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ENGINEERING MATHEMATICS -II

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(Regulation 2017)
Common to all branches of B.E

UNIT- III
ANALYTIC FUNCTIONS

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3.1 INTRODUCTION

The theory of functions of a complex variable is the most important in solving a large number of Engineering and Science problems. Many complicated intergrals of real function are solved with the help of a complex variable.

3.1 (a) Complex Variable

$x + iy$ is a complex variable and it is denoted by z.

$(i.e.) z = x + iy \; where \; i = \sqrt{-1}$

3.1 (b) Function of a complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of z in a given region R of complex plane there corresponds one or more values of w is said to be a function z and is denoted by $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$where $u(x, y)$and $v(x, y)$are real functions of the real variables x and y.

Note:

(i) single-valued function

If for each value of z in R there is correspondingly only one value of w, then w is called a single valued function of z.

Example: $w = z^2, w = \frac{1}{z}$

$w = z^2$					$w = \frac{1}{z}$				
z	1	2	-2	3	z	1	2	-2	3
w	1	4	4	9	w	1	$\frac{1}{2}$	$\frac{1}{-2}$	$\frac{1}{3}$

(ii) Multiple – valued function

If there is more than one value of w corresponding to a given value of z then w is called multiple – valued function.

Example: $w = z^{1/2}$

$w = z^{1/2}$			
z	4	9	1
w	$-2,2$	$3,-3$	$1,-1$

(iii) The distance between two points $z$and z_o is $|z - z_o|$

(iv) The circle C of radius δ with centre at the point $z_o$can be represented by $|z - z_o| = \delta$.

(v) $|z - z_o| < \delta$ represents the interior of the circle excluding its circumference.

(vi) $|z - z_o| \leq \delta$ represents the interior of the circle including its circumference.

(vii) $|z - z_o| > \delta$ represents the exterior of the circle.

(viii) A circle of radius 1 with centre at origin can be represented by $|z| = 1$

3.1 (c) Neighbourhood of a point z_o

Neighbourhood of a point z_o, we mean a sufficiently small circular region [excluding the points on the boundary] with centre at z_o.

$$(i.e.)\ |z - z_o| < \delta$$

Here, δ is an arbitrary small positive number.

3.1 (d) Limit of a Function

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_o.

Then the limit of $f(z)$ as z approaches $z_o is\ w_o$.

$$(i.e.)\ \lim_{z \to z_o} f(z) = w_o$$

3.1 (e) Continuity

If $f(z)$ is said to continuous at $z = z_o$ then

$$\lim_{z \to z_o} f(z) = f(z_o)$$

If two functions are continuous at a point their sum, difference and product are also continuous at that point, their quotient is also continuous at any such point $[dr \neq 0]$

Example: 3.1 State the basic difference between the limit of a function of a real variable and that of a complex variable. [A.U M/J 2012]

Solution:

In real variable, $x \to x_0$ implies that x approaches x_0 along the X-axis (or) a line parallel to the X-axis.

In complex variables, $z \to z_0$ implies that z approaches z_0 along any path joining the points $z\ and\ z_0$ that lie in the z-plane.

3.1 (f) Differentiability at a point

A function $f(z)$ is said to be differentiable at a point, $z = z_0$ if the limit

$$f(z_0) = \underset{\Delta z \to 0}{\mathrm{Lt}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists.}$$

This limit is called the derivative of $f(z)$at the point $z = z_0$

If $f(z)$ is differentiable at z_0, then $f(z)$is continuous at z_0. This is the necessary condition for differentiability.

Example: 3.2 If $f(z)$ is differentiable at z_0, then show that it is continuous at that point.

Solution:

As $f(z)$ is differentiable at z_0, both $f(z_0)$ and $f'(z_0)$exist finitely.

Now, $\lim_{z \to z_o} |f(z) - f(z_o)| = \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0} (z - z_0)$

$$= \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0} \lim_{z \to z_o} (z - z_0)$$

$$= f'(z_0).0 = 0$$

Hence, $\lim_{z \to z_o} f(z) = \lim_{z \to z_o} f(z_o) = f(z_o)$

As $f(z_0)$ is a constant.

This is exactly the statement of continuity of $f(z)$ at z_0.

Example: 3.3 Give an example to show that continuity of a function at a point does not imply the existence of derivative at that point.

Solution:

Consider the function $w = |z|^2 = x^2 + y^2$

This function is continuous at every point in the plane, being a continuous function of two real variables. However, this is not differentiable at any point other than origin.

Example: 3.4 Show that the function $\mathbf{f(z)}$ is discontinuous at $\boldsymbol{z = 0}$, given that $\boldsymbol{f(z) = \frac{2xy^2}{x^2+3y^4}}$, when $\boldsymbol{z \neq 0}$ and $\mathbf{f(0) = 0}$.

Solution:

Given $f(z) = \frac{2xy^2}{x^2+3y^4}$,

Consider $\lim_{z \to z_o} [f(z)] = \lim_{\substack{y=mx \\ x \to 0}} [f(z)] = \lim_{x \to 0} \frac{2x(mx)^2}{x^2+3(mx)^4} = \lim_{x \to 0} \left[\frac{2m^2x}{1+3m^4x^2}\right] = 0$

$$\lim_{\substack{y^2=x \\ x \to 0}} [f(z)] = \lim_{x \to 0} \frac{2x^2}{x^2 + 3x^2} = \lim_{x \to 0} \frac{2x^2}{4x^2} = \frac{2}{4} = \frac{1}{2} \neq 0$$

$\therefore f(z)$ is discontinuous

Example: 3.5 Show that the function $\boldsymbol{f(z)}$ is discontinuous at the origin $\boldsymbol{(z = 0)}$, given that

$\boldsymbol{f(z) = \frac{xy(x-2y)}{x^3+y^3}}$**, when $\boldsymbol{z \neq 0}$**

$\boldsymbol{= 0}$ **, when $\boldsymbol{z = 0}$**

Solution:

Consider $\lim_{z \to z_o} [f(z)] = \lim_{\substack{y=mx \\ x \to 0}} [f(z)] = \lim_{x \to 0} \frac{x(mx)(x-2(mx))}{x^3+(mx)^3}$

$$= \lim_{x \to 0} \frac{m(1-2m)x^3}{(1+m^3)x^3} = \frac{m(1-2m)}{1+m^3}$$

Thus $\lim_{z \to 0} f(z)$ depends on the value of m and hence does not take a unique value.

$\therefore \lim_{z \to 0} f(z)$does not exist.

$\therefore f(z)$ is discontinuous at the origin.

3.2 ANALYTIC FUNCTIONS – NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYTICITY IN CARTESIAN AND POLAR CO-ORDINATES

Analytic [or] Holomorphic [or] Regular function

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire Function: [Integral function]

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$.

Example: $e^z, \sin z, \cos z, sinhz, \cosh z$

3.2 (i) The necessary condition for $f = (z)$ to be analytic. [Cauchy – Riemann Equations]

The necessary conditions for a complex function $f = (\mathrm{z}) = u(x, y) + iv(x, y)$ to be analytic in a region R are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $i.e., u_x = v_y$ and $v_x = -u_y$

[OR]

Derive C – R equations as necessary conditions for a function $\mathbf{w = f(z)}$ to be analytic.

[Anna, Oct. 1997] [Anna, May 1996]

Proof:

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function at the point z in a region R. Since $f(z)$ is analytic, its derivative $f^{'}(z)$ exists in R

$$f'(z) = \mathrm{Lt}\frac{f(z+\Delta z)-f(z)}{\Delta_z}$$

Let $z = x + iy$

$$\Rightarrow \Delta z = \Delta_x + i\Delta_y$$

$$z + \Delta_z = (x + \Delta_x) + i(y + \Delta_y)$$

$$f(z) = u(x, y) + iv(x, y)$$

$$f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y$$

$$f(z + \Delta z) - f(z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]$$

$$= [u(x + \Delta x, y + \Delta y) - u(x, y)] + i[v(x + \Delta x, y + \Delta y) - v(x, y)]$$

$$f'(z) = \underset{\Delta z \to 0}{\mathrm{Lt}} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\mathrm{Lt}} \frac{u(x+\Delta x,y+\Delta y)-u(x,y)]+i[v(x+\Delta x,y+\Delta y)-v(x,y)]}{\Delta x+i\Delta y}$$

Case (i)

If $\Delta z \to 0$,firsts we assume that $\Delta y = 0$ and $\Delta x \to 0$.

$$\therefore\ f'(z) = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{[u(x+\Delta x,y)-u(x,y)]+i[v(x+\Delta x,y)-v(x,y)]}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \ldots(1)$$

Case (ii)

If $\Delta z \to 0$ Now, we assume that $\Delta x = 0$ and $\Delta y \to 0$

$$\therefore\ f'(z) = \underset{\Delta y \to 0}{\mathrm{Lt}} \frac{[u(x,y+\Delta y)-u(x,y)]+i[v(x,y+\Delta y)-v(x,y)]}{i\Delta y}$$

$$= \frac{1}{i}\underset{\Delta y \to 0}{\mathrm{Lt}} \frac{u(x,y+\Delta y)-u(x,y)}{\Delta y} + \underset{\Delta y \to 0}{\mathrm{Lt}} \frac{v(x,y+\Delta y)-v(x,y)}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \ldots(2)$$

From (1) and (2), we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\,i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}$$

$(i.e.)\ u_x = v_y, \quad v_x = -u_y$

The above equations are known as Cauchy – Riemann equations or C-R equations.

Note: (i) The above conditions are not sufficient for $f(z)$ to be analytic. The sufficient conditions are given in the next theorem.

(ii) Sufficient conditions for $f(z)$ to be analytic.

If the partial derivatives $u_{x\,,}u_{y,}v_x$ and v_y are all continuous in D and $u_{x\,,} = v_y$ and $u_y = -v_{x'}$ then the function $f(z)$ is analytic in a domain D.

(ii) Polar form of C-R equations

In Cartesian co-ordinates any point z is $z = x + iy$.

In polar co-ordinates, $z = re^{i\theta}$ where r is the modulus and θ is the argument.

Theorem: If $\mathrm{f}(\mathrm{z}) = u(r,\theta) + iv(r,\theta)$ is differentiable at $z = re^{i\theta}$, then $u_r = \frac{1}{r}v_\theta, v_r = -\frac{1}{r}u_\theta$

(OR) $\boldsymbol{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$

Proof:

Let $z = re^{i\theta}$ and $w = f(z) = u + iv$

$(\,i.e.)\ u + iv = f(z) = f(re^{i\theta})$

Diff. p.w. r. to r, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\ e^{i\theta} \qquad \ldots(1)$$

Diff. p.w. r. to θ, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\ e^{i\theta} \qquad \ldots(2)$$

$$= ri\left[f'(re^{i\theta})\ e^{i\theta}\right]$$

$$= ri\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{by} \quad (1)$$

$$= ri\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -\,i\frac{\partial v}{\partial r}, \;\; \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$(i.e.)\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \;\; \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial v}{\partial \theta}$$

Problems based on Analytic functions – necessary conditions Cauchy – Riemann equations

Example: 3.6 Show that the function $\mathbf{f(z) = xy + iy}$ is continuous everywhere but not differentiable anywhere.

Solution:

Given $f(z) = xy + iy$

$(i.e.)\;\; u = xy, v = y$

x and y are continuous everywhere and consequently $u(x, y) = xy$ and $v(x, y) = y$ are continuous everywhere.

Thus $f(z)$ is continuous everywhere.

But

$u = xy$	$v = y$
$u_x = y$ $u_y = x$ $u_x \neq v_y$	$v_x = 0$ $v_y = 1$ $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not differentiable anywhere though it is continuous everywhere .

Example: 3.7 Show that the function $\boldsymbol{f(z) = \bar{z}}$ is nowhere differentiable. [A.U N/D 2012]

Solution:

Given $f(z) = \bar{z} = x - iy$

i.e.,

$u = x$	$v = -y$
$\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = 0$	$\frac{\partial v}{\partial x} = 0$ $\frac{\partial v}{\partial y} = -1$

$\therefore u_x \neq v_y$

C−R equations are not satisfied anywhere.

Hence, $f(z) = \bar{z}$ is not differentiable anywhere (or) nowhere differentiable.

Example: 3.8 Show that $\mathbf{f(z) = |z|^2}$ is differentiable at $\boldsymbol{z = 0}$ but not analytic at $\boldsymbol{z = 0}$.

Solution:

Let $z = x + iy$

$\bar{z} = x - iy$

$|z|^2 = z\,\bar{z} = x^2 + y^2$

$(i.e.)\ f(z) = |z|^2 = (x^2 + y^2) + i0$

$u = x^2 + y^2$	$v = 0$
$u_x = 2x$ $u_y = 2y$	$v_x = 0$ $v_y = 0$

So, the C−R equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at $z = 0$.

So, f(z) may be differentiable only at $z = 0$.

Now, $u_x = 2x$, $u_y = 2y$,$v_x = 0$ and $v_y = 0$ are continuous everywhere and in particular at (0,0).

Hence, the sufficient conditions for differentiability are satisfied by $f(z)$ at $z = 0$.

So, $f(z)$ is differentiable at $z = 0$ only and is not analytic there.

Inverse function

Let $w = f(z)$ be a function of z and $z = F(w)$ be its inverse function.

Then the function $w = f(z)$ will cease to be analytic at $\frac{dz}{dw} = 0$ and $z = F(w)$ will be so, at point where $\frac{dw}{dz} = 0$.

Example: 3.9 Show that $f(z) = log\ z$ analytic everywhere except at the origin and find its derivatives.

Solution:

Let $z = re^{i\theta}$

$$f(z) = log\ z$$

$$= \log(re^{i\theta}) = logr + \log(e^{i\theta}) = logr + i\theta$$

Note : $e^{-\infty} = 0$

$\log e^{-\infty} = \log 0\ ; -\infty = log0$

But, at the origin, $r = 0$. Thus, at the origin,

$$f(z) = log0 + i\theta = -\infty + i\theta$$

So, $f(z)$is not defined at the origin and hence is not differentiable there.

