frac{x-1}{(x+3)^9}\right) + C$

2. $\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\, dx$ **Ans:** $\frac{7}{6} + log\left(\frac{2}{3}\right)$

3. $\int \frac{x^2+1}{(x-3)(x-2)^2}\, dx$ **Ans:** $10\, log\,(x-3) - 9\, log\,(x-2) + 5\,\frac{1}{(x-2)} + C$

4. $\int_3^4 \frac{x^3-4x-10}{x^2-x-6}\, dx$ **Ans :** $\frac{3}{2}\, log\left(\frac{27}{2}\right)$

5. $\int \frac{x^3+4}{x^2+4}\, dx$ **Ans:** $\frac{x^2}{2} + 2\, tan^{-1}\frac{x}{2} - 2log\,(x^2+1) + C$

6. $\int \frac{2x^2-x+4}{x^3+4x}\, dx$ **Ans:** $logx + \frac{1}{2}\, log(x^2+4) - \frac{1}{2}\, tan^{-1}\left(\frac{x}{2}\right) + C$

7. $\int_0^1 \frac{x^4}{1+x^2}\, dx$ **Ans:** $\frac{\pi}{4} - \frac{2}{3}$

8. $\int \frac{x^{27}}{x^{14}+4}\, dx$ **Ans:** $\frac{1}{7}\left[\frac{x^{14}}{2} - 2log(x^{14}+4) + C\right]$

9. $\int \frac{x+4}{x^2+2x+5}\, dx$ **Ans:** $\frac{1}{2}\, log(x^2+2x+15) + \frac{3}{2}\, tan^{-1}\left(\frac{x+1}{2}\right) + C$

10. $\int \frac{x^2+3x+5}{(x+1)(x+3)(x+5)}\, dx$ **Ans:** $\frac{3}{8}\, log\,(x+1) - \frac{5}{8}(x+3) + \frac{15}{8}\, log(x+5) + C$

11. $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx$ **Ans:** $log\,(x^2+1) + tan^{-1}(x) - 2\, log\,(x-1) - \frac{1}{x-1} + C$

12. $\int \frac{1}{3x^2-4x-5}\, dx$ **Ans:** $\frac{1}{2\sqrt{19}}\, log\left(\frac{3x-2-\sqrt{19}}{3x-2+\sqrt{19}}\right) + C$

13. $\int \frac{1}{3x^2+13x-10}\, dx$ **Ans:** $\frac{1}{17}\, log\left(\frac{3x-2}{3x+15}\right) + C$

14. $\int \frac{x^2}{x^4-1}\, dx$ **Ans:** $\frac{1}{2}\left[tan^{-1}x + \frac{1}{2}\, log\left(\frac{x+1}{x-1}\right)\right] + C$

15. $\int \frac{1}{x^3+1}\, dx$ **Ans:** $\frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\, tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

3.5Integration of irrational functions

We have already discussed the methods to integrate rational functions, However in the case of irrational functions it is very difficult to integrate. Here we developed some systematic procedure to integrate the irrational functions.

The following are some of the standard forms of irrational functions.

Type I

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\, dx$$

Formulae used in this topic

$$\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}(x/a) + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = cosh^{-1}(x/a) + C = log\left(x + \sqrt{x^2 - a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = sinh^{-1}(x/a) + C = log\left(x + \sqrt{a^2 + x^2}\right) + C$$

Example:3.77

(i)Evaluate $\int \frac{1}{\sqrt{8+3x-x^2}}\, dx$

Solution:

$$\int \frac{1}{\sqrt{8+3x-x^2}}\, dx = \int \frac{1}{\sqrt{-\left[x^2-3x-8\right]}}\, dx$$

$$= \int \frac{1}{\sqrt{-\left[\left(x-\frac{3}{2}\right)^2 - 8 - \frac{9}{4}\right]}}\, dx$$

$$= \int \frac{1}{\sqrt{-\left[\left(x-\frac{3}{2}\right)^2 - \frac{41}{4}\right]}}\, dx$$

$$= \int \frac{1}{\sqrt{\left[\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2\right]}}\, dx$$

$$= sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right) + C$$

(ii)Evaluate $\int \frac{1}{\sqrt{[x^2-x-2]}}\, dx$

Solution:

Consider, $[x^2 - x - 2] = x^2 - x + \frac{1}{4} - \frac{9}{4}$

$$= \left[\left(x-\frac{1}{2}\right)^2 - \frac{9}{4}\right] \qquad = \left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

$$\int \frac{1}{\sqrt{[x^2-x-2]}}\, dx = \int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}}\, dx$$

$$= \log\left(x-\frac{1}{2}\right) + \sqrt{[x^2-x-2]} + C$$

(iii)Evaluate $\int \frac{\mathbf{1}}{\sqrt{\mathbf{2x^2-x+5}}}\, \boldsymbol{dx}$

Solution:

Consider, $\quad 2x^2 - x + 5 = 2\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{5}{2}\right]$

$$= 2\left[x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \frac{1}{16} + \frac{5}{2}\right]$$

$$= 2\left[\left(x-\frac{1}{4}\right)^2 + \left(\frac{\sqrt{39}}{4}\right)^2\right]$$

$$\int \frac{1}{\sqrt{2x^2-x+5}}\, dx = \int \frac{1}{\sqrt{2\left[\left(x-\frac{1}{4}\right)^2+\left(\frac{\sqrt{39}}{4}\right)^2\right]}}\, dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left[\left(x-\frac{1}{4}\right)^2+\left(\frac{\sqrt{39}}{4}\right)^2\right]}}\, dx$$

$$= \frac{1}{\sqrt{2}}\log\left(\left(x-\frac{1}{4}\right) + \frac{1}{\sqrt{2}}\sqrt{2x^2-x+5}\right) + C$$

TYPE II

$$\int \frac{\boldsymbol{px+q}}{\sqrt{\boldsymbol{ax^2+bx+c}}}\, \boldsymbol{dx}$$

Put $px + q = A\frac{d}{dx}(ax^2+bx+c) + B$ and then proceed.