At points other than the origin, we have

$u(r,\theta) = log\ r$	$v(r,\theta) = \theta$
$u_r = \frac{1}{r}$ $u_\theta = 0$	$v_r = 0$ $v_\theta = 1$

So, $logz$ satisfies the C−R equations.

Further $\frac{1}{r}$ is not continuous at $z = 0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z = 0$. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin. The derivative is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to ‘r’, we get

$$(i.e.)\ e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

Example: 3.10 Check whether $w = \bar{z}$ is analytics everywhere. [Anna, Nov 2001] [A.U M/J 2014]

Solution:

Let $w = f(z) = \bar{z}$

$u + iv = x - iy$

$u = x$	$v = -y$
$u_x = 1$ $u_y = 0$	$v_x = 0$ $v_y = -1$

$u_x \neq v_y$ at any point p(x,y)

Hence, C−R equations are not satisfied.

∴The function $f(z)$ is nowhere analytic.

Example: 3.11 Test the analyticity of the function $w = sin\, z$.

Solution:

Let $w = f(z) = sinz$

$$u + iv = \sin(x + iy)$$

$$u + iv = \sin x \cos iy + \cos x \sin iy$$

$$u + iv = \sin x \cosh y + i \cos x \sin hy$$

Equating real and imaginary parts, we get

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$ $u_y = \sin x \sinh y$	$v_x = -\sin x \sinh y$ $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

C −R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: 3.12 Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not. [Anna, May 2001]

Solution:

Let $f(z) = 2xy + i(x^2 - y^2)$

$(i.e.)$

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not an analytic function.

Example: 3.13 Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Solution:

Given $f(z) = \cosh z = \cos(iz) = \cos[i(x+iy)]$

$$= \cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$$

$$u + iv = \cosh x \cos y + i \sinh x \sin y$$

$u = \cosh x \cos y$	$v = \sinh x \sin y$
u_x =sinh x cos y $u_y = -\cosh x \sin y$	v_x =cosh x sin y v_y =sinh x cos y

$\therefore u_x, u_y, v_x$ and v_y exist and are continuous.

$u_x = v_y$ and $u_y = -v_x$

C−R equations are satisfied.

$\therefore f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

$$= \sinh x \cos y + i \cosh x \sin y$$

$$= \sinh(x+iy) = \sinh z$$

Example: 3.14 If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$, and prove that $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$. [Anna, Nov 2001]

Solution:

Let $w = u(x,y) + iv(x,y)$

As $f(z)$ is analytic, we have $u_x = v_y, u_y = -v_x$

Now, $\frac{dw}{dz} = f'(z) = u_x + iv_x = v_y - iu_y = i(u_y + iv_y)$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}(u+iv) = -i\frac{\partial}{\partial y}(u+iv)$$

$$= \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

We know that, $\frac{\partial w}{\partial z} = 0$

$$\therefore \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$$

Also $\frac{\partial^2 w}{\partial \bar{z} \partial z} = 0$

Example: 3.15 Prove that every analytic function $w = u(x,y) + iv(x,y)$ can be expressed as a function of z alone. [A.U. M/J 2010, M/J 2012]

Proof:

Let $z = x + iy$ and $\bar{z} = x - iy$

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z+\bar{z}}{2i}$$

Hence, u and v and also w may be considered as a function of z and $\bar{z}$

Consider $\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$

$$= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right)$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$

$= 0$ by C−R equations as w is analytic.

This means that w is independent of $\bar{z}$

$(i.e.)$ w is a function of z alone.

This means that if $w = u(x,y) + iv(x,y)$ is analytic, it can be rewritten as a function of $(x + iy)$.

Equivalently a function of $\bar{z}$ cannot be an analytic function of z.

Example: 3.16 Find the constants a, b, c if $\mathbf{f(z) = (x + ay) + i(bx + cy)}$ is analytic.

Solution:

$$f(z) = u(x,y) + iv(x,y)$$

$$= (x + ay) + i(bx + cy)$$

$u = x + ay$	$v = bx + cy$
$u_x = 1$	$v_x = b$
$u_y = a$	$v_y = c$

Given $f(z)$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ and $a = -b$

Example: 3.17 Examine whether the following function is analytic or not $\mathbf{f(z) = e^{-x}(\cos y - i \sin y)}$.

Solution:

Given $f(z) = e^{-x}(\cos y - i\sin y)$

$\Rightarrow u + iv = e^{-x}\cos y - ie^{-x}\sin y$

$u = e^{-x}\cos y$	$v = -e^{-x}\sin y$
$u_x = -e^{-x}\cos y$	$v_x = e^{-x}\sin y$
$u_y = -e^{-x}\sin y$	$v_y = -e^{-x}\cos y$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C$-$R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: 3.18 Test whether the function $\mathbf{f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}\left(\frac{y}{x}\right)}$ is analytic or not.

Solution:

Given $f(z) = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$(i.e.)u + iv = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$u = \frac{1}{2}\log(x^2 + y^2)$	$v = \tan^{-1}\left(\frac{y}{x}\right)$
$u_x = \frac{1}{2}\frac{1}{x^2 + y^2}(2x)$ $= \frac{x}{x^2 + y^2}$ $u_y = \frac{1}{2}\frac{1}{x^2 + y^2}(2y)$ $= \frac{y}{x^2 + y^2}$	$v_x = \frac{1}{1 + \frac{y^2}{x^2}}\left[-\frac{y}{x^2}\right]$ $= \frac{-y}{x^2 + y^2}$ $v_y = \frac{1}{1 + \frac{y^2}{x^2}}\left[\frac{1}{x}\right]$ $= \frac{x}{x^2 + y^2}$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C$-$R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: 3.19 Find where each of the following functions ceases to be analytic.

(i) $\frac{z}{(z^2 - 1)}$ **(ii)** $\frac{z + i}{(z - i)^2}$

Solution:

(i) Let $f(z) = \frac{z}{(z^2 - 1)}$

$$f'(z) = \frac{(z^2 - 1)(1) - z(2z)}{(z^2 - 1)^2} = \frac{-(z^2 + 1)}{(z^2 - 1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist.

$(i.e.)\ f'(z) \to \infty$

$(i.e.)(z^2 - 1)^2 = 0$

$(i.e.)\quad z^2 - 1 = 0$

$z = 1$

$z = \pm 1$

$\therefore f(z)$ is not analytic at the points $z = \pm 1$

(ii) Let $f(z) = \frac{z + i}{(z - i)^2}$

$$f'(z) = \frac{(z - i)^2(1)(z + i)[2(z - i)]}{(z - i)^4} = \frac{(z + 3i)}{(z - i)^3}$$

$$f'(z) \to \infty, at\ z = i$$

$\therefore f(z)$ is not analytic at $z = i$.

Exercise: 3.1

1. Examine the following function are analytic or not

1. $f(z) = e^x(\cos y + i \sin y)$ [**Ans:** analytic]
2. $f(z) = e^x(\cos y - i \sin y)$ [**Ans:** not analytic]
3. $f(z) = z^3 + z$ [**Ans:** analytic]
4. $f(z) = \sin x \cos y + i \cos x \sinh y$ [**Ans:** analytic]
5. $f(z) = (x^2 - y^2 + 2xy) + i(x^2 - y^2 - 2xy)$ [**Ans:** not analytic]
6. $f(z) = 2xy + i(x^2 - y^2)$ [**Ans:** not analytic]
7. $f(z) = \cosh z$ [**Ans:** analytic]
8. $f(z) = y$ [**Ans:** not analytic]
9. $f(z) = (x^2 - y^2 - 2xy) + i(x^2 - y^2 + 2xy)$ [**Ans:** analytic]
10. $f(z) = \frac{x-iy}{x^2+y^2}$ [**Ans:** analytic]

2. For what values of z, the function ceases to be analytic.

1. $\frac{1}{z^2-4}$ [**Ans:** $z = \pm 1$]
2. $\frac{z^2-4}{z^2+1}$ [**Ans:** $z = \pm 1$]

3. Verify C-R equations for the following functions.

1. $f(z) = ze^z$
2. $f(z) = lz + m$
3. $f(z) = \cos z$

4. Prove that the following functions are nowhere differentiable.

1. $f(z) = e^x(\cos y - i \sin y)$
2. $f(z) = |z|$
3. $f(z) = z - \bar{z}$

5. Find the constants a, b, c so that the following are differentiable at every point.

1. $f(z) = x + ay - i(bx + cy)$ [**Ans.** $a = b, c = -1$]
2. $f(z) = ax^2 - by^2 + i\, cxy$ [**Ans.** $a = b = \frac{c}{2}$]

3.3 PROPERTIES – HARMONIC CONJUGATES

3.3 (a) Laplace equation

$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ is known as Laplace equation in two dimensions.

3.3 (b) Laplacian Operator

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian operator and is denoted by ∇^2.

Note: (i) $\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$ is known as Laplace equation in three dimensions.

Note: (ii) The Laplace equation in polar coordinates is defined as

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}=0$$

Properties of Analytic Functions

Property: 1 Prove that the real and imaginary parts of an analytic function are harmonic functions.

Proof:

Let $f(z)=u+iv$ be an analytic function

$u_x=v_y\ldots(1)$ and $u_y=-v_x$ $\ldots(2)$ by C−R

Differentiate (1) & (2) p.w.r. to *x*, we get

$u_{xx}=v_{xy}\ldots(3)$ and $u_{xy}=-v_{xx}$ $\ldots(4)$

Differentiate (1) & (2) p.w.r. to *x*, we get

$u_{yx}=v_{yy}\ldots(5)$ and $u_{yy}=-v_{yx}$ $\ldots(6)$

(3) +(6) ⇒ $u_{xx}+u_{yy}=0$ $[\because v_{xy}=v_{yx}]$

(5) – (4) ⇒ $v_{xx}+v_{yy}=0$ $[\because u_{xy}=u_{yx}]$

$\therefore$ u and v satisfy the Laplace equation.

3.3 (c) Harmonic function (or) [Potential function]

A real function of two real variables *x* and *y* that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function.

Note: A harmonic function is also known as a potential function.

3.3 (d) Conjugate harmonic function

If u and v are harmonic functions such that $u+iv$ is analytic, then each is called the conjugate harmonic function of the other.

Property: 2 If w $=\boldsymbol{u(x,y)+iv(x,y)}$ is an analytic function the curves of the family $\boldsymbol{u(x,y)}=\mathbf{c_1}$ and the curves of the family $\boldsymbol{v(x,y)}=\mathbf{c_2}$ cut orthogonally, where $\mathbf{c_1}$ and$\mathbf{c_2}$ are varying constants.

Proof: **[A.U D15/J16 R-13] [A.U N/D 2016 R-13] [A.U A/M 2017 R-08]**

Let $f(z)=u+iv$ be an analytic function

$\Rightarrow u_x=v_y\ldots(1)$ and $u_y=-v_x$ $\ldots(2)$ by C−R

Given $u=\mathrm{c}_1$ and $v=\mathrm{c}_2$

Differentiate p.w.r. to *x*, we get

$$u_x+u_y\frac{dy}{dx}=0 \text{ and } \quad v_x+v_y\frac{dy}{dx}=0$$

$$\Rightarrow\frac{dy}{dx}=\frac{-u_x}{u_y} \text{ and } \frac{dy}{dx}=\frac{-v_x}{v_y}$$

$$\Rightarrow m_1=\frac{-u_x}{u_y} \qquad \Rightarrow m_2=\frac{-v_x}{v_y}$$

$$m_1.m_2 = \left(\frac{-u_x}{u_y}\right)\left(\frac{-v_x}{v_y}\right) = \left(\frac{u_x}{u_y}\right)\left(\frac{u_y}{u_x}\right) = -1 \text{ by (1) and (2)}$$

Hence, the family of curves form an orthogonal system.

Property: 3 An analytic function with constant modulus is constant. [AU. A/M 2007] [A.U N/D 2010]

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x$ $\ldots(2)$ by C−R

Given $|f(z)| = \sqrt{u^2 + v^2} = c \neq 0$

$\Rightarrow |f(z)| = u^2 + v^2 = c^2$ (say)

$(i.e)\ u^2 + v^2 = c^2 \quad \ldots(3)$

Differentiate (3) p.w.r. to *x* and *y*; we get

$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \quad \ldots(4)$

$2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \quad \ldots(5)$

$(4) \times u \quad \Rightarrow u^2u_x + uv\, v_x = 0 \quad \ldots(6)$

$(5) \times v \quad \Rightarrow uv\, u_y + v^2v_y = 0 \quad \ldots(7)$

$(6)+(7) \quad \Rightarrow u^2u_x + v^2v_y + uv\,[v_x + u_y] = 0$

$\Rightarrow u^2u_x + v^2\,u_x + uv\,[-u_y + u_y] = 0$ by (1) & (2)

$\Rightarrow (u^2+v^2)\,u_x = 0$

$\Rightarrow u_x = 0$

Similarly, we get $v_x = 0$

We know that $f'(z) = u_x + v_x = 0 + i0 = 0$

Integrating w.r.to z, we get, $f(z) = c$ [Constant]

Property: 4 An analytic function whose real part is constant must itself be a constant. [A.U M/J 2016]

Proof :

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x$ $\ldots(2)$ by C−R

Given $u = c$ [Constant]

$\Rightarrow u_x = 0, \quad u_y = 0$

$\Rightarrow u_x = 0, \quad v_x = 0$ by (2)

We know that $f'(z) = u_x + iv_x = 0 + i0 = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 5 Prove that an analytic function with constant imaginary part is constant. [A.U M/J 2005]

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x$ $\ldots(2)$ by C−R

Given $v = c$ [Constant]

$\Rightarrow v_x = 0, \qquad v_y = 0$

We know that $f'(z) = u_x + iv_x$

$= v_y + iv_x$ by (1) $= 0 + i0$

$\Rightarrow f'(z) = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 6 If $\mathbf{f(z)}$ and $\overline{\mathbf{f(z)}}$are analytic in a region D, then show that $\mathbf{f(z)}$ is constant in that region D.