Example:3.78

(i)Evaluate $\int \frac{\boldsymbol{x}}{\sqrt{\mathbf{3-2x-x^2}}}\, \boldsymbol{dx}$

Solution:

Let $x = A\ \frac{d}{dx}(3-2x-x^2) + B$

$$= A(-2-2x) + B$$

$$x = -2Ax + (-2A+B)$$

Equating the coefficients of x and constant terms on both sides

$$1 = -2A \Rightarrow A = -\frac{1}{2}\ and\ 0 = -2A + B \Rightarrow B = -1$$

$$= \int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \frac{1}{2}\int \frac{\frac{d}{dx}(3-2x-x^2)}{\sqrt{3-2x-x^2}}\,dx - \int \frac{1}{\sqrt{3-2x-x^2}}\,dx$$

$$= \frac{1}{2} \times 2\sqrt{3-2x-x^2} - \int \frac{1}{\sqrt{3-2x-x^2}}\,dx$$

Now we complete square

$$3-2x-x^2 = 4-1-2x-x^2 = 4-(x+1)^2$$

$$= \int \frac{x}{\sqrt{3-2x-x^2}}\,dx = -\sqrt{3-2x-x^2} - \int \frac{1}{\sqrt{3-2x-x^2}}\,dx$$

$$= -\sqrt{3-2x-x^2} - \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx$$

$$= -\sqrt{3-2x-x^2} - sin^{-1}\left(\frac{x+1}{2}\right)\,dx$$

(ii)Evaluate $\int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx$

Solution:

Given $\int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx$

$$2x-1 = A\frac{d}{dx}(x^2+5x+6)+B$$

$$2x-1 = A(2x+5)+B$$

Equating the coefficients of x we get

$$2 = 2A \quad \Rightarrow A = 1$$

Put $x = 0$ we get

$$-1 = 5A+B$$

$$-1 = 5+B \Rightarrow B = -6$$

$$\therefore \int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx = \int \frac{2x-5}{\sqrt{x^2+5x+6}}\,dx + \int \frac{-6}{\sqrt{x^2+5x+6}}\,dx$$

$$= 2\sqrt{x^2+5x+6} - 6\int \frac{1}{\sqrt{\left[\left(x+\frac{5}{2}\right)^2+6-\frac{25}{4}\right]}}\,dx$$

$$= 2\sqrt{x^2+5x+6} - 6\int \frac{1}{\sqrt{\left[\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}}\,dx$$

$$= 2\sqrt{x^2+5x+6} - 6\log\left[\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right] + C$$

TYPE III

$$\int \sqrt{\boldsymbol{ax^2+bx+c}}\,\boldsymbol{dx}$$

Formulae used

$$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right) + C$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right) + C$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right) + C$$

Example:3.79

(i)Evaluate $\int \sqrt{x^2+2x+5}\,dx$

Solution:

Consider $\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{(x+1)^2+\ 4}\ dx$

$$= \frac{x+1}{2}\sqrt{x^2+2x+5} + \frac{4}{2}\log\left((x+1)+\sqrt{x^2+2x+5}\right) + C$$

(ii)Evaluate $\int \sqrt{5+4x-x^2}\ dx$

Solution:

Consider $5+4x-\ x^2 = 9-\ (x^2-4x+4)$

$$= 9-\ (x-2)^2$$

Hence $\int \sqrt{5+4x-x^2}\ dx = \int \sqrt{9-\ (x-2)^2}\ dx$

$$= \frac{9}{2}sin^{-1}\left(\frac{x-2}{3}\right) + \left(\frac{x-2}{2}\right)\sqrt{5+4x-x^2} + C$$

Type IV

$$\int (px+q)\sqrt{ax^2+bx+c}\ dx$$

Put $px+q = A\,d(ax^2+bx+c\,) +\ B$

$= A\ \int \sqrt{ax^2+bx+c}\ \ d\ (ax^2+bx+c) + B\int \sqrt{ax^2+bx+c}\ \ dx$ and then proceed.

Example:3.80

(i)Evaluate $\int (3x-2)\sqrt{x^2+x+1}\ dx$

Solution:

Given $\int (3x-2)\sqrt{x^2+x+1}\ dx$

Put $3x-2 = A\,d(x^2+x+1\,) +\ B$

$3x-2 = A\,(2x+1\,) +\ B$

Equating the coefficients of x we get

$$3 = 2A \quad \Rightarrow A = \frac{3}{2}$$

Put $x = 0$ we get

$$-2 = A + B = \frac{3}{2} + B$$

$$B = -2 - \frac{3}{2} = \frac{-7}{2}$$

$$\int (3x-2)\sqrt{x^2+x+1}\ dx$$

$$= \frac{3}{2}\int \sqrt{x^2+x+1}\, d\,(x^2+x+1) + \left(\frac{-7}{2}\right)\int \sqrt{x^2+x+1}\, dx$$

$$= \frac{3}{2}\frac{(x^2+x+1)^{3/2}}{3/2} - \frac{7}{2}\sqrt{\left[\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}\right]}\, dx$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2}\sqrt{\left[\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]\ dx}$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2}\left[\frac{x+\frac{1}{2}}{2}\sqrt{x^2+x+1} + \frac{\left(\frac{3}{4}\right)}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right] + C$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1} + \frac{3}{8}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right] + C$$

(ii)Evaluate $\int (x+1)\sqrt{x^2-2x+3}\ dx$

Solution:

Let $x + 1 = A\frac{d}{dx}(x^2-2x+3) + B$

$$= A(\ 2x-2) + B$$

$$x + 1 = 2Ax + (-2A + B)$$

Equating the coefficients of x and constant terms on both sides.

$1 = 2A \Rightarrow A = \frac{1}{2}$

and $1 = -2A + B \Rightarrow B = 2$

$$\int (x+1)\sqrt{x^2-2x+3}\, dx = \frac{1}{2}\int \sqrt{x^2-2x+3}\frac{d}{dx}(x^2-2x+3) + 2\sqrt{x^2-2x+3}\, dx$$

$$= \frac{1}{2}\frac{(x^2-2x+3)^{3/2}}{3/2} + 2\int \sqrt{x^2-2x+3}\, dx$$

$$= \frac{1}{3}(x^2-2x+3)^{3/2} + 2\int \sqrt{x^2-2x+3}\, dx$$

Now we complete the square

$$(x^2-2x+3) = x^2-2x+3 = (x-1)^2+2$$

$$\int (x+1)\sqrt{x^2-2x+3}\,dx = \frac{1}{3}(x^2-2x+3)^{3/2} + 2\int \sqrt{(x-1)^2+2}\,dx$$

$$= \frac{1}{3}(x^2-2x+3)^{3/2} + (x-1)\sqrt{x^2-2x+3} + 2\log(x-1) + \sqrt{x^2-2x+3} + C$$

TYPE V

$$\int \frac{\boldsymbol{dx}}{(\boldsymbol{px+q})\sqrt{\boldsymbol{ax^2+bx+c}}}$$

Put $px + q = \frac{1}{t}$ and then proceed

Example:3.81

(i)Evaluate $\int \frac{\boldsymbol{dx}}{(\boldsymbol{x-1})\sqrt{\boldsymbol{x^2-2x+3}}}\,\boldsymbol{dx}$