Proof:

Let $f(z) = u(x,y) + iv\,(x,y)$ be an analytic function.

$\overline{f(z)} = u(x,y) - iv(x,y) = u(x,y) + i[-v(x,y)]$

Since, $f(z)$ is analytic in D, we get $u_x = v_y$ and $u_y = -v_x$

Since, $\overline{f(z)}$ is analytic in D, we have $u_x = -v_y$ and $u_y = v_x$

Adding, we get $u_x = 0$ and $u_y = 0$ and hence, $v_x = v_y = 0$

$\therefore\ f(z) = u_x + iv_x = 0 + i0 = 0$

$\therefore\ f(z)$ is constant in D.

Problems based on properties

Theorem: 1 If $\mathbf{f(z)} = \boldsymbol{u + iv}$ is a regular function of z in a domain D, then $\boldsymbol{\nabla^2|f(z)|^2 = 4|f'(z)|^2}$

Solution:

Given $f(z) = u + iv$

$\Rightarrow\ |f(z)| = \sqrt{u^2 + v^2}$

$\Rightarrow |f(z)|^2 = u^2 + v^2$

$\Rightarrow \nabla^2|f(z)|^2 = \nabla^2(u^2 + v^2)$

$= \nabla^2(u^2) + \nabla^2(v^2) \qquad \ldots(1)$

$$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 += \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} \qquad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$

$$(2)\Rightarrow\ \nabla^2(u^2) = 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad [\therefore u \text{ is harmonic}]$$

$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$

Similarly, $\nabla^2(v^2) = 2v_x^2 + 2v_y^2$

$(1)\Rightarrow\ \nabla^2|f(z)|^2 = 2\left[u_x^2 + u_y^2 + v_x^2 + v_y^2\right]$

$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2] \qquad [\because u_x = v_y; u_y = -v_x]$

$$= 4[u_x^2 + v_x^2]$$

$$(i.e.)\nabla^2|f(z)|^2 = 4|f'(z)|^2$$

Note : $f(z) = u + iv; f'(z) = u_x + iv_x$;

(or) $f'(z) = v_y + iu_y$; $|f'(z)| = \sqrt{u_x^2 + v_x^2}$; $|f'(z)|^2 = u_x^2 + v_x^2$

Theorem: 2 If $\mathbf{f(z) = u + iv}$ is a regular function of z in a domain D, then $\mathbf{\nabla^2 \log |f(z)| = 0}$ if $\mathbf{f(z)\ f'(z) \neq 0}$ in D. i.e., $\mathbf{\log |f(z)|}$ is harmonic in D. [A.U A/M 2017 R-13]

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\log |f(z)| = \frac{1}{2} \log (u^2 + v^2)$$

$$\nabla^2 \log|f(z)| = \frac{1}{2} \nabla^2 \log (u^2 + v^2) = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log(u^2 + v^2)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2 + v^2)] \qquad \ldots(1)$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] = \frac{1}{2}\frac{\partial^2}{\partial x}\left[\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{uu_x+vv_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)[uu_{xx}+u_xu_x+vv_{xx}+v_xv_x]-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[uu_{xx}+vv_{xx}+u_x^2+v_x^2]-2(uu_x+vv_x)^2}{(u^2+v^2)^2}$$

Similarly, $\frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2 + v^2)] = \frac{(u^2+v^2)[uu_{yy}+vv_{yy}+u_y^2+v_y^2]-2(uu_y+vv_y)^2}{(u^2+v^2)^2}$

(1)⇒ $\nabla^2 \log|f(z)| = \frac{(u^2+v^2)[u(u_{xx}+u_{yy})+\mathrm{v}(v_{xx}+v_{yy}) +(u_x^2+u_y^2) +(v_x^2+v_y^2)]-2[uu_x+vv_x]^2-2[uu_y+vv_y]^2}{(u^2+v^2)^2}$

$$= \frac{(u^2+v^2)[u(0)+(u_x^2+v_x^2)+u_y^2+v_y^2)--2[u^2u_x^2+v^2v_x^2+2uv\ u_xv_x+u^2\ u_y^2+v^2v_y^2+2uv\ u_y\,v_y]}{(u^2+v^2)^2}$$

$$[\because u_{xx} + u_{yy} = 0, v_{xx} + vv_{yy} = 0]$$

$$= \frac{(u^2+v^2)[|f'(z)^2|+|f'(z)^2|-2[u^2(u_x^2+ u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_x\ v_x+u_y\,v_y]}{(u^2+v^2)^2}$$

$[\because f'(z) = u + iv, |f'(z)| = u_x + iv_x$ (or) $f'(z) = v_y - iu_{y,}|f'(z)|^2 = u_x^2 + v_x^2$

(or) $|f'(z)|^2 = u_y^2 + v_y^2$

$$= \frac{2(u^2+v^2)[|f'(z)^2|-2[u^2|f'(z)|^2+v^2|f'(z)^2|+2uv(0)]}{(u^2+v^2)^2}$$

$$[\because u_x = v_{y,}\ u_y = -v_x]$$

$$\Rightarrow u_x\ v_x + u_yv_y = 0$$

$$\Rightarrow u_x^2 + u_y^2 = u_x^2 + v_x^2 = |f'(z)|^2$$

$$\Rightarrow v_x^2 + v_y^2 = u_y^2 + v_y^2 = |f'(z)|^2$$

$$= \frac{2(u^2+v^2)|f(z)|^2-2(u^2+v^2)|f'(z)|^2}{(u^2+v^2)^2}$$

$$(i.e.)\ \nabla^2 \log|f(z)| = 0$$

Theorem: 3 If $\mathbf{f(z) = u + iv}$ is a regular function of z in a domain D, then

$$\boldsymbol{\nabla^2(u^p) = p(p-1)\,u^{p-2}|f'(z)|^2}$$

Solution:

$$\nabla^2(u^p) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^p)$$

$$= \frac{\partial^2}{\partial x^2}(u^p) + \frac{\partial^2}{\partial y^2}(u^p)$$

$$\frac{\partial^2}{\partial x^2}(u^p) = \frac{\partial}{\partial x}\left[pu^{p-1}\frac{\partial u}{\partial x}\right] = pu^{p-1}u_{xx} + p(p-1)u^{p-2}(u_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^p) = pu^{p-1}u_{yy} + p(p-1)u^{p-2}(u_y)^2$

$$(1) \Rightarrow \nabla^2(u^p) = pu^{p-1}(u_{xx} + u_{yy}) + p(p-1)u^{p-2}[u_x^2 + u_y^2]$$

$$= pu^{p-1}(0) + p(p-1)u^{p-2}|f'(z)|^2$$

$[\because u_{xx} + u_{yy} = 0, f(z) = u + iv, f'(z) = u_x + iv_x,\ |f'(z)|^2 = u_x^2 + u_y^2$

$$\therefore\ \nabla^2(u^p) = p(p-1)\,u^{p-2}|f'(z)|^2$$

Theorem: 4 If $\mathbf{f(z) = u + iv}$ is a regular function of z, then $\boldsymbol{\nabla^2|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2}$.

[A.U N/D 2015 R-13]

Solution:

Let $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2} \qquad \ldots(a)$$

$$|f(z)|^p = (u^2 + v^2)^{p/2} \qquad \ldots(b)$$

$$\nabla^2|f(z)|^p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2)^{p/2}$$

$$= \frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} + \frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2}$$

$$\frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} = \frac{\partial}{\partial x}\left[\frac{p}{2}(u^2 + v^2)^{\frac{p}{2}-1}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]\right]$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_xu_x + vv_{xx} + v_xv_x]$$

$$+p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)(2uu_x + 2vv_x)$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2} = p(u^2 + v^2)^{\frac{p}{2}-1}\left[uu_{yy} + u_y^2 + vv_{yy} + v_y^2\right]$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_y + vv_y)^2$$

$$\Rightarrow\ \nabla^2|f(z)|^p = p(u^2 + v^2)^{\frac{p}{2}-1}\left[u(u_{xx} + u_{yy}) + v(v_{vx} + v_{yy}) + u_x^2 + u_y^2 + v_x^2 + v_y^2\right] +$$

$$2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}\left[u^2u_x^2 + v^2v_x^2 + 2uv\,u_xv_x + u^2u_y^2 + v^2v_y^2 + 2uv\,u_yv_y\right]$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}\left[u(0) + v(0) + 2(u_x^2 + u_y^2)\right]$$

$$+2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}\left[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_xv_x+u_yv_y)\right]$$

$$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2+2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}[u^2|f'(z)|^2+v^2|f'(z)|^2+2uv(0)]$$

$$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2+2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}(u^2+v^2)|f'(z)|^2$$

$$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2+2p\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2$$

$$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2\left[1+\frac{p}{2}-1\right]$$

$$= 2p(u^2+v^2)^{\frac{p}{2}-1}|f'(z)|^2 \quad = p^2(u^2+v^2)^{\frac{p-2}{2}}|f'(z)|^2$$

$$= p^2(\sqrt{u^2+v^2})^{p-2}|f'(z)|^2$$

$= p^2|f(z)|^{p-2}|f'(z)|^2$ by (a) & (b)

Theorem: 5 If $\mathbf{f(z) = u + iv}$ is a regular function of z, in a domain D, then

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$$ **[A.U A/M 2015 R8]**

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}\left[\sqrt{u^2+v^2}\right]$$

$$= \frac{1}{2\sqrt{u^2+v^2}}[2uu_x+2vv_x] \quad = \frac{uu_x+vv_x}{\sqrt{u^2+v^2}}$$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 = \frac{(uu_x+vv_x)^2}{u^2+v^2} = \frac{u^2u_x^2+v^2v_x^2+2uv\ u_xv_x}{u^2+v^2}$$

Similarly, $\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{u^2u_y^2+v^2v_y^2+2uv\ u_yv_y}{u^2+v^2}$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2+\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{u^2[u_x^2+u_y^2]+v^2[v_x^2+v_y^2]+2uv\ [u_xv_x+u_yv_y]}{u^2+v^2}$$

$$= \frac{u^2|f'(z)|^2+v^2|f'(z)|^2+2u\ v\ (0)}{u^2+v^2}\,[\because u_x = v_y;\ u_y = -v_x]$$

$$= \frac{(u^2+v^2)|f(z)|^2}{u^2+v^2} = |f'(z)|^2[\because u_xv_x+u_yv_y = 0]$$

Theorem: 6 If $\mathbf{f(z)} = \boldsymbol{u+iv}$ is a regular function of z, then $\mathbf{\nabla^2|Re\ f(z)|^2 = 2|f'(z)|^2}$

Solution:

Let $f(z) = u + iv$

$$\text{Re}f(z) = u$$

$$|\text{Re}\,f'(z)|^2 = u^2$$

$$\nabla^2|\text{Re}\,f'(z)|^2 = \nabla^2u^2$$

$$= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= \left(\frac{\partial^2}{\partial x^2}\right)(u^2)+\left(\frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= 2[u_x^2+u_y^2]$$

$$= 2\,|f'(z)|^2$$

Theorem: 7 If $\mathbf{f(z)} = \boldsymbol{u} + \boldsymbol{iv}$ is a regular function of z, then prove that $\boldsymbol{\nabla^2}|\mathbf{Im\ f(z)}|^2 = \mathbf{2}|\mathbf{f'(z)}|^2$

Proof:

Let $f(z) = u + iv$

$$\mathrm{I\,m} f(z) = v$$

$$|Im\ f(z)|^2 = v^2$$

$$\frac{\partial}{\partial x}(v^2) = 2vv_x$$

$$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x v_x] = 2[vv_{xx} + v_x^2]$$

Similarly, $\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Im\ f(z)|^2 = 2[v(v_{xx} + v_{yy}) + v_x^2 + v_y^2]$$

$$= 2[v(0) + u_x^2 + v_x^2] \qquad \text{by C}-\text{R equation}$$

$$= 2|\mathrm{f}'(\mathrm{z})|^2$$

Theorem: 8 Show that $\boldsymbol{\frac{\partial^2}{\partial x^2}} + \boldsymbol{\frac{\partial^2}{\partial y^2}} = \mathbf{4}\boldsymbol{\frac{\partial^2}{\partial z \partial \bar{z}}}$ (or) S T $\nabla^2 = \mathbf{4}\boldsymbol{\frac{\partial^2}{\partial z \partial \bar{z}}}$

Proof:

Let *x* & *y* are functions of *z* and $\bar{z}$

that is $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y} \qquad \ldots(1)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{-1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \left(2\frac{\partial}{\partial z}\right)\left(2\frac{\partial}{\partial \bar{z}}\right) \text{ by (1) \& (2)}$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Theorem: 9 If $\boldsymbol{f(z)}$ is analytic, show that $\boldsymbol{\nabla^2}|\,\mathbf{f(z)}|^2 = \mathbf{4}|\mathbf{f'(z)}|^2$

Solution:

We know that, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$|f(z)|^2 = f(z)\overline{f(z)}$$

$$\nabla^2|\,\mathrm{f(z)}|^2 = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}[f(z)\overline{f(z)}]$$

$$= 4\left[\frac{\partial}{\partial z}f(z)\right]\left[\frac{\partial}{\partial \bar{z}}\overline{f(z)}\right]$$

[∵ $f(z)$ is independent of $\bar{z}$ and $\overline{f(z)}$ is independent of z]

$$\therefore\ \nabla^2|\,f(z)|^2 = 4[f'(z)\left[\overline{\frac{\partial}{\partial z}f(z)}\right] = 4f'(z)\overline{f'(z)}$$

$$= 4|\,f'(z)|^2 \quad [\because z\bar{z} = |\,(z)|^2]$$

Example: 3.20 Give an example such that u and v are harmonic but $u + iv$ is not analytic.

[A.U. N/D 2005]

Solution:

$$u = x^2 - y^2,\ \ v = \frac{-y}{x^2+y^2}$$

Example: 3.21 Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic. [A.U N/D 2016 R-13]

Solution:

Given $u = 2x^2 - my^2 + 3x$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\ [\because \text{u is harmonic}] \qquad \ldots(1)$$

$$\begin{array}{l|l} \dfrac{\partial u}{\partial x} = 4x + 3 & \dfrac{\partial u}{\partial y} = -2my \\ \dfrac{\partial^2 u}{\partial x^2} = 4 & \dfrac{\partial^2 u}{\partial y^2} = -2m \end{array}$$

$$\therefore (1) \Rightarrow (4) + (-2m) = 0$$

$$\Rightarrow m = 2$$

3.4 CONSTRUCTION OF ANALYTIC FUNCTION

There are three methods to find $f(z)$.

Method: 1 Exact differential method.

(i) Suppose the harmonic function $u(x, y)$ is given.

Now, $dv = v_x dx + v_y dy$ is an exact differential

Where, v_x and v_y are known from u by using C−R equations.

$\therefore v = \int v_x\, dx + \int v_y dy = -\int u_y dx + \int u_x dy$

(ii) Suppose the harmonic function $v(x, y)$ is given.

Now, $du = u_x dx + u_y dy$ is an exact differential

Where, u_x and u_y are known from v by using C−R equations.

$$u = \int u_x\, dx + \int u dy$$

$$= \int v_y dx + \int -v_x dy$$

$$= \int v_y dx - \int v_x dy$$

Method: 2 Substitution method

$$f(z) = 2u\left[\frac{1}{2}(z+a), \frac{-i}{2}(z-a)\right] - [u(a,0), -iv(a,0)]$$

Here, $u(a,0), -iv(a,0)$ is a constant

Thus $f(z) = 2u\left[\frac{1}{2}(z+a), \frac{-i}{2}(z-a)\right] + C$

By taking $a = 0$, that is, if $f(z)$ is analytic $z = 0 + i0$,

We have the simpler formula for $f(z)$

$$f(z) = 2\left[u\frac{z}{2}, \frac{-iz}{2}\right] + C$$

Method: 3 [Milne – Thomson method]

(i) $\boxed{\text{To find } f(z) \text{ when } u \text{ is given}}$

Let $f(z) = u + iv$

$$f'(z) = u_x + iv_x$$

$$= u_x - iv_y \text{ [by C−R condition]}$$

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

(ii) $\boxed{\text{To find } f(z) \text{ when v is given}}$

Let $f(z) = u + iv$

$$f'(z) = u_x + iv_x$$

$$= v_y + iv_x \quad \text{[by C−R condition]}$$

$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

Example: 3.22 Construct the analytic function $f(z)$ for which the real part is $e^x \cos y$.

Solution:

Given $u = e^x \cos y$

$\Rightarrow u_x = e^x \cos y$ $\quad [\because \cos 0 = 1]$

$\Rightarrow u_x(z,0) = e^x$

$\Rightarrow u_y = e^x \cos y$ $\quad [\because \sin 0 = 0]$

$\Rightarrow u_y(z,0) = 0$

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$\therefore f(z) = \int e^z\,dz - i\int 0\,dz + C$$

$$= e^z + C$$

Example: 3.23 Determine the analytic function $w = u + iv$ if $u = e^{2x}(x\cos 2y - y\sin 2y)$

Solution:

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$u_x = e^{2x}[\cos 2y] + (xcos2y - y\sin 2y)[2\,e^{2x}]$$

$$u_x(z,0) = e^{2z}[1]+[z(1)-0][2e^{2z}]$$
$$= e^{2z} + 2ze^{2z}$$
$$= (1+2z)e^{2x}$$
$$u_y = e^{2x}[-2x\sin 2y - (y2cos2y\ + \sin 2y)]$$
$$u_y(z,0) = e^{2z}[-0-(0+0)] = 0$$

$\therefore f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int(1+2z)e^{2z}dz - i\int 0 + dz + C$$
$$= \int(1+2z)e^{2z}dz + C$$
$$= (1+2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} + C \quad [\because \int uv\,dz = uv_1 - u'v_2 + u''v_3 - \cdots]$$
$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + C$$
$$= ze^{2z} + C$$

Example: 3.24 Determine the analytic function where real part is

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. [Anna, May 2001]

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$u_x = 3x^2 - 3y^2 + 6x$$
$$\Rightarrow u_x(z,0) = 3z^2 - 0 + 6z$$
$$u_y = 0 - 6xy + 0 - 6y$$
$$\Rightarrow u_y(z,0) = 0$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int(3z^2+6z)dz - i\int 0 + dz + C$$
$$= 3\frac{z^2}{3} + 6\frac{z^2}{2} + C$$
$$= z^3 + 3z^2 + C$$

Example: 3.25 Determine the analytic function whose real part in $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

[Anna, May 1996][A.U Tvli. A/M 2009][A.U N/D 2012]

Solution:

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)[2\cos 2x] - \sin 2x[2\sin 2x]}{[\cosh 2y - \cos 2x]^2}$$
$$u_x(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{[\cosh 0 - \cos 2z]^2}$$
$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z-2[\cos^2 2z+\sin^2 2z]}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z-2}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z-2}{(1-\cos 2z)}$$

$$= \frac{-2}{2sin^2 2}$$

$$= -cosec^2 z$$

$$u_y = \frac{(\cosh 2y-\cos 2x)\,(0)-\sin 2x[2\sin 2y]}{[\cosh 2y-\cos 2x]^2}$$

$$\Rightarrow\ u_y(z,0)=0$$

$f(z)=\int u_x\,(z,0)dz-i\int u_y(z,0)dz+C$ [by Milne−Thomson rule],

where C is a complex constant.

$$f(z)=\int(-cosec^2 z)dz-i\int 0\,dz+C$$

$$= cot\ z+C$$

Example: 3.26 Show that the function $u=\frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find $f(z)$ [A.U A/M 2008, A.U A/M 2017 R8]

Solution:

Given $u=\frac{1}{2}\log(x^2+y^2)$

$$u_x=\frac{1}{2}\frac{1}{(x^2+y^2)}(2x)=\frac{x}{x^2+y^2},$$

$$\Rightarrow u_x(z,0)=\frac{z}{z^2}=\frac{1}{z}$$

$$u_{xx}=\frac{(x^2+y^2)[1]-x[2x]}{[x^2+y^2]^2}=\frac{x^2+y^2-2x^2}{[x^2+y^2]^2}=\frac{y^2-x^2}{[x^2+y^2]^2}\qquad\ldots(1)$$

$$u_y=\frac{1}{2}\frac{1}{x^2+y^2}(2y)=\frac{y}{x^2+y^2}$$

$$\Rightarrow u_y(z,0)=0$$

$$u_{yy}=\frac{(x^2+y^2)[1]-y[2y]}{[x^2+y^2]^2}=\frac{x^2-y^2}{[x^2+y^2]^2}\qquad\ldots(2)$$

To prove u is harmonic:

$$\therefore u_{xx}+u_{yy}=\frac{(y^2-x^2)+(x^2-y^2)}{[x^2+y^2]^2}=0\qquad by\ (1)\&(2)$$

$\Rightarrow u$ is harmonic.

To find $f(z)$:

$f(z)=\int u_x\,(z,0)dz-i\int u_y(z,0)dz+C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z)=\int\frac{1}{z}dz-i\int 0\,dz+C$$

$$= log\ z+C$$

To find v :

$$f(z) = log\,(re^{i\theta}) \qquad [\because z = re^{i\theta}]$$

$$u + iv = \log r + \log\, e^{i\theta} = \log r + i\theta$$

$$\Rightarrow u = logr, v = \theta$$

Note: $z = x + iy$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\log r = \frac{1}{2}\log(x^2 + y^2)$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) \qquad i.e., v = tan^{-1}\left(\frac{y}{x}\right)$$

Example: 3.27 Construct an analytic function $f(z) = u + iv$, given that $u = e^{x^2-y^2}\cos 2xy$. Hence find v. [A.U D15/J16, R-08]

Solution:

Given $u = e^{x^2-y^2}\cos 2xy = e^{x^2}e^{-y^2}\cos 2xy$

$$u_x = e^{-y^2}\left[e^{x^2}(-2y\sin 2xy) + \cos 2xy\; e^{x^2}2x\right]$$

$$u_x(z,0) = 1\left[e^{z^2}(0) + 2ze^{z^2}\right] = 2ze^{z^2}$$

$$u_y = e^{x^2}\left[e^{-y^2}(-2x\sin 2xy) + \cos 2xye^{-y^2}(-2y)\right]$$

$$u_y(z,0) = e^{z^2}[0 + 0] = 0$$

$$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

$$= \int 2z\,e^{z^2}dz + C$$

$$= 2\int z\,e^{z^2}dz + C$$

put $t = z^2$, dt= $2z\,dz$

$$= \int e^t\,dt + C$$

$$= e^t + C$$

$$f(z) = e^{z^2} + C$$

To find v :

$$u + iv = e^{(x+iy)^2} = e^{x^2-y^2+i\,2xy} = e^{x^2-y^2}e^{i2\,xy}$$

$$= e^{x^2-y^2}\,[\cos(2xy) + isin\,(2xy)]$$

$$v = e^{x^2-y^2}\sin 2xy \quad [\because \text{equating the imaginary parts}]$$

Example: 3.28 Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$. [Anna, May 1996] [A.U M/J 2014]

Solution:

Given $v = e^{-x}(x\cos y + y\sin y)$

$$v_x = e^{-x}[\cos y] + (x\cos y + y\sin y)[-e^{-x}]$$

$$v_x(z,0) = e^{-z} + (z)(-e^{-z}) = (1-z)e^{-z}$$

$$v_y = e^{-x}[-x\sin y + (y\cos y \ + \sin y\ (1))]$$

$$v_x(z,0) = e^{-z}[0+0+0] = 0$$

$\therefore f(z) = \int v_y\,(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

$$f(z) = \int 0dz + i\int(1-z)e^{-z}dz + C$$

$$= i\int(1-z)e^{-z}dz + C$$

$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$$

$$= i[-(1-z)e^{-z} + e^{-z}] + C$$

$$= ize^{-z} + C$$

Example: 3.29 In a two dimensional flow, the stream function is $\psi = tan^{-1}\left(\frac{y}{x}\right)$. Find the velocity potential $\boldsymbol{\varphi}$. **[A.U M/J 2016 R13]**

Solution:

Given $\psi = tan^{-1}(y/x)$

We should denote, ϕ by u and $\psi$by v

$$\therefore v = tan^{-1}(y/x)$$

$v_x = \frac{1}{1+(y/x)^2}\left[\frac{-y}{x^2}\right] = \frac{-y}{x^2+y^2}$, $\qquad v_x(z,0) = 0$

$v_y = \frac{1}{1+(y/x)^2}\left[\frac{1}{x}\right] = \frac{x}{x^2+y^2}$ $\qquad v_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$

$$\therefore f(z) = \int v_y\,(z,0)dz + i\int v_x(z,0)dz + C$$

$$f(z) = \int\frac{1}{z}dz + i\int 0\,dz + C \ = \log z + C$$

To find $\boldsymbol{\varphi}$:

$f(z) = log\,(re^{i\theta})$ $\qquad [\because z = re^{i\theta}]$

$$u + iv = \log r + \log\, e^{i\theta}$$

$$u + iv = \log r + i\theta$$

$$\Rightarrow u = \log r$$

$$\boxed{z = x + iy, |z| = \sqrt{x^2+y^2}}$$

$$\Rightarrow\ u = log\sqrt{x^2+y^2}$$

$$= \frac{1}{2}\log(\,x^2+y^2)$$

So, the velocity potential φ is

$$\varphi = \frac{1}{2}\log(\,x^2+y^2)$$

Note: In two dimensional steady state flows, the complex potential $f(z) = \varphi(x,y) + i\psi(x,y)$ is analytic.

Example: 3.30 If $\boldsymbol{w = u + iv}$ is an analytic function and $\boldsymbol{v = x^2 - y^2 + \frac{x}{x^2+y^2}}$, find $\boldsymbol{u}$.