Solution:

Given $\int \frac{dx}{(x-1)\sqrt{x^2-2x+3}}\,dx$

Put $x - 1 = \frac{1}{t}$; $x = 1 + \frac{1}{t}$; $dx = \frac{-1}{t^2}\,dt$

$$= \int \frac{1}{\frac{1}{t}\sqrt{\left[\left(1+\frac{1}{t}\right)^2 - 2\left(1+\frac{1}{t}\right) + 3\right]}}\left(\frac{-1}{t^2}\right)dt$$

$$= -\int \frac{1}{t\sqrt{\frac{(1+t)^2}{t^2} - 2\frac{(t+1)}{t} + 3}}\,dt$$

$$= -\int \frac{1}{\sqrt{(1+t)^2 - 2t\,(t+1) + 3t^2}}\,dt$$

$$= \int \frac{1}{\sqrt{1+t^2+2t-2t^2-2t+3t^2}}\,dt$$

$$= -\int \frac{1}{\sqrt{2t^2+1}}\,dt$$

$$= -\int \frac{1}{\sqrt{u^2+1}}\frac{du}{\sqrt{2}} \qquad put\ u = t\sqrt{2} \Rightarrow du = \sqrt{2}\,dt$$

$$= -\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{u^2+1}}\,du$$

$$= -\frac{1}{\sqrt{2}} sinh^{-1}u + C = -\frac{1}{\sqrt{2}} sinh^{-1}\left(\sqrt{2}\,t\right) + C$$

$$= -\frac{1}{\sqrt{2}} sinh^{-1}\left(\frac{\sqrt{2}}{x-1}\right) + C$$

(ii)Evaluate $\int \frac{\boldsymbol{dx}}{(\boldsymbol{x-1})\sqrt{\boldsymbol{x^2+2x-8}}}\,\boldsymbol{dx}$

Solution:

Given $\int \frac{dx}{(x-1)\sqrt{x^2+2x-8}}\,dx$

Put $x-1=\frac{1}{t}$; $x=1+\frac{1}{t}$; $dx=\frac{-1}{t^2}\,dt$

$$=\int \frac{1}{\frac{1}{t}\sqrt{\left[\left(1+\frac{1}{t}\right)^2+2\left(1+\frac{1}{t}\right)-8\right]}}\left(\frac{-1}{t^2}\right)\,dt$$

$$=-\int \frac{1}{t\sqrt{\frac{(1+t)^2}{t^2}+2\frac{(t+1)}{t}-8}}\,dt$$

$$=-\int \frac{1}{\sqrt{(1+t)^2+2t\,(t+1)-8t^2}}\,dt$$

$$=-\int \frac{1}{\sqrt{1+t^2+2t+2t^2+2t-8t^2}}\,dt$$

$$=-\int \frac{1}{\sqrt{1+4t-5t^2}}\,dt$$

$$=-\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{-t^2+\frac{4}{5}t+\frac{1}{5}}}dt$$

$$=-\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{-t^2+\frac{4}{5}t+\left(\frac{2}{5}\right)^2-\left(\frac{2}{5}\right)^2+\frac{1}{5}}}dt$$

$$=-\frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{\frac{9}{25}-\left(t-\frac{2}{5}\right)^2}}\,dt$$

$$=-\frac{1}{\sqrt{5}}sin^{-1}\left(\frac{t-\frac{2}{5}}{\frac{3}{5}}\right)+C$$

$$=-\frac{1}{\sqrt{5}}sin^{-1}\left(\frac{5t-2}{\frac{3}{5}}\right)+C$$

$$=-\frac{1}{\sqrt{5}}sin^{-1}\left(\frac{\frac{5}{x-1}-2}{3}\right)+C$$

Type VI

$$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$$

Put $x=\frac{1}{t}$ and then proceed.

Example:3.82

(i)Prove that $\int_0^1 \frac{dx}{1+x^2\sqrt{x^2+2}} = {}^{\pi}/_{6}$

Solution:

Given $\int \frac{dx}{1+x^2\sqrt{x^2+2}}$

Put $x=\frac{1}{t}$, $\quad dx=\frac{-1}{t^2}\,dt$

$$\int \frac{dx}{1+x^2\sqrt{x^2+2}} = \int \frac{\left(\frac{-1}{t^2}\right)dt}{\left(1+\frac{1}{t^2}\right)\sqrt{\frac{1}{t^2}+2}}$$

$$= \int \frac{-dt}{\left(t^2+1\right)\sqrt{\frac{1+2t^2}{t^2}}}$$

Put $u^2 = 1 + 2t^2$

$$= \int \frac{-t\,dt}{(t^2+1)\sqrt{1+2t^2}}$$

$2udu = 4t\,dt$

$$= \int \frac{-\frac{u}{2}du}{\left(\frac{u^2+1}{2}\right)\sqrt{1+u^2-1}}udu$$

$\frac{u}{2}\,du = t\,dt$

$$= \int \frac{-\frac{u}{2}du}{\left(\frac{u^2+1}{2}\right)u}$$

$u^2 = 1 + 2t^2$

$$= \int \frac{-du}{u^2+1}$$

$t^2 = \frac{u^2-1}{2}$

$$= -\int \frac{du}{u^2+1}$$

$1 + t^2 = 1 + \frac{u^2-1}{2}$

$$= -tan^{-1}u$$

$= \frac{u^2+1}{2}$

$$= tan^{-1}\sqrt{1+2t^2} = -tan^{-1}\sqrt{1+\frac{2}{x^2}}$$

$$\int_0^1 \frac{dx}{1+x^2\sqrt{x^2+2}} = \left[-tan^{-1}\sqrt{1+\frac{2}{x^2}}\right]_0^1$$

$$= \left(-tan^{-1}\sqrt{3}\right) - (-tan^{-1}\infty)$$

$$= \left(-\frac{\pi}{3}\right) + \frac{\pi}{2} = \frac{\pi}{6}$$

(ii)Evaluate: $\int \frac{dx}{x^2\sqrt{4+x^2}}$

Solution:

Given $\int \frac{dx}{x^2\sqrt{4+x^2}}$

Put $x = \frac{1}{t} \quad \Rightarrow \quad dx = -\frac{1}{t^2}\,dt$

$$\int \frac{dx}{x^2\sqrt{4+x^2}} = \int \frac{\frac{-1}{t^2}\,dt}{\frac{1}{t^2}\sqrt{4+\frac{1}{t^2}}}$$

$$= -\int \frac{t\,dt}{\sqrt{4t^2+1}}$$

$$= -\frac{1}{8}\int \frac{8t\,dt}{\sqrt{4t^2+1}}$$

$$= -\frac{1}{8}\int \frac{d\,(4t^2+1)}{\sqrt{4t^2+1}}$$

$$= -\frac{1}{8}\left[2\sqrt{4t^2+1}\right]$$

$$= -\frac{1}{4}\sqrt{\frac{4}{x^2}+1}$$

$$= -\frac{\sqrt{4+x^2}}{4x}$$

Type VII

An expression involves only one irrational quantity of the form $(ax+b)^{1/n}$ or $x^{1/n}$, then put $+b = t^n$, where n is the L.C.M of denominators of the various fractional powers.