Solution: **[Anna, May 1999]**

Given $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$v_x = 2x - 0 + \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad v_x(z,0) = 2z + \frac{(-z^2)}{(z^2)}$$

$$\Rightarrow v_x(z,0) = 2z - \frac{1}{z^2}$$

$$v_y = 0 - 2y + \frac{0 - x(2y)}{(x^2+y^2)^2}$$

$$= 0 - 2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \ v_y(z,0) = 0$$

$\therefore f(z) = \int v_y\,(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

$$f(z) = \int 0dz + i\int\left(2z - \frac{1}{z^2}\right)dz + C$$

$$= i\left[2\frac{z^2}{2} + \frac{1}{z}\right] + C \qquad \left[\because \int \frac{-1}{z^2}dz = \frac{1}{z}\right]$$

$$= i\left[z^2 + \frac{1}{z}\right] + C$$

Example: 3.31 If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ interms of z. **[A.U Dec. 1997]**

Solution:

Given $u - v = e^x(\cos y - \sin y)$, $\quad \ldots(A)$

Differentiate (A) p.w.r. to x, we get

$$u_x - v_x = e^x(\cos y - \sin y),$$

$$u_x(z,0) - v_x(z,0) = e^z \qquad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$u_y - v_y = e^x(-\sin y - \cos y)$$

$$u_y(z,0) - v_y(z,0) = e^z[-1]$$

$$\text{i.e., } u_y(z,0) - v_y(z,0) = -e^z$$

$-v_x(z,0) - u_x(z,0) = -e^z \quad \ldots(2)$ [by C−R conditions]

$$(1) + (2) \Rightarrow -2v_x(z,0) = 0$$

$$\Rightarrow v_x(z,0) = 0$$

$$(1) \Rightarrow u_x(z,0) = e^z$$

$f(z) = \int u_x\,(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

$$f(z) = \int e^z dz + i0 + C$$

$$= e^z + C$$

Example: 3.32 Find the analytic functions $f(z) = u + iv$ given that

(i) $\boldsymbol{2u + v = e^x(\cos y - \sin y)}$

(ii) $\boldsymbol{u - 2v = e^x(\cos y - \sin y)}$ **[A.U A/M 2017 R-13]**

Solution:

Given (i) $2u + v = e^x(\cos y - \sin y) \quad \ldots(A)$

Differentiate (A) p.w.r. to x, we get

$$2u_x + v_x = e^x(\cos y - \sin y)$$

$$2u_x - u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - u_y(z,0) = e^z \quad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$2u_y + v_y = e^x[-\sin y - \cos y]$$

$$2u_y + u_x = e^x\,[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$2u_y(z,0) + u_x(z,0) = e^z(-1) = -e^z \quad \ldots(2)$$

$$(1) \times (2) \quad \Rightarrow 4u_x(z,0) - 2u_y(z,0) = 2e^z \quad \ldots(3)$$

$$(2) + (3) \quad \Rightarrow 5u_x(z,0) = e^z$$

$$\Rightarrow u_x(z,0) = \tfrac{1}{5}e^z$$

$$(1) \quad \Rightarrow u_y(z,0) = \tfrac{2}{5}e^z - e^z = -\tfrac{3}{5}e^z$$

$$\Rightarrow u_y(z,0) = -\tfrac{3}{5}e^z$$

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

Where, C is a complex constant.

$$f(z) = \int \tfrac{1}{5}e^z dz - i\int -\tfrac{3}{5}e^z dz + C$$

$$= \tfrac{2}{5}e^z + \tfrac{3}{5}ie^z + C$$

$$= \tfrac{1+3i}{5}e^z + C$$

(ii) $u - 2v = e^x(\cos y - \sin y) \quad \ldots(B)$

Differrentiate (B) p.w.r. to x, we get

$$u_x - 2v_x = e^x(\cos y - \sin y)$$

$$u_x + 2u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$u_x(z,0) + 2u_y(z,0) = e^z \quad \ldots(1)$$

Differentiate (B) p.w.r. to y, we get

$$u_y - 2v_y = e^x[-\sin y - \cos y]$$

$$u_y - 2u_x = e^x\,[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$u_y(z,0) - 2u_x(z,0) = -e^z \quad \ldots(2)$$

$$(1) \times (2) \quad \Rightarrow 2u_x(z,0) + 4u_y(z,0) = 2e^z \quad \ldots(3)$$

$$(2) + (3) \quad \Rightarrow 5u_y(z,0) = e^z$$

$$\Rightarrow u_y(z,0) = \tfrac{1}{5}e^z$$

$$(1) \quad \Rightarrow u_x(z,0) = -\frac{2}{5}e^z + e^z$$

$$= \tfrac{3}{5}e^z$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

$$f(z) = \int \tfrac{3}{5}e^z dz - i\int \tfrac{1}{5}e^z dz + C$$

$$= \tfrac{3}{5}e^z - i\tfrac{1}{5}e^z + C = \tfrac{3-i}{5}e^z + C$$

Example: 3.33 Determine the analytic function $\boldsymbol{f(z) = u + iv}$ given that $\boldsymbol{3u + 2v = y^2 - x^2 + 16xy}$ [A.U. N/D 2007]

Solution:

Given $3u + 2v = y^2 - x^2 + 16xy \qquad \ldots(A)$

Differentiate (A) p.w.r. to x, we get

$$3u_x + 2v_x = -2x + 16y$$

$3u_x - 2u_y = -2x + 16y$ [by C−R condition]

$$3u_x(z,0) - 2u_y(z,0) = -2z \qquad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$3u_y + 2v_y = 2y + 16x$$

$3u_y + 2u_x = 2y + 16x$ [by C−R condition]

$$3u_y(z,0) + 2u_x(z,0) = 16z \qquad \ldots(2)$$

$$(1)\times(2) \quad \Rightarrow 6u_x(z,0) - 4u_y(z,0) = -4z \qquad \ldots(3)$$

$$(2)\times(3) \quad \Rightarrow 9u_y(z,0) + 6u_x(z,0) = 48z$$

$$(3)-(4) \quad \Rightarrow -13u_y(z,0) = -52z$$

$$\Rightarrow u_y(z,0) = 4z$$

$$(1) \quad \Rightarrow 3u_x(z,0) = 8z - 2z = 6z$$

$$\Rightarrow u_x(z,0) = 2z$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

where C is a complex constant.

$$f(z) = \int 2z dz - i\int 4z dz + C$$

$$= 2\,\tfrac{z^2}{2} - i\tfrac{4z^2}{2} + C$$

$$= z^2 - i2z^2 + C$$

$$= (1 - i2)z^2 + C$$

Example:3.34 Find an analytic function $f(z) = u + iv$ given that $2u + 3v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$ [A.U. A/M 2017 R-8]

Solution:

Given $2u + 3v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Differentiate p.w.r. to x, we get

$$2u_x + 3v_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$2u_x - 3u_y = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - 3u_y(z,0) = \frac{2\cos 2z(1 - \cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z}$$

$$= \frac{-2}{2\sin^2 z} = -cosec^2 z$$

$$2u_x(z,0) - 3u_y(z,0) = -cosec^2 z \qquad \ldots(1)$$

Differentiate p.w.r. to y, we get

$$2u_y + 3v_y = \frac{0 - \sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}(2)$$

$$2u_y + 3u_x = \frac{0 - \sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}(2) \qquad \text{[by C − R condition]}$$

$$2u_y(z,0) + 3u_x(z,0) = 0 \qquad \ldots(2)$$

Solving (1) & (2) we get,

$$\Rightarrow u_x(z,0) = -\frac{2}{13}\text{cosec}^2 z$$

$$\Rightarrow u_y(z,0) = -\frac{2}{13}\text{cosec}^2 z$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant

$$f(z) = \int \left(\frac{-2}{13}\right)\text{cosec}^2 z\,dz - i\int\left(\frac{3}{13}\right)\text{cosec}^2 z dz + C$$

$$= \left(\frac{2}{13}\right)\cot z + \left(\frac{3}{13}\right)\cot z + C$$

$$= \frac{2+3i}{i3}\cot z + C$$

Example: 3.35 Find the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(cos\,y - sin\,y)$ [A.U A/M 22017 R-13]

Solution:

Given $2u + 3v = e^x(cos\,y - sin\,y)$

Differentiate p.w.r. to x, we get

$$2u_x + 3v_x = e^x(\cos y - \sin y)$$

$$2u_x - 3u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - 3u_y(z,0) = e^z \qquad \ldots(1)$$

Differentiate p.w.r. to y, we get

$$2u_y + 3v_y = e^x[-\sin y - \cos y]$$

$$2u_y + 3u_x = -e^x\,[\sin y + \cos y] \quad \text{[by C−R condition]}$$

$$2u_y(z,0) + 3u_x(z,0) = -e^z \qquad \ldots(2)$$

$$(1)\times(3) \quad \Rightarrow 6u_x(z,0) - 9u_y(z,0) = 3e^z \qquad \ldots(3)$$

$$(2)\times 2 \quad \Rightarrow 6u_x(z,0) + 4u_y(z,0) = -2e^z \qquad \ldots(4)$$

$$(3)-(4) \quad \Rightarrow -13u_y(z,0) = 5e^z$$

$$\Rightarrow u_y(z,0) = -\frac{5}{13}e^z$$

$$(1) \quad \Rightarrow 2u_x(z,0) + \frac{15}{13}e^z = e^z$$

$$2u_x(z,0) = e^z - \frac{15}{13}e^z = -\frac{2}{13}e^z$$

$$\Rightarrow u_x(z,0) = -\frac{1}{13}e^z$$

$$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$$

$$\therefore f(z) = \int \frac{-1}{13}e^z\;dz - i\int\left(\frac{-5}{13}\right)dz + C$$

$$= \frac{-1}{13}e^z + \frac{5}{13}e^z i + C = \frac{-1+5i}{13}e^z + C$$

Exercise: 3.4

Construction of an analytic function

1. Show that the function $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.

   [**Ans:** $v(x,y) = 3x^2y + 4xy - x^3 + C$, $f(z) = -iz^3 + 2z^2 + iC$ where C is a real constant.]

2. If $f(z) = u + iv$ is an analytic function of z, and if $u = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$, find v.

   [**Ans**: $v = \frac{-2\sinh 2y}{e^{2y}+e^{-2y}-2\cos 2x} + C$]

3. Find v such that $w = u + iv$ is an analytic function of z, given that $u = e^{x^2-y^2}\cos 2xy$. Hence find w.

   [**Ans:** $v = e^{x^2-y^2}\sin 2xy + C \quad w = e^{z^2} + C$]

4. Find the analytic function $w = u + iv$ if $w = e^{2^x}(x\cos 2y - y\sin 2y)$.Hence find u.

   [**Ans**: $w = ize^{2z} + C, u = -(x\sin 2y + y\cos 2y)\,e^{2x} + C$]

5. If $v = \frac{x-y}{x^2+y^2}$ find u such $u + iv$ is an analytic function. What is the harmonic conjugate of v?

   [**Ans**: $u = \frac{x+y}{x^2+y^2} + C$ Harmonic conjugate of v is $-u = \frac{-(x+y)}{x^2+y^2}$, $f(z) = \frac{1+i}{z} + C$

6. Find the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y+\cos 2x}$

   [**Ans:** $f(z) = \tan z + C$]

7. Find the analytic function whose imaginary part is $-e^{-2xy}\cos(x^2-y^2)$

[**Ans:** $f(z) = -ie^{iz^2} + C$]

6. Prove that $u = 2^x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function. [**Ans**: $v = 2y - 3x^2y + y^3 + C, f(z) = 2z - z^3 + iC$]
7. Find the real part of the analytic function whose imaginary part is $e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$. Construct the analytic function.

   [Ans: $u = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y]\,, f(z) = z^2e^{-z} + C$]
8. Find the analytic function $f(z) = u + iv$ given that $2u + v = e^{2x}[(2x+y)\cos 2y + (x-2y)\sin 2y]$

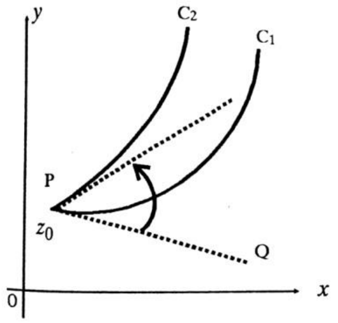
[**Ans**: $f(z) = ze^{2z} + C$]

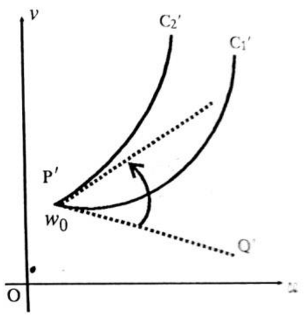
9. Prove that $u = x^2 - y^2$ and $v = -\frac{y}{x^2+y^2}$ are harmonic functions but not harmonic conjugates.

3.5 CONFORMAL MAPPING

Definition: Conformal Mapping

A transformation that preserves angels between every pair of curves through a point, both in magnitude and sense, is said to be conformal at that point.





Definition: Isogonal

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be an isogonal at that point.

Note: 3.4 (i) A mapping $w = f(z)$ is said to be conformal at $z = z_0$, if $f'(z_0) \neq 0$.

Note: 3.4 (ii) The point, at which the mapping $w = f(z)$ is not conformal,

$(i.e.) f'(z) = 0$ is called a **critical point** of the mapping.

If the transformation $w = f(z)$ is conformal at a point, the inverse transformation $z = f^{-1}(w)$ is also conformal at the corresponding point.

The critical points of $z = f^{-1}(w)$ are given by $\frac{dz}{dw} = 0$. hence the critical point of the transformation $w = f(z)$ are given by $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$,

Note: 3.4 (iii) Fixed points of mapping.

Fixed or invariant point of a mapping $w = f(z)$ are points that are mapped onto themselves, are "Kept fixed" under the mapping. Thus they are obtained from $w = f(z) = z$.

The identity mapping $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has infinitely many fixed points.

$w = \frac{1}{z}$ has two fixed points, a rotation has one and a translation has none in the complex plane.

Some standard transformations

Translation:

The transformation $w = C + z$, where C is a complex constant, represents a translation.

Let $z = x + iy$

$w = u + iv$ and $C = a + ib$

Given $w = z + C$,

$(i.e.)\ u + iv = x + iy + a + ib$

$\Rightarrow u + iv = (x + a) + i(y + b)$

Equating the real and imaginary parts, we get $u = x + a, v = y + b$

Hence the image of any point $p(x, y)$in the z−plane is mapped onto the point $p'(x + a, y + b)$ in the w-plane. Similarly every point in the z−plane is mapped onto the w plane.

If we assume that the w−plane is super imposed on the z-plane, we observe that the point (x, y) and hence any figure is shifted by a distance $|C| = \sqrt{a^2 + b^2}$ in the direction of C i.e., translated by the vector representing C.

Hence this transformation transforms a circle into an equal circle. Also the corresponding regions in the z and w planes will have the same shape, size and orientation.

Problems based on $w = z + k$

Example: 3.36 What is the region of the w plane into which the rectangular region in the Z plane bounded by the lines $x = 0, y = 0,\ x = 1\ and\ y = 2$ is mapped under the transformation $w = z + (2 - i)$

Solution:

Given $w = z + (2 - i)$

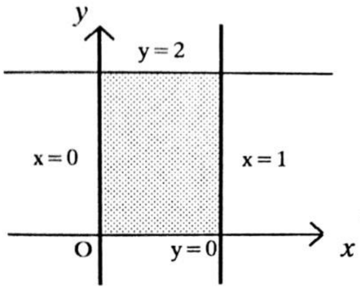
$(i.e.)\ u + iv = x + iy + (2 - i) = (x + 2) + i(y - 1)$

Equating the real and imaginary parts

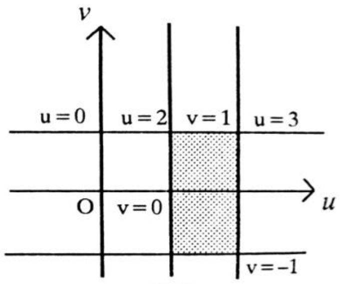
$$u = x + 2,\ v = y - 1$$

Given boundary lines are	transformed boundary lines are
$x = 0$	$u = 0 + 2 = 2$
$y = 0$	$v = 0 - 1 = -1$
$x = 1$	$u = 1 + 2 = 3$
$y = 2$	$v = 2 - 1 = 1$

Hence, the lines $x = 0, y = 0, x = 1,$ and $y = 2$ are mapped into the lines $u = 2, v = -1,$ $u = 3, and\ v = 1$ respectively which form a rectangle in the w plane.



z-plane



w-plane

Example: 3.37 Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$

Solution:

Given $w = z + 2 + 4i$

$$(i.e.)\ u + iv = x + iy + 2 + 4i$$

$$= (x + 2) + i(y + 4)$$

Equating the real and imaginary parts, we get

$$u = x + 2, v = y + 4,$$

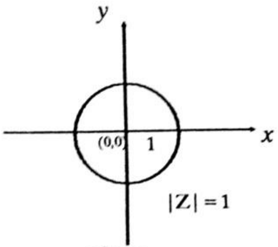
$$x = u - 2, y = v - 4,$$

Given $|z| = 1$

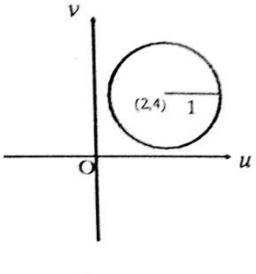
$$(i.e.)\ x^2 + y^2 = 1$$

$$(u - 2)^2 + (v - 4)^2 = 1$$

Hence, the circle $x^2 + y^2 = 1$is mapped into $(u - 2)^2 + (v - 4)^2 = 1$ in w plane which is also a circle with centre (2, 4)and radius 1.



z-Plane



w-Plane

2. Magnification and Rotation

The transformation $w = cz$, where c is a complex constant, represents both magnification and rotation.

This means that the magnitude of the vector representing z is magnified by $a = |c|$ and its direction is rotated through angle $\alpha = amp\ (c)$. Hence the transformation consists of a magnification and a rotation.

Problems based on $\mathbf{w = cz}$

Example: 3.38 Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x = 0, y = 0, x + y = 1$ is transformed under the transformation $w = 2z$.

Solution:

Let $w = u + iv$

$$z = x + iy$$

Given $w = 2z$

$$u + iv = 2(x + iy)$$

$$u + iv = 2x + i2y$$

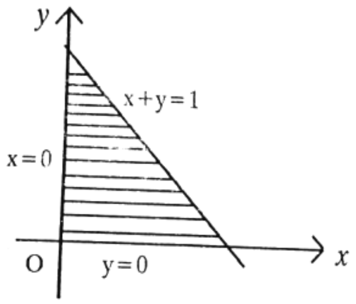
$$u = 2x \Rightarrow x = \frac{u}{2}, v = 2y \Rightarrow y = \frac{v}{2}$$

Given region (D) whose boundary lines are		Transformed region D' whose boundary lines are
$x = 0$	$\Rightarrow$	$u = 0$
$y = 0$	$\Rightarrow$	$v = 0$
$x + y = 1$	$\Rightarrow$	$\frac{u}{2} + \frac{v}{2} = 1 [\because x = \frac{u}{2}, y = \frac{v}{2}]$ $(i.e.)\ u + v = 2$

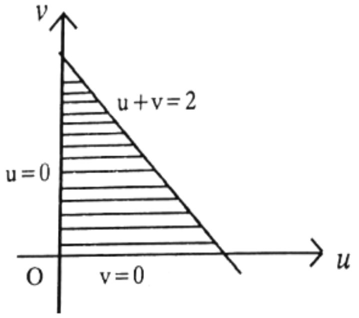
In the z plane the line $x = 0$ is transformed into $u = 0$in the w plane.

In the z plane the line $y = 0$ is transformed into $v = 0$in the w plane.

In the z plane the line $x + y =$ is transformed into$u + v = 2$ in the w plane.



z-plane



w-plane

Example: 3.39 Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$.

Solution:

Given $w = 5z$

$$|w| = 5|z|$$

i.e., $|w| = 5\lambda \quad [\because |z| = \lambda]$

Hence, the image of $|z| = \lambda$ in the z plane is transformed into $|w| = 5\lambda$ in the w plane under the transformation $w = 5z$.

Example: 3.40 Find the image of the circle $|z| = 3$ under the transformation $w = 2z$

[A.U N/D 2012] [A.U N/D 2016 R-13]

Solution:

Given $w = 2z, \ |z| = 3$

$$|w| = (2)|z|$$

$$= (2)(3), \quad \text{Since } |z| = 3$$

$$= 6$$

Hence, the image of $|z| = 3$ in the z plane is transformed into $|w| = 6\, w$ plane under the transformation $w = 2z$.

Example: 3.41 Find the image of the region $y > 1$ under the transformation $\boldsymbol{w = (1 - i)z}$. [Anna, May – 1999]

Solution:

Given $w = (1 - i)z$.

$$u + v = (1 - i)(x + iy)$$

$$= x + iy - ix + y$$

$$= (x + y) + i(y - x)$$

$$\text{i.e.,} \quad u = x + y, \quad v = y - x$$

$$u + v = 2y \qquad u - v = 2x$$

$$y = \frac{u+v}{2} \qquad x = \frac{u-v}{2}$$

Hence, image region $y > 1$ is $\frac{u+v}{2} > 1$ i.e., $u + v > 2$ in the w plane.

3. Inversion and Reflection

The transformation $w = \frac{1}{z}$ represents inversion w.r.to the unit circle $|z| = 1$, followed by reflection in the real axis.

$$\Rightarrow w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u+iv}$$

$$\Rightarrow x + iy = \frac{1}{u^2+v^2}$$

$$\Rightarrow x = \frac{1}{u^2+v^2} \qquad \ldots(1)$$

$$\Rightarrow y = \frac{-v}{u^2+v^2} \qquad \ldots(2)$$

We know that, the general equation of circle in z plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots(3)$$

Substitute, (1) and (2) in (3)we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$\Rightarrow c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad \ldots(4)$$

which is the equation of the circle in w plane

Hence, under the transformation $w=\frac{1}{z}$ a circle in z plane transforms to another circle in the w plane. When the circle passes through the origin we have $c=0$ in (3). When $c=0$, equation (4) gives a straight line.

Problems based on $w=\frac{1}{z}$

Example: 3.42 Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$

[Anna – May 1999, May 2001] [A.U N/D 2016 R-18]

Solution:

Given $|z-2i|=2$ .....(1) is a circle.

Centre = (0,2)

radius = 2

Given $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$$(1)\quad \Rightarrow \left|\frac{1}{w}-2i\right|=2$$

$$\Rightarrow |1-2wi|=2|w|$$

$$\Rightarrow |1-2(u+iv)i|=2|u+iv|$$

$$\Rightarrow |1-2ui+2v|=2|u+iv|$$

$$\Rightarrow |1+2v-2ui|=2|u+iv|$$

$$\Rightarrow \sqrt{(1+2v)^2+(-2u)^2}=2\sqrt{u^2+v^2}$$

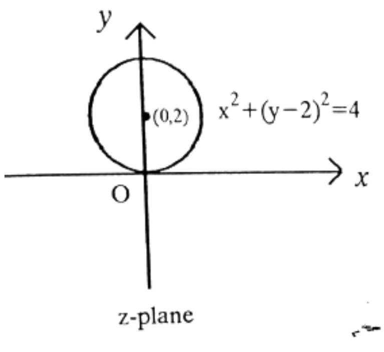
$$\Rightarrow (1+2v)^2+4u^2=4(u^2+v^2)$$

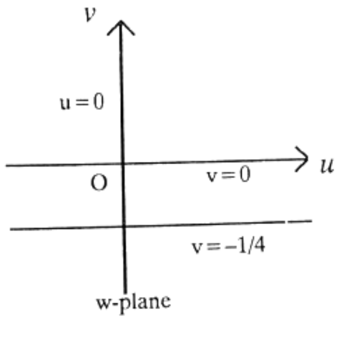
$$\Rightarrow 1+4v^2+4v+4u^2=4(u^2+v^2)$$

$$\Rightarrow 1+4v=0$$

$$\Rightarrow v=-\frac{1}{4}$$

Which is a straight line in w plane.





Example: 3.43 Find the image of the circle $|z-1|=1$ in the complex plane under the mapping $w=\frac{1}{z}$

[A.U N/D 2009] [A.U M/J 2016 R-8]

Solution:

Given $|z-1|=1$ .....(1) is a circle.

Centre =(1,0)

radius = 1

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

(1) $\Rightarrow \left|\frac{1}{w} - 1\right| = 1$

$\Rightarrow |1 - w| = |w|$

$\Rightarrow |1 - (u + iv)| = |u + iv|$

$\Rightarrow |1 - u + iv| = |u + iv|$

$\Rightarrow \sqrt{(1-u)^2 + (-v)^2} = \sqrt{u^2 + v^2}$

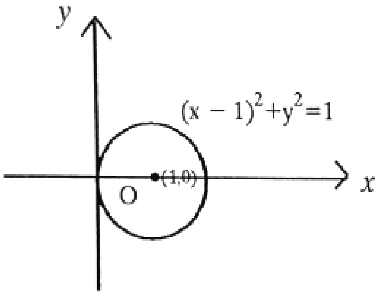
$\Rightarrow (1-u)^2 + v^2 = u^2 + v^2$

$\Rightarrow 1 + u^2 - 2v + v^2 = u^2 + v^2$

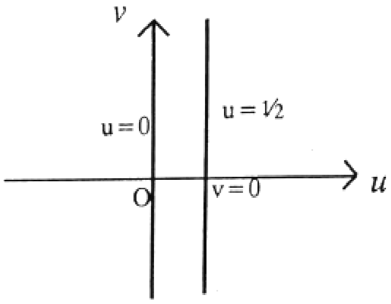
$\Rightarrow \ 2u = 1$

$\Rightarrow \ u = \frac{1}{2}$

which is a straight line in the w- plane



z-plane



w-plane

Example: 3.44 Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$ (ii) $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution :

Given $w = \frac{1}{z}$ (given)

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$x = \frac{u}{u^2+v^2} \ldots.(1), y = \frac{-v}{u^2+v^2} \ldots.(2)$$

(i) Given strip is $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$

$\frac{1}{4} = \frac{-v}{u^2+v^2}$ by (2)

$\Rightarrow \ u^2 + v^2 = -4v$

$$\Rightarrow u^2+v^2+4v=0$$

$$\Rightarrow u^2+(v+2)^2=4$$

which is a circle whose centre is at $(0,-2)$ in the w plane and radius is 2k.

when $y=\frac{1}{2}$

$$\frac{1}{2}=\frac{-v}{u^2+v^2} \qquad \text{by (2)}$$

$$\Rightarrow u^2+v^2=-2v$$

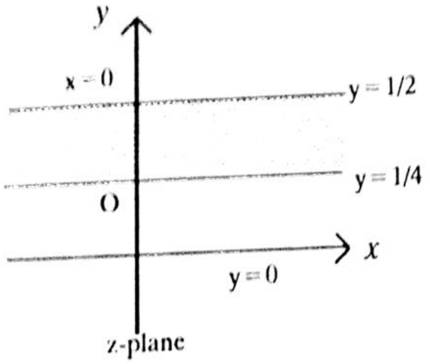
$$\Rightarrow u^2+v^2+2v=0$$

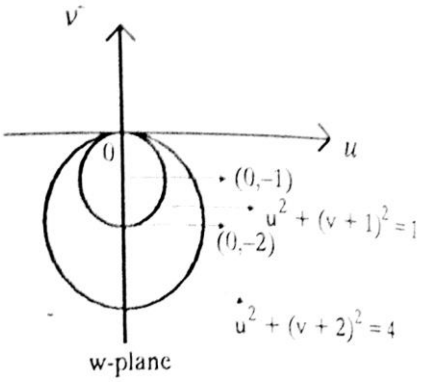
$$\Rightarrow u^2+(v+1)^2=0$$

$$\Rightarrow u^2+(v+1)^2=1 \qquad \ldots\ldots(3)$$

which is a circle whose centre is at $(0,-1)$ in the w plane and unit radius

Hence the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ is transformed into the region in between circles $u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the w plane.