Example:3.83

(i)Evaluate $\int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}}$

Solution:

Given $\int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}}$

Since L.C.M of 2 and 3 is 6.

Let $1+x = t^6 \quad \Rightarrow \quad dx = 6t^5\,dt$

$$\int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}} = \int \frac{6\,t^5}{t^3-t^2}\,\text{dt} = \int \frac{6\,t^5}{t^2(t-1)}\,\text{dt}$$

$$= \int \frac{6\,t^3}{(t-1)}\,\text{dt}$$

$$= 6\int \frac{(t^3-1+1)}{t-1}\,dt$$

$$t^3-1^3 = (t-1)(t^2+t+1)$$

$$= 6\int \left[\frac{t^3-1}{t-1}+\frac{1}{t-1}\right]dt$$

$$= 6\int \left\{(t^2+t+1)+\frac{1}{t-1}\right\}dt$$

$$= 6\left[\frac{t^3}{3}+\frac{t^2}{2}+t+\log(t-1)\right]$$

$$= 2\,(1+x)^{1/2}+3\,(1+x)^{1/3}+6(1+x)^{1/6}+6\log\left\{\left((1+x)^{1/6}\right)-1\right\}$$

(ii)Evaluate $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Solution:

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx = \int \frac{1}{x^{1/2}-x^{1/3}}\,dx$$

Since L.C.M of 2 and 3 is 6.

Let $u^6 = x$ $\quad\quad 6u^5du = dx$

$$\int \frac{1}{x^{1/2}-x^{1/3}}\,dx = \int \frac{1}{u^3-u^2}\ 6u^5d$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$= \int \frac{1}{u^2(u-1)}\ 6u^5du$$

$$= \int \frac{6u^3}{u-1}\,du$$

$$= 6\int \frac{u^3-1+1}{u-1}\,du$$

$$= 6\int \left[\frac{(u-1)(u^2+u+1)}{u-1} + \frac{1}{u-1}\right] du$$

$$= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u\right] + 6\log(u-1) + C$$

$$= 2u^3 + 3u^2 + 6u + 6\log(u-1) + C \quad \text{where } u = x^{1/6}$$

$$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\log\left(\sqrt[6]{x}-1\right) + C$$

Type VIII

$$\int \frac{\mathbf{1}}{\sqrt{(\boldsymbol{x}-\boldsymbol{a})(\mathbf{x}-\mathbf{b})}}\ \textbf{(or)}\int \sqrt{\frac{\boldsymbol{x}-\boldsymbol{a}}{\boldsymbol{b}-\boldsymbol{x}}}, \boldsymbol{b} > a$$

Put $x = acos^2\theta + bsin^2\theta$

$$= \int (u^2+u+1)\,du + 6\int \frac{1}{u-1}\,du$$

Example:3.84

(i)Evaluate $\int \sqrt{\frac{5-x}{x-2}}\,dx$

Solution:

Let $x = 2sin^2\theta + 5cos^2\theta$ $\quad\quad dx = (4\sin\theta\cos\theta - 10\cos\theta\sin\theta)d\theta$

$$dx = (-6\sin\theta\cos\theta)d\theta$$

$$\therefore 5 - x = 5(sin^2\theta + cos^2\theta) - (2sin^2\theta + 5cos^2\theta) = 3sin^2\theta$$

$$and\ x - 2 = 2sin^2\theta + 5cos^2\theta - 2(sin^2\theta + cos^2\theta) = 3\,cos^2\theta$$

$$\int \sqrt{\frac{5-x}{x-2}}\,dx = \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\cos\theta}(-6\sin\theta\cos\theta\,d\theta)$$

$$= -6\int sin^2\theta\,d\theta$$

$$= -6\int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= -3\left[\theta - \frac{\sin 2\theta}{2}\right]$$

$$= -3\theta + \frac{3}{2}\sin 2\theta$$

$$= -3\, sin^{-1}\left(\sqrt{\frac{5-x}{3}}\right) + 3\sin\theta\cos\theta$$

$$= -3\, sin^{-1}\left(\sqrt{\frac{5-x}{3}}\right) + 3\sqrt{\frac{5-x}{3}}\sqrt{\frac{x-2}{3}}$$

$$= -3\, sin^{-1}\left(\sqrt{\frac{5-x}{3}}\right) + \sqrt{(5-x)(x-2)}$$

(ii)Evaluate $\int \frac{\boldsymbol{dx}}{\sqrt{(\boldsymbol{x}-\boldsymbol{\alpha})(\boldsymbol{x}-\boldsymbol{\beta})}}$ $\quad \boldsymbol{\beta} > \alpha$

Solution:

Given $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ $\quad \beta > \alpha$

Put $x = \alpha\, sin^2\theta + \beta\, cos^2\theta \quad \Rightarrow dx = (2\alpha\sin\theta\cos\theta - 2\beta\cos\theta\sin\theta)\, d\theta$

$$\Rightarrow dx = 2(\alpha - \beta)\sin\theta\cos\theta\ d\theta$$

$\therefore x - \alpha = (\alpha sin^2\theta + \beta cos^2\theta) - \alpha(sin^2\theta + cos^2\theta) = (\beta - \alpha)cos^2\theta$

$and\ \beta - x = \beta(sin^2\theta + cos^2\theta) - (\alpha sin^2\theta + \beta cos^2\theta) = (\beta - \alpha)\, sin^2\theta$

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} = \int \frac{2(\alpha-\beta)\sin\theta\cos\theta\, d\theta}{\sqrt{(\beta-\alpha)\cos^2\theta\,(\beta-\alpha)\, sin^2\theta}}$$

$$= 2(\alpha - \beta)\int \frac{d\theta}{\sqrt{(\beta-\alpha)^2}}$$

$$= -\frac{2(\alpha - \beta)}{\beta - \alpha}\int d\theta$$

$$= -2\int d\theta$$

$$= -2\theta$$

$$= -2\, sin^{-1}\sqrt{\frac{\beta - x}{\beta - \alpha}}$$

Type IX

$$\int \frac{\boldsymbol{dx}}{\boldsymbol{a + b\, cosx}}\ \boldsymbol{or}\ \int \frac{\boldsymbol{dx}}{\boldsymbol{a + b\, sinx}}\ \textbf{or} \int \frac{\boldsymbol{dx}}{\boldsymbol{a\, sinx + b\, cosx + C}}$$