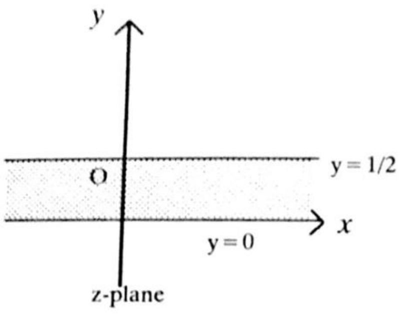
ii) Given strip is $0<y<\frac{1}{2}$

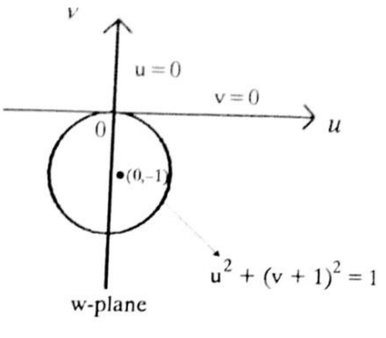
when $y=0$

$\Rightarrow v=0$ by (2)

when $y=\frac{1}{2}$ we get $u^2+(v+1)^2=1$ by (3)

Hence, the infinite strip $0<y<\frac{1}{2}$ is mapped into the region outside the circle $u^2+(v+1)^2=1$ in the lower half of the w plane.





Example: 3.45 Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$. [Anna – May 1998]

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \ldots.(1), y = \frac{-v}{u^2+v^2} \qquad \ldots(2)$$

Given $x = 2$ in the z plane.

$$\therefore 2 = \frac{u}{u^2+v^2} \qquad \text{by } (1)$$

$$2(u^2+v^2) = u$$

$$u^2 + v^2 - \frac{1}{2}u = 0$$

which is a circle whose centre is $\left(\frac{1}{4}, 0\right)$ and radius $\frac{1}{4}$

$\therefore x = 2$ in the z plane is transformed into a circle in the w plane.

Example: 3.46 What will be the image of a circle containing the origin(i.e., circle passing through the origin) in the XY plane under the transformation $w = \frac{1}{z}$? [Anna – May 2002]

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \qquad \ldots(1),$$

$$y = \frac{-v}{u^2+v^2} \qquad \ldots(2)$$

Given region is circle $x^2 + y^2 = a^2$ in z plane.

Substitute, (1) and (2), we get

$$\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] = a^2$$

$$\left[\frac{u^2+v^2}{(u^2+v^2)^2}\right] = a^2$$

$$\frac{1}{(u^2+v^2)} = a^2$$

$$u^2 + v^2 = \frac{1}{a^2}$$

Therefore the image of circle passing through the origin in the XY −plane is a circle passing through the origin in the $w-$ plane.

Example: 3.47 Determine the image of $1 < x < 2$ under the mapping $w = \frac{1}{z}$

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \quad \ldots.(1), \qquad y = \frac{-v}{u^2+v^2} \quad \ldots.(2)$$

Given $1 < x < 2$

When $x = 1$

$$\Rightarrow 1 = \frac{u}{u^2+v^2} \quad \text{by } \ldots.(1)$$

$$\Rightarrow u^2 + v^2 = u$$

$$\Rightarrow u^2 + v^2 - u = 0$$

which is a circle whose centre is $\left(\frac{1}{2}, 0\right)$ and is $\frac{1}{2}$

When $x = 2$

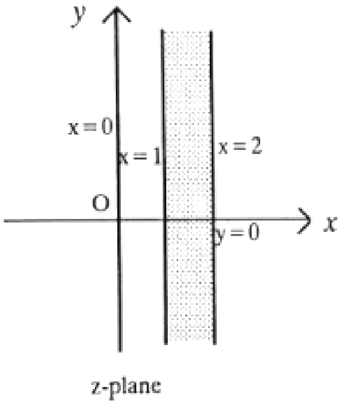
$$\Rightarrow 2 = \frac{u}{u^2+v^2} \quad \text{by } \ldots.(1)$$

$$\Rightarrow u^2 + v^2 = \frac{u}{2}$$

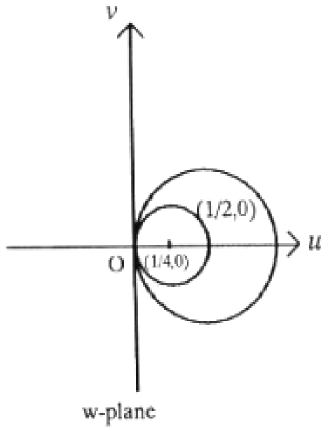
$$\Rightarrow u^2 + v^2 - \frac{u}{2} = 0$$

which is a circle whose centre is $\left(\frac{1}{4}, 0\right)$ and is $\frac{1}{4}$

Hence, the infinite strip $1 < x < 2$ is transformed into the region in between the circles in the $w-$ plane.



z-plane



w-plane

Example: 3.48 Show the transformation $w=\frac{1}{z}$ transforms all circles and straight lines in the $z-$plane into circles or straight lines in the $w-$plane.

[A.U N/D 2007, J/J 2008, N/D 200] [A.U N/D 2016 R-13]

Solution:

Given $w=\frac{1}{z}$

i.e., $z=\frac{1}{w}$

Now, $w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{u+iv+u-iv}=\frac{u-iv}{u^2+v^2}$$

$$\text{i.e.,}\quad x+iy=\frac{u}{u^2+v^2}+i\frac{v}{u^2+v^2}$$

$$x=\frac{u}{u^2+v^2}\quad \ldots.(1),\qquad y=\frac{-v}{u^2+v^2}\quad \ldots(2)$$

The general equation of circle is

$$a(x^2+y^2)+2gx+2fy+c=0\qquad \ldots(3)$$

$$a\left[\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right]+2g\left[\frac{u}{u^2+v^2}\right]+2f\left[\frac{-v}{u^2+v^2}\right]+c=0$$

$$a\frac{(u^2+v^2)}{(u^2+v^2)^2}+2g\frac{u}{u^2+v^2}-2f\frac{v}{u^2+v^2}+c=0$$

The transformed equation is

$$c(u^2+v^2)+2gu-2fv+a=0\qquad \ldots(4)$$

(i) $a\neq 0, c\neq 0$ ⇒ circles not passing through the origin in $z-$plane map into circles not passing through the origin in the $w-$plane.

(ii) $a\neq 0, c=0$ ⇒ circles through the origin in $z-$plane map into straight lines not through the origin in the $w-$plane.

(iii) $a=0, c\neq 0$ ⇒ the straight lines not through the origin in $z-$plane map onto circles through the origin in the $w-$plane.

(iv) $a=0, c=0$ ⇒ straight lines through the origin in $z-$plane map onto straight lines through the origin in the $w-$plane.

Example: 3.49 Find the image of the hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$.

[A.U M/J 2010, M/J 2012]

Solution:

Given $w=\frac{1}{z}$

$$x+iy=\frac{1}{Re^{i\phi}}$$

$$x+iy=\frac{1}{R}e^{-i\phi}=\frac{1}{R}[\cos\phi-i\sin\phi]$$

$$x=\frac{1}{R}\cos\phi,\ y=-\frac{1}{R}\sin\phi$$

Given $x^2-y^2=1$

$$\Rightarrow \left[\frac{1}{R}\cos\phi\right]^2 - \left[\frac{-1}{R}\sin\phi\right]^2 = 1$$

$$\frac{cos^2\phi - sin^2\phi}{R^2} = 1$$

$$\cos 2\phi = R^2 \qquad i.e., R^2 = \cos 2\phi$$

which is lemniscate

4. Transformation $w = z^2$

Problems based on $w = z^2$

Example: 3.50 Discuss the transformation $w = z^2$. [Anna – May 2001]

Solution:

Given $w = z^2$

$$u + iv = (x + iy)^2 = x^2 + (iy)^2 + i2xy = x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \qquad \ldots.(1), \qquad v = 2xy \qquad \ldots.(2)$$

Elimination:

$$(2) \Rightarrow x = \frac{v}{2y}$$

$$(1) \Rightarrow u = \left(\frac{v}{2y}\right)^2 - y^2$$

$$\Rightarrow u = \frac{v^2}{4y^2} - y^2$$

$$\Rightarrow 4uy^2 = v^2 - 4y^4$$

$$\Rightarrow 4uy^2 + 4y^4 = v^2$$

$$\Rightarrow y^2[4u + 4y^2] = v^2$$

$$\Rightarrow 4y^2[u + y^2] = v^2$$

$$\Rightarrow v^2 = 4y^2(y^2 + u)$$

when $y = c\ (\neq 0)$, we get

$$v^2 = 4c^2(u + c^2)$$

which is a parabola whose vertex at $(-c^2, 0)$ and focus at (0,0)

Hence, the lines parallel to X-axis in the z plane is mapped into family of confocal parabolas in the w plane.

when $y = 0$, we get $v^2 = 0$ i.e., $v = 0, u = x^2$ i.e., $u > 0$

Hence, the line $y = 0$, in the z plane are mapped into $v = 0$, in the w plane.

Elimination:

$$(2) \Rightarrow y = \frac{v}{2x}$$

$$(1) \Rightarrow u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow u = x^2 - \frac{v^2}{4x^2}$$

$$\Rightarrow \frac{v^2}{4x^2} = x^2 - u$$

$$\Rightarrow\ v^2 = (4x^2)(x^2-u)$$

when $x = c\ (\neq 0)$, we get $v^2 = 4c^2(c^2-u) = -4c^2(u-c^2)$

which is a parabola whose vertex at $(c^2, 0)$ and focus at $(0,0)$ and axis lies along the u −axis and which is open to the left.

Hence, the lines parallel to y axis in the z plane are mapped into confocal parabolas in the w plane when $x = 0$, we get $v^2 = 0$. i.e., $v = 0, u = -y^2$ i.e., $u < 0$

i.e., the map of the entire y axis in the negative part or the left half of the u −axis.

Example: 3.51 Find the image of the hyperbola $x^2 - y^2 = 10$ under the transformation $w = z^2$ if $w = u + iv$ [Anna – May 1997]

Solution:

Given $w = z^2$

$$u + iv = (x+iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \quad \dots\dots(1)$$

$$v = 2xy \qquad \dots\dots(2)$$

Given $x^2 - y^2 = 10$

i.e., $u = 10$

Hence, the image of the hyperbola $x^2 - y^2 = 10$ in the z plane is mapped into $u = 10$ in the w plane which is a straight line.

Example: 3.52 Determine the region of the w plane into which the circle $|z-1| = 1$ is mapped by the transformation $w = z^2$.

Solution:

In polar form $z = re^{i\theta}, w = Re^{i\phi}$

Given $|z-1| = 1$

i.e., $|re^{i\theta} - 1| = 1$

$$\Rightarrow |r\cos\theta + i\, r\sin\theta| = 1$$

$$\Rightarrow |(r\cos\theta - 1) + i\, r\sin\theta| = 1$$

$$\Rightarrow (r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1^2$$

$$\Rightarrow r^2\cos^2\theta + 1 - 2\, r\cos\theta + r^2 sin^2\theta = 1$$

$$\Rightarrow r^2[\cos^2\theta + sin^2\theta = 2r\cos\theta$$

$$\Rightarrow r^2 = 2r\cos\theta$$

$$\Rightarrow\ r = 2\cos\theta \quad \dots(1)$$

Given $w = z^2$

$$Re^{i\phi} = (re^{i\theta})^2$$

$$Re^{i\phi} = r^2 e^{i2\theta}$$

$$\Rightarrow R = r^2, \qquad \phi = 2\theta$$

$$(1) \qquad \Rightarrow r^2 = (2\cos\theta)^2$$

$$\Rightarrow r^2 = 4cos\,^2\theta$$

$$= 4\left[\frac{1+\cos 2\theta}{2}\right]$$

$$r^2 = 2[1+\cos 2\,\theta]$$

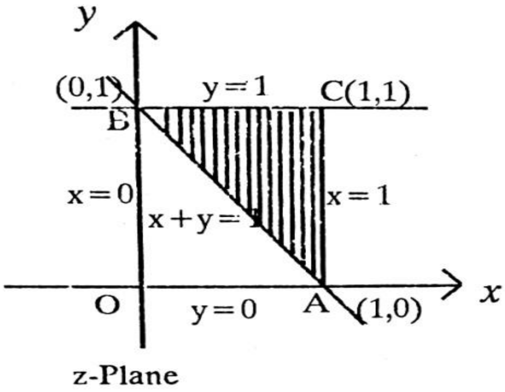
$$R = 2[1+\cos\phi] \qquad by\ (2),$$

which is a Cardioid

Example: 3.53 Find the image under the mapping $w = z^2$ of the triangular region bounded by $y = 1$, $x = 1$, and $x + y = 1$ and plot the same. [Anna, Oct., - 1997]

Solution :

In Z-plane given lines are $y = 1$, $x = 1$, $x + y = 1$



Given $w = z^2$

$$u + iv = (x+iy)^2$$

$$u + iv = x^2 - y^2 + 2xyi$$

Equating the real and imaginary parts, we get

$$u = x^2 - y^2 \quad \ldots.(1)$$

$$v = 2xy \qquad \ldots(2)$$

When $x = 1$	When $y = 1$
$(1) \Rightarrow u = 1 - y^2 \quad \ldots.(3)$ $(2) \Rightarrow v = 2y \quad \ldots(4)$	$(1) \Rightarrow u = x^2 - 1 \quad \ldots.(5)$ $(2) \Rightarrow v = 2x \quad \ldots(6)$
$(4) \Rightarrow v^2 = 4y^2$ $v^2 = 4(1-u)\ by\ (3)$ i.e., $v^2 = -4(u-1)$	$(6) \Rightarrow v^2 = 4x^2$ $= 4(u+1) by\ (5)$

when $x + y = 1$

$$(1) \Rightarrow u = (x+y)(x-y)$$

$$u = x - y \qquad [\because x + y = 1]$$

$$u = \sqrt{(x+y)^2 - 4xy}$$

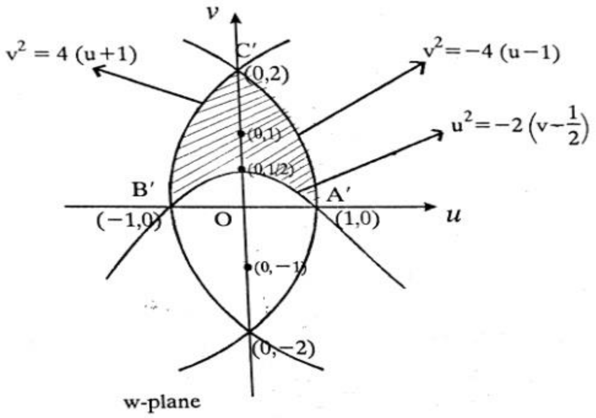
$$u = \sqrt{1-2v}$$

$$u^2 = 1 - 2v \; = -2\left(v - \tfrac{1}{2}\right)$$

$\therefore$ The image of $x = 1$ is $v^2 = -4(u-1)$

The image of $y = 1$ is $v^2 = 4(u+1)$

The image of $x + y = 1$ is $u^2 = -2\left(v - \frac{1}{2}\right)$



$v^2 = -4(u-1)$		
u	0	1
v	± 2	0

$v^2 = 4(u+1)$		
u	0	-1
v	± 2	0

$u^2 = -2\left(v - \frac{1}{2}\right)$			
u	0	1	-1
v	1/2	0	0

Problems based on critical points of the transformation

Example: 3.54 Find the critical points of the transformation $w^2 = (z-\alpha)\,(z-\beta)$.