Put $t = tan\, {}^{x}/_{2} \ \Rightarrow\ dt = \frac{1}{2}sec^2\, {}^{x}/_{2}\, dx$

$$= \frac{1}{2}\left[1 + tan^2\, {}^{x}/_{2}\right]dx$$

$$= \frac{1}{2}(1 + t^2)dx$$

$$dx = \frac{2}{1+t^2}\ dt$$

$$sinx = \frac{2t}{1+t^2},\quad cosx = \frac{1-t^2}{1+t^2}$$

Example:3.85

(i)Evaluate $\int_0^{\pi} \frac{dx}{5+3cosx}$

Solution:

Consider $\int \frac{dx}{5+3cosx}$

Put $t = tan\frac{x}{2}$

$dt = \frac{1}{2}(1+t^2)dx$

$dx = \frac{2}{1+t^2}\ dt$

$sinx = \frac{2t}{1+t^2}$

$cosx = \frac{1-t^2}{1+t^2}$

$$\int \frac{dx}{5+3cosx} = \left(\int \frac{\frac{2dt}{1+t^2}}{5+3\left(\frac{1-t^2}{1+t^2}\right)}\right)$$

$$= \int \frac{2dt}{8+2t^2} = \int \frac{dt}{4+t^2}$$

$$= \frac{1}{2}\left(tan^{-1}\left(\frac{t}{2}\right)\right)$$

$$= \frac{1}{2}\left(tan^{-1}\left(\frac{tan(x/2)}{2}\right)\right)$$

$$\int_0^{\pi} \frac{dx}{5+3cosx} = \left[\frac{1}{2}\left(tan^{-1}\left(\frac{tan(x/2)}{2}\right)\right)\right]_0^{\pi}$$

$$= \frac{1}{2}tan^{-1}\left(\infty - \frac{1}{2}\ 0\right)$$

$$= \frac{1}{2}tan^{-1}\infty$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

Type X

Example:3.86

Prove that $\int_0^{\pi} \frac{dx}{a^2cos^2x+b^2sin^2x} = \frac{\pi}{ab}$

Solution:

Consider $\int \frac{dx}{a^2cos^2x+b^2sin^2x}$

$$= \int \frac{dx}{\cos^2 x\, [a^2+b^2\tan^2 x]}$$

$$= \int \frac{\sec^2 x dx}{a^2+b^2\tan^2 x}$$

put $t = \tan x \;\; \Rightarrow dt = \; \sec^2 x dx$

$$= \int \frac{dt}{a^2+b^2 t^2}$$

$$= \int \frac{dt}{b^2\left[t^2+\left(\frac{a}{b}\right)^2\right]}$$

$$= \frac{1}{ab}\left(\tan^{-1}\left(\frac{bt}{a}\right)\right)$$

$$= \frac{1}{ab}\left(\tan^{-1}\left(\frac{b}{a}\tan x\right)\right)$$

$$\int_0^{\pi} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = 2\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$= 2\left[\frac{1}{ab}\left(\tan^{-1}\left(\frac{b\tan x}{a}\right)\right)\right]_0^{\pi/2}$$

$$= \frac{2}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{ab}$$

Exercise 3.5

Evaluate

1. $\int \frac{1}{\sqrt{x^2 - 6x + 13}}\, dx$ **Ans:** $\log\left(x - 3\sqrt{x^2 - 6x + 13}\right) + C$

2. $\int \frac{1}{\sqrt{3x - 2x^2}}\, dx$ **Ans:** $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x-3}{3}\right) + C$

3. $\int \frac{2x - 3}{\sqrt{2x^2 - 7x + 5}}\, dx$ **Ans:** $\sqrt{2x^2 - 7x + 5} + \frac{1}{2\sqrt{2}}\left[\log(2x - 7) + \sqrt{2x^2 - 7x + 5}\right] + C_1$

$$where\ C_1 = C - \frac{1}{2\sqrt{2}}\log 2$$

4. $\int \sqrt{3 - 2x - x^2}\, dx$ **Ans:** $2\sin^{-1}\left(\frac{x+1}{2}\right) + \left(\frac{x+1}{2}\right)\sqrt{3 - 2x - x^2} + C$

5. $\int \sqrt{x^2 + 2x + 10}\, dx$ **Ans:** $\left(\frac{x+1}{2}\right)\sqrt{x^2 + 2x + 10} + \frac{9}{2}\log\left[x + 1 + \sqrt{x^2 + 2x + 10}\right] + C$

6. $\int (x - 1)\sqrt{x^2 + 2x - 6}\, dx$

Ans: $\frac{1}{3}(x^2 + 2x - 8)^{3/2} - (x + 1)\sqrt{x^2 + 2x - 8} - 9\log\left[(x + 1) + \sqrt{x^2 + 2x - 8}\right] + C$

7. $\int (x + 1)\sqrt{x^2 - 2x + 2}\, dx$

Ans: $\frac{1}{3}(x^2 - 2x + 2)^{3/2} - (x - 1)\sqrt{x^2 - 2x + 2} + \log\left[(x - 1) + \sqrt{x^2 - 2x + 2}\right] + C$

8. $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}\, dx$ **Ans:** $-\log\left[\frac{1-x}{2\,(1+x)} + \sqrt{\frac{3}{4} + \left(\frac{1-x}{2\,(1+x)}\right)^2}\right] + C$

9. $\int \frac{1}{(x^2-1)\sqrt{x^2+1}}\, dx$ **Ans:** $\frac{1}{2\sqrt{2}}\log\left[\frac{\sqrt{x^2+1} - x\sqrt{2}}{\sqrt{x^2+1} + x\sqrt{2}}\right] + C$

10. $\int \frac{1}{x^{3/2} - \sqrt{x}}\, dx$ **Ans:** $\log\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + C$

11. $\int \frac{dx}{(1+x)^{3/2} + (1+x)^{1/2}}$ **Ans:** $2\, tan^{-1}\sqrt{1+x}$

12. $\int \frac{dx}{\sin x}$ **Ans:** $\log\left(\tan\frac{x}{2}\right)$

13. $\int \frac{dx}{13 + 12\cos x}$ **Ans:** $\frac{2}{5}\, tan^{-1}\left(\frac{1}{5}\tan\frac{x}{2}\right)$

14. $\int \sqrt{\frac{x-\alpha}{x-\beta}}\, dx \quad \beta > \alpha$ **Ans:** $(\beta - \alpha)\, sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} - \sqrt{(x-\alpha)(x-\beta)} + C$

3.6 Improper Integrals:

The Integral I=$\int_a^b f(x)dx$ is said to be proper or definite only when the limits $a\ and\ b$ are finite and the integrand $f(x)$ is continuous in the interval $[a, b]$

Now we extend the definition that if the limits are infinite or the integrand $f(x)$ is discontinuous then the integral is said to be improper integral or infinite integral

Types of Improper Integrals

There are two types of improper integrals

1. With infinite limits of integration
2. The integrand is discontinuous.

Type I (Infinite limits of integration)

1. $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$
2. $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$
3. $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$, $'a'$ is a real number.

Provided both the limits on right side exist.

Type II (Discontinuous of the integrand)

1. If f is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

2. If f is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

3. If f is discontinuous at c, in $[a, b]$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx$$

Provided both the integral's on right exists.

Note:

The improper integral is said to be convergent if the limit exists and is divergent if the limit does not exist.

Problems on Type I

Example: 3.87

Determine whether the integral $\int_1^\infty \frac{1}{x}dx$ is convergent or divergent.

Solution:

The given integral is $\int_1^\infty \frac{1}{x}dx$

an improper integral , since upper limit of integration is infinite then,

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx$$

$$= \lim_{t\to\infty}[logx]_1^t$$

$$= \lim_{t\to\infty}[logt - log1]$$

$$= \lim_{t\to\infty}[logt - 0] = \infty$$

The given integral is divergent and it diverges to ∞.

Example:3.88

Determine whether the integral $\int_0^\infty \frac{1}{1+x^2}dx$ is convergent or divergent.

Solution:

The given integral is $\int_0^\infty \frac{1}{1+x^2}dx$ an improper integral, since upper limit of integration is infinite then,

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx$$

$$= \lim_{t\to\infty}[tan^{-1}x]_0^t$$

$$= \lim_{t\to\infty}[tan^{-1}t - tan^{-1}0]$$

$$= \lim_{t\to\infty} tan^{-1}t$$

$$= tan^{-1}\infty = \frac{\pi}{2}$$

The given integral is convergent.

Example:3.89

For what values of p the integral $\int_1^\infty \frac{1}{x^p} dx$ convergent?

Solution:

If $p \neq 1, \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\, dx = \lim_{t\to\infty} \int_1^t x^{-p} dx$

$$= \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$$

$$= \lim_{t\to\infty}\frac{1}{p-1}\left[1 - \frac{1}{t^{p-1}}\right]$$

$$= \begin{cases} \dfrac{1}{p-1}, p > 1, converges \\ \infty, \qquad p \leq 1, diverges \end{cases}$$

Example:3.90

Evaluate $\int_1^\infty \frac{logx}{x} dx$

Solution:

Take $I = \int \frac{logx}{x} dx$

Put $u = logx \qquad dv = \frac{1}{x}dx \qquad du = \frac{1}{x}dx \qquad v = logx$

$$I = \int \frac{logx}{x} dx = (logx)^2 - \int logx\left(\frac{1}{x}\right)dx$$

$$I = (logx)^2 - I \Rightarrow 2I = (logx)^2 \Rightarrow I = \frac{1}{2}(logx)^2$$

$$\int_1^\infty \frac{logx}{x} dx = \lim_{t\to\infty}\int_1^t \frac{logx}{x} dx = \lim_{t\to\infty}\left(\frac{1}{2}(logx)^2\right)_1^t$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}(logt)^2 - \frac{1}{2}(log1)^2\right]$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}(logt)^2\right] = \infty \qquad [\log 1 = 0, \log\infty = \infty]$$

The given integral is divergent.

Example:3.91

Evaluate $\int_{-\infty}^{\infty} xe^{-x^2} dx$

Solution:

Consider $\int xe^{-x^2}dx$

Put $u = x^2$, $du = 2xdx$

$$\int xe^{-x^2}dx = \int e^{-u}\frac{du}{2} = \frac{1}{2}\left[\frac{e^{-u}}{-1}\right]$$

$$= -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2} \ldots (1)$$

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx \ldots (2)$$

Take $\int_{-\infty}^{0} xe^{-x^2}dx = \lim_{t\to-\infty}\int_{t}^{0} xe^{-x^2}dx = \lim_{t\to-\infty}\left[\frac{-1}{2}e^{-x^2}\right]_t^0 by\ (1)$

$$= \lim_{t\to-\infty}\left[\frac{-1}{2} + \frac{1}{2}e^{-t^2}\right] = \frac{-1}{2}$$

Take $\int_{0}^{\infty} xe^{-x^2}dx = \lim_{t\to\infty}\int_{0}^{t} xe^{-x^2}dx = \lim_{t\to\infty}\left[\frac{-1}{2}e^{-x^2}\right]_0^t$ by (1)

$$= \lim_{t\to\infty}\left[\frac{-1}{2}e^{-t^2} + \frac{1}{2}\right] = \frac{1}{2}$$

$$\therefore (2) \Rightarrow \int_{-\infty}^{\infty} xe^{-x^2}dx = \frac{-1}{2} + \frac{1}{2} = 0$$

Example:3.92

Evaluate: $\int_{-\infty}^{0} xe^{x}dx$

Solution:

$$\int_{-\infty}^{0} xe^{x}dx = \lim_{t\to-\infty}\int_{t}^{0} xe^{x}dx = \lim_{t\to-\infty}\int_{t}^{0} xd(e^{x})$$

$$= \lim_{t\to-\infty}\left[[xe^{x}]_t^0 - \int_{t}^{0} e^{x}dx\right]$$

$$= \lim_{t\to-\infty}[(0 - te^{t}) - [e^{x}]_t^0]$$

$$= \lim_{t\to-\infty}[-te^{t} - (1 - e^{t})]$$

$$= \lim_{t\to-\infty}[-te^{t} - 1 + e^{t}]$$

$$= 0 - 1 + 0 = -1 \ (Finite)$$

The given integral is convergent.

Example:3.93

Evaluate $\int_0^{\frac{\pi}{2}} secxdx$

Solution:

The given Integral $\int_0^{\frac{\pi}{2}} secxdx$ is an improper integral of second kind since $f(x) = secx \ 0 \le x \le \frac{\pi}{2}$ is not defined at $x = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} secxdx = \lim_{t \to \pi/2} \int_0^t secxdx$$

$$= \lim_{t \to \pi/2} [log(secx + tanx)]_0^t$$

$$= \lim_{t \to \pi/2} [\log(sec(t) + tan(t) - \log(sec0 + tan0)] \quad = \infty$$

The given integral is divergent.

Example:3.94

Determine whether the integral $\int_1^{\infty} \frac{logx}{x^2} dx$ is convergent or divergent.

Solution:

Consider $\int \frac{logx}{x^2} dx$

Put $u = logx$ $\qquad dv = \frac{1}{x^2} dx$

$du = \frac{1}{x} d$ $\qquad xv = \int \frac{1}{x^2} dx = \frac{-1}{x}$

$$\int udv = uv - \int vdu$$

$$\therefore \int \frac{logx}{x^2} dx = (logx)\left(\frac{-1}{x}\right) - \int \left(\frac{-1}{x}\right)\left(\frac{1}{x}\right) dx$$

$$= -\frac{1}{x} logx + \int \frac{1}{x^2} dx \quad = \frac{-1}{x} logx - \frac{1}{x} \quad = \frac{-1}{x}[logx + 1]$$

$$\therefore \int_1^{\infty} \frac{logx}{x^2} dx = \lim_{t \to \infty} \int_1^t \frac{logx}{x^2} dx$$

$$= \lim_{t \to \infty} \left[\frac{-1}{x}(logx + 1)\right]_1^t$$

$$= \lim_{t \to \infty} \left\{\left[\frac{-1}{t}(logt + 1)\right] - [-(\log 1 + 1)\right]$$

$$= \lim_{t \to \infty} \left[\frac{-1}{t}(logt + 1) + 1\right] = \lim_{t \to \infty} \left[\frac{-logt}{t} - \frac{1}{t} + 1\right]$$

$$= \lim_{t\to\infty}\left(\frac{-logt}{t}\right) - 0 + 1 \qquad ...(1)$$

Take $\lim_{t\to\infty}\left(\frac{-logt}{t}\right)$

$\lim_{t\to\infty}\left(\frac{-logt}{t}\right) = \frac{\infty}{\infty}$ (form)

Apply L' Hospital's rule

$$= \lim_{t\to\infty}\left[\frac{\frac{-1}{t}}{1}\right] = 0$$

(1) $\Rightarrow \int_1^\infty \frac{logx}{x^2}dx = 0 + 1 = 1\ (finite)$

The given integral $\int_1^\infty \frac{logx}{x^2}dx$ is convergent.

Example:3.95

Evaluate $\int_3^\infty \frac{1}{(x-2)^{3/2}}dx$

Solution:

Consider $\int \frac{1}{(x-2)^{3/2}}dx$…(1)

Put $u = x - 2 \quad \Rightarrow du = dx$

(1) $\Rightarrow \int \frac{1}{(x-2)^{3/2}}dx = \int \frac{1}{u^{3/2}}du = \int u^{-3/2}\,du = \frac{u^{-3/2+1}}{-3/2+1} = \frac{u^{-1/2}}{-1/2}$

$$\frac{-2}{\sqrt{u}} = \frac{-2}{\sqrt{x-2}}$$

$$\int_3^\infty \frac{1}{(x-2)^{3/2}}dx = \lim_{t\to\infty}\left[\int_3^t \frac{1}{(x-2)^{3/2}}dx\right] = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x-2}}\right]_3^t$$

$$= \lim_{t\to\infty}\left[\left(\frac{-2}{\sqrt{t-2}}\right) - \left(\frac{-2}{\sqrt{1}}\right)\right]$$

$$= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}}\right) + 2 = 0 + 2 = 2 \text{ (finite)}$$

The given integral $\int_3^\infty \frac{1}{(x-2)^{3/2}}dx$ is convergent.

Example:3.96

Evaluate : $\int_{-\infty}^\infty \frac{1}{x^2+a^2}dx$

Solution:

Given $\int_{-\infty}^\infty \frac{1}{x^2+a^2}dx$

$$= \int_{-\infty}^0 \frac{1}{x^2+a^2}dx + \int_0^\infty \frac{1}{x^2+a^2}dx \qquad \dots (1)$$

Take $\int_{-\infty}^{0} \frac{1}{x^2+a^2} dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{x^2+a^2} dx = \lim_{t\to-\infty} \left[\frac{1}{a} tan^{-1} \frac{x}{a}\right]_t^0$

$$= \lim_{t\to-\infty} \left[\frac{1}{a} 0 - \frac{1}{a} tan^{-1} \frac{t}{a}\right]$$

$$= -\frac{1}{a} tan^{-1} \left(\frac{-\infty}{a}\right)$$

$$= \frac{1}{a}\frac{\pi}{2} \qquad [\because \tan(-\theta) = -tan\theta]$$

Take $\int_{0}^{\infty} \frac{1}{x^2+a^2} dx \lim_{t\to\infty} \int_{0}^{t} \frac{1}{x^2+a^2} dx = \lim_{t\to\infty} \left[\frac{1}{a} tan^{-1} \frac{x}{a}\right]_0^t$

$$= \lim_{t\to\infty} \left[\frac{1}{a} tan^{-1} \frac{t}{a} - 0\right] = \frac{1}{a}\frac{\pi}{2}$$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} dx = \frac{1}{a}\frac{\pi}{2} + \frac{1}{a}\frac{\pi}{2} = \frac{1}{a}[\pi]$$

Problems on Type II

Example:3.97

Evaluate $\int_0^2 \frac{1}{\sqrt{x}} dx$

Solution:

Here, infinite discontinuity occurs at x=0

$$\therefore \int_0^2 \frac{1}{\sqrt{x}} dx = \lim_{t\to0^+} \int_t^2 x^{-1/2} dx$$

$$= \lim_{t\to0^+} \left[\frac{x^{1/2}}{1/2}\right]_t^2$$

$$= \lim_{t\to0^+} \left[2\sqrt{x}\right]_t^2$$

$$= \lim_{t\to0^+} \left[2\sqrt{2} - 2\sqrt{t}\right]$$

$$= 2\sqrt{2} \text{ (finite)}$$

The given integral $\int_0^2 \frac{1}{\sqrt{x}} dx$ is convergent.

Example:3.98

Evaluate $\int_0^3 \frac{1}{x-1} dx$

Solution:

Here, infinite discontinuity occurs at $x = 1$

$$\therefore \int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

Take $\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} [\log(x-1)]_0^t$

$$= \lim_{t\to 1^-} \log(t-1) = -\infty$$

$\int_0^1 \frac{1}{x-1}\,dx$ is divergent.

$\Rightarrow \int_1^3 \frac{1}{x-1}\,dx$ is also divergent.

The given integral $\int_0^3 \frac{1}{x-1}\,dx$ isdivergent

Example:3.99

Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

Solution:

The infinite discontinuity occurs at $x = 2$

$$\therefore \int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_2^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+} \left[2\sqrt{x-2}\right]_t^5$$

$$= \lim_{t\to 2^+} \left(2\sqrt{3} - 2\sqrt{t-2}\right)$$

$$= 2\sqrt{3} \text{ (finite)}$$

The given integral $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ is convergent.

Example:3.100

Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$

Solution:

Here infinite discontinuity occurs at $x = 1$

1) $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx \quad \ldots(1)$

Take $\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^{2/3}}\,dx$

$$= \lim_{t\to 1^-} \left[3(x-1)^{1/3}\right]_0^t$$

$$= \lim_{t\to 1^-} \left[3(t-1)^{1/3} + 3\right]$$

$= 3$

Take $\int_1^3 \frac{1}{(x-1)^{2/3}} dx = \lim_{t \to 1^+} \int_t^t \frac{1}{(x-1)^{2/3}} dx$

$= \lim_{t \to 1^+} \left[3(x-1)^{1/3}\right]_t^3$

$= \lim_{t \to 1^+} \left[3\left[2^{1/3} - (t-1)^{1/3}\right]\right]$

$= 3\left(2^{1/3}\right)$

(1) $\Rightarrow \int_0^3 \frac{1}{(x-1)^{2/3}} dx = 3 + 3\left(2^{1/3}\right)$

$= 3 \left[1 + 2^{1/3}\right]$

Example:3.101

Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution:

Here infinite discontinuity occurs at $x = 1$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{t \to 1^-} [sin^{-1}(x)]_0^t$$

$$= \lim_{t \to 1^-} [sin^{-1}(t) - sin^{-1}(0)]$$

$$= sin^{-1}(1) - 0 = \pi/2$$

The given integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent.

Example:3.102

Evaluate $\int_0^2 \frac{1}{(x-1)^2} dx$

Solution:

Here infinite discontinuity occurs at $x = 2$

$$\int_0^2 \frac{1}{(x-1)^2} dx = \lim_{t \to 2^-} \int_0^t \frac{dx}{(x-1)^2}$$

$$= \lim_{t \to 2^-} \int_0^t (x-2)^{-2} dx$$

$$= \lim_{t \to 2^-} \left[\frac{(x-2)^{-2+1}}{-2+1}\right]_0^t$$

$$= \lim_{t\to 2^-} \left[\frac{-1}{(x-2)}\right]_0^t$$

$$= -\left\{\lim_{t\to 2^-} \left[\frac{1}{(t-2)} - \frac{1}{(0-2)}\right]\right\}$$

$$= -\lim_{t\to 2^-} \left[\frac{1}{(t-2)}\right] - \frac{1}{2} = \infty$$

Thus the given integral is divergent

Comparison test for improper integrals

Let $\int_a^b f(x)dx$ be an improper integral.

i) If there exists a $g(x)$ such that $|f(x)| \leq g(x)$ for all x in $[a,b]$ and $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ also converges.

ii) If there exists function $g(x)$ such that $f(x) \geq |g(x)|$ for all x in $[a,b]$ and $\int_a^b g(x)dx$ diverges then $\int_a^b f(x)dx$ also diverges.

Limit form of comparison Tests.

Let $f(x) > 0$ and $g(x) > 0$and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$ where $k \neq 0$

Then, the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge together.

If $k = 0$, only the convergence of $\int_a^\infty g(x)dx$ implies that of $\int_a^\infty f(x)dx$

Absolute Convergence

The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f(x)|dx$ is convergent.

Note:

1) The same definition holds for $\int_a^\infty f(x)dx$ also
2) When the improper integral changes sign within the limits of the integration, then the above test is applied.

Example: 3.103

Discuss the convergence of $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}\, dx$

Solution:

Let $f(x) = \frac{x\tan^{-1}x}{\sqrt{4+x^3}} = \frac{\tan^{-1}x}{\sqrt{x}\sqrt{1+4x^{-3}}}$ and $g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{tan^{-1}x}{\sqrt{1+4x^{-3}}}$$
$$= \frac{\pi}{2}$$

Hence, by comparision test 2, the integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together, Now $\int_1^\infty g(x)dx$ is divergent.

$\therefore$ $\int_1^\infty f(x)dx$ is also divergent.

Example :3.104

Discuss the convergence of $\int_1^\infty \frac{sinx}{x^4}dx$

Solution:

$$\left|\int_1^\infty \frac{sinx}{x^4}dx\right| \leq \int_1^\infty \left|\frac{sinx}{x^4}\right| dx \leq \int_1^\infty \frac{dx}{x^4}$$

$\Rightarrow$ convergent

$\int_1^\infty \frac{sinx}{x^4}dx$ is absolutely convergent and hence convergent.

Example: 3.105

Test the convergence of $\int_0^\infty e^{-x^2}\ dx$

Solution:

The given integral $\int_0^\infty e^{-x^2}\ dx$is an improper integral of first kind and the integral can be written as $\int_0^\infty e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^\infty e^{-x^2}\ dx$

The first integral in the right hand side $\int_0^1 e^{-x^2}\ dx$ is proper integral. So it is enough to check the second one.

We have that,

$$x \geq 1$$
$$x^2 \geq x$$
$$-x^2 \leq -x$$
$$e^{-x^2} \leq e^{-x}$$
$$\int_1^\infty e^{-x^2}\ dx \leq \int_1^\infty e^{-x}\ dx$$
$$= \lim_{b\to\infty}\int_1^b e^{-x}\ dx \qquad = \lim_{b\to\infty}[-e^{-x}]_1^b$$

$$= \lim_{b\to\infty}[e^{-1} - e^{-b}]$$

$$= [e^{-1} - 0] = \frac{1}{e}$$

Hence by comparison test the given integral is convergent.

Exercise 3.6

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\int_1^\infty \frac{1}{x^2}\, dx$ Ans: convergent, 1
2. $\int_{-\infty}^0 \frac{1}{3-4x}\, dx$ Ans: divergent
3. $\int_0^\infty e^{-x}\, dx$ Ans: convergent, 1
4. $\int_\infty^\infty \frac{x^2}{9+x^6}\, dx$ Ans: convergent, $\frac{\pi}{9}$
5. $\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\, dx$ Ans: divergent
6. $\int_{-1}^1 \frac{1}{x^{2/3}}\, dx$ Ans: convergent, 6
7. $\int_0^\infty \frac{x}{\sqrt{1-x^2}}\, dx$ Ans: convergent, 1
8. $\int_0^3 \frac{1}{x^2-6x+5}\, dx$ Ans: divergent
9. $\int_0^1 \log x\, dx$ Ans: convergent, -1
10. $\int_{-2}^3 \frac{1}{x^4}\, dx$ Ans: divergent