[A.U Oct., 1997] [A.U N/D 2014] [A.U M/J 2016 R-13]

Solution:

Given $w^2 = (z-\alpha)\,(z-\beta)$ ...(1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow 2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$= 2z - (\alpha+\beta)$$

$$\Rightarrow \frac{dw}{dz} = \frac{2z-(\alpha+\beta)}{2w} \qquad \ldots(2)$$

Case $(i)\,\frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z-(\alpha+\beta)}{2w} = 0$$

$$\Rightarrow 2z - (\alpha+\beta) = 0$$

$$\Rightarrow 2z = \alpha+\beta$$

$$\Rightarrow z = \frac{\alpha+\beta}{2}$$

Case $(ii)\,\frac{dz}{dw} = 0$

$$\Rightarrow \frac{2w}{2z-(\alpha+\beta)} = 0$$

$$\Rightarrow \frac{w}{z-\frac{\alpha+\beta}{2}} = 0$$

$$\Rightarrow w = 0 \Rightarrow (z-\alpha)(z-\beta) = 0$$

$$\Rightarrow z = \alpha, \beta$$

∴ The critical points are $\frac{\alpha+\beta}{2}, \alpha$ and β.

Example: 3.55 Find the critical points of the transformation $w = z^2 + \frac{1}{z^2}$. [A.U A/M 2017 R-13]

Solution:

Given $w = z^2 + \frac{1}{z^2} \qquad \ldots(1)$

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = 2z - \frac{2}{z^3} = \frac{2z^4-2}{z^3}$$

Case $(i)\,\frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z^4-2}{z^3} = 0 \Rightarrow 2z^4 - 2 = 0$$

$$\Rightarrow z^4 - 1 = 0$$

$$\Rightarrow z = \pm 1, \pm i$$

Case $(ii)\,\frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{2z^4-2} = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0$$

∴ The critical points are ± 1 , $\pm i$, 0

Example: 3.56 Find the critical points of the transformation $w = z + \frac{1}{z}$

Solution:

Given $w = z + \frac{1}{z}$ ...(1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

Case $(i)\ \frac{dw}{dz} = 0$

$$\Rightarrow \frac{z^2-1}{z^3} = 0 \Rightarrow z^2 - 1 = 0 \Rightarrow z = \pm 1$$

Case $(ii)\ \frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{z^2-1} = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0$$

∴ The critical points are $0, \pm 1$.

Example: 3.57 Find the critical points of the transformation $w = 1 + \frac{2}{z}$. [A.U N/D 2013 R-08]

Solution:

Given $w = 1 + \frac{2}{z}$ ... (1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = \frac{-2}{z^2}$$

Case $(i)\ \frac{dw}{dz} = 0$

$$\Rightarrow \frac{-2}{z^2} = 0$$

Case $(ii)\ \frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^2}{2} = 0 \Rightarrow z = 0$$

∴ The critical points is $z = 0$

Example: 3.58 Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of the z plane into the upper half of the w plane. What is the image of the circle $|z| = 1$ under this transformation.

[Anna, May – 2001]

Solution:

Given $|z| = 1$ is a circle

Centre = (0,0)

Radius = 1

Given $w = \frac{z}{1-z}$

$$\Rightarrow z = \frac{w}{w+1}$$

$$\Rightarrow |z| = \left|\frac{w}{w+1}\right| = \frac{|w|}{|w+1|}$$

Given $|z| = 1$

$$\Rightarrow \frac{|w|}{|w+1|} = 1$$

$$\Rightarrow |w| = |w+1|$$

$$\Rightarrow |u+iv| = |u+iv+1|$$

$$\Rightarrow \sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$$

$$\Rightarrow u^2+v^2 = (u+1)^2+v^2$$

$$\Rightarrow u^2+v^2 = u^2+2u+1+v^2$$

$$\Rightarrow 0 = 2u+1$$

$$\Rightarrow u = \frac{-1}{2}$$

Further the region $|z| < 1$ transforms into $u > \frac{-1}{2}$

Exercise: 3.5

1. Define Critical point of a transformation.
2. Find the image the circle $|z| = a$ under the following transformations.
   (i) $w = z + 2 + 3i$
   (ii) $w = 2z$ [A.U N/D 2016 R-13]
3. Find the image of the circle $|z+1| = 1$ in the complex plane under the mapping $= \frac{1}{z}$.
4. Find the image of $|z-3i| = 3$ under the mapping $w = \frac{1}{z}$
5. Consider the transformation $w = 3z$, corresponding to the region R of $z-$ plane bounded by $x = 0, y = 0, x + y = 2$.
6. Verify the transformation $w = 1 + \frac{iz}{1+z}$ maps the exterior of the circle $|z| = 1$ into the upper half plane $v > 0$.
7. Find the image of $|z-2i| = 3$ under $w = \frac{1}{z}$
   (i) the circle $|z-2i| = 2$
   (ii) the strip $1 < x < 2$
8. Show that the transformation $w = \frac{iz+1}{z+i}$ tranforms the exterior and interior regions of the circle $|z| = 1$ into the upper and lower half of the w plane respectively.
9. Show that $w = \frac{z-i}{z+i}$ maps the real axis in the z plane onto $|w| = 1$ in the w plane. Show also that the upper half of the z plane, $Im(z) \geq 0$, goes onto the circular disc $|w| \leq 1$.
10. Prove that $w = \frac{1+iz}{i+z}$ maps the line segment joining -1 and 1 onto a semi circle in the w plane.
11. Show that the transformation $w = \frac{z-i}{z+i}$ maps the circular disc $|z| \leq 1$ onto the lower half of the w plane.

12. Prove that $w = \frac{z}{1-z}$ maps the upper half of the z plane onto the upper half of the w plane. What is the image of the circle $|z| = 1$ under this transformation.
13. Show that the transformation $w = \frac{i-z}{i+z}$ maps the circle $|z| = 1$ onto the imaginary axis of the w plane. Find also the images of the interior and exterior of the circle.
14. Plot the image under the mapping $w = z^2$ of the rectangular region bounded by
    (i) $x = -1, x = 2, y = 1 \; and \; y = 2.$
    (ii) $x = 1, \quad x = 3, y = 1 \; and \; y = 2.$
    (iii) $u = 1, \quad u = 3, v = 1 \; and \; v = 2.$
15. Under the mapping $w = e^z$ discuss the transforms of the lines.
    (i) $y = 0$, (ii) $y = \frac{\pi}{2}$, (iii) $y = 2\pi$.

3.6. BILINEAR TRANSFORMATION

- **3.5.a. Introduction**

The transformation $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

This transformation was first introduced by A.F. Mobius, So it is also called Mobius transformation.

A bilinear transformation is also called a linear fractional transformation because $\frac{az+b}{cz+d}$ is a fraction formed by the linear fuctions $az - b$ and $cz + d$.

Theorem: 1 Under a bilinear transformation no two points in z plane go to the same point in w plane.

Proof:

Suppose z_1 and z_2 go to the same point in the w plane under the transformation $w = \frac{az+b}{cz+d}$.

Then $\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$

$$\Rightarrow (az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d)$$

$$i.e., \quad (az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d) = 0$$

$$\Rightarrow acz_1 z_2 + adz_1 + bcz_2 + bd - acz_1 z_2 - adz_2 - bcz_1 - bd = 0$$

$$\Rightarrow (ad - bc)(z_1 - z_2) = 0$$

$$\text{or} \quad z_1 = z_2 \qquad [\because ad - bc \neq 0]$$

This implies that no two distinct points in the z plane go to the same point in w plane. So, each point in the z plane go to a unique point in the w plane.

Theorem: 2 The bilinear transformation which transforms z_1, z_2, z_3, into w_1, w_2, w_3 is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Proof:

If the required transformation $w = \frac{az+b}{cz+d}$.

$$\Rightarrow w - w_1 = \frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d} = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

$$\Rightarrow\ (cz+d)(cz_1+d)(w-w_1) = (ad-bc)(z-z_1)$$

$$\Rightarrow (cz_2+d)(cz_3+d)\ (w_2-w_3) = (ad-bc)(z_2-z_3)$$

$$\Rightarrow (cz+d)\ (cz_3+d)(w-w_3) = (ad-bc)(z-z_3)$$

$$\Rightarrow (cz_2+d)(cz_1+d)(w_2-w_1) = (ad-bc)(z_2-z_1)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{\left[\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\right]\left[\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}\right]}{\left[\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}\right]\left[\frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}\right]}$$

$$= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Now, $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ ... (1)

Let : $A = \frac{w_2-w_3}{w_2-w_1}, B = \frac{z_2-z_3}{z_2-z_1}$

$$(1) \Rightarrow \frac{w-w_1}{w-w_3}A = \frac{z-z_1}{z-z_3}B$$

$$\frac{wA-w_1A}{w-w_3} = \frac{zB-z_1B}{z-z_3}$$

$$\Rightarrow wAz - wAz_3 - w_1Az + w_1Az_3 = wBz - wz_1B - w_3zB + w_3z_1B$$

$$\Rightarrow w[(A-B)z + (Bz_1 - Az_3)] = (Aw_1 - Bw_3)z + (Bw_3z_1 - Aw_1z_3)$$

$$\Rightarrow w = \frac{(Aw_1-Bw_3)z + (Bw_3z_1 - Aw_1z_3)}{(A-B)z + (Bz_1 - Az_3)}$$

$\frac{az+b}{cz+d}$, Hence a $= Aw_1 - Bw_3, b = Bw_3z_1 - Aw_1z_3\ , c = A - B, d = Bz_1 - Az_3$

Cross ratio

Definition:

Given four point z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z-z_1)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

Note: (1) $w = \frac{az+b}{cz+d}$ can be expressed as $cwz + dw - (az+b) = 0$

It is linear both in $w\ and\ z$ that is why, it is called bilinear.

Note: (2) This transformation is conformal only when $\frac{dw}{dz} \neq 0$

$$i.e., \frac{ad-bc}{(cz+d)^2} \neq 0$$

$$i.e., ad - bc \neq 0$$

If $ad - bc \neq 0$, every point in the z plane is a critical point.

Note: (3) Now, the inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

Note: (4) Each point in the plane except $z = \frac{-d}{c}$ corresponds to a unique point in the w plane.

The point $z = \frac{-d}{c}$ corresponds to the point at infinity in the w plane.

Note: (5) The cross ratio of four points

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$ is invariant under bilinear transformation.

Note: (6) If one of the points is the point at infinity the quotient of those difference which involve this points is replaced by 1.

Suppose $z_1 = \infty$, then we replace $\frac{z - z_1}{z_2 - z_1}$ by 1 (or)Omit the factors involving ∞

Example: 3.59 Find the fixed points of $w = \frac{2zi+5}{z-4i}$.

Solution:

The fixed points are given by replacing w by z

$$z = \frac{2zi+5}{z-4i}$$

$$z^2 - 4iz = 2zi + 5\ ;\ z^2 - 6iz - 5 = 0$$

$$z = \frac{6i \pm \sqrt{-36+20}}{2} \qquad \therefore z = 5i, i$$

Example: 3.60 Find the invariant points of $w = \frac{1+z}{1-z}$

Solution:

The invariant points are given by replacing w by z

$$z = \frac{1+z}{1-z}$$

$$\Rightarrow z - z^2 = 1 + z$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

Example: 3.61 Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$. [Anna, May 1996]

Solution:

The invariant points are given by

$$z = 2 - \frac{2}{z}\ ; \qquad z = \frac{2z-2}{z}$$

$$z^2 = 2z - 2; \qquad z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Example: 3.62 Find the fixed point of the transformation $w = \frac{6z-9}{z}$. [A.U N/D 2005]

Solution:

The fixed points are given by replacing $w = z$

i.e., $w = \frac{6z-9}{z} \Rightarrow z = \frac{6z-9}{z}$

$$\Rightarrow z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)^2 = 0$$

$$\Rightarrow z = 3,3$$

The fixed points are $3, 3$.

Example: 3.63 Find the invariant points of the transformation $w = \frac{2z+6}{z+7}$. [A.U M/J 2009]

Solution:

The invariant (fixed) points are given by

$$w = \frac{2z+6}{z+7}$$

$$\Rightarrow z^2 + 7z = 2z + 6$$

$$\Rightarrow z^2 + 5z - 6 = 0$$

$$\Rightarrow (z+6)(z-1) = 0$$

$$\Rightarrow z = -6, z = 1$$

Example: 3.64 Find the invariant points of$f(z) = z^2$. [A.U M/J 2014 R-13]

Solution:

The invariant points are given by $z = w = f(z)$

$$\Rightarrow z = z^2$$

$$\Rightarrow z^2 - z = 0$$

$$\Rightarrow z(z-1) = 0$$

$$\Rightarrow z = 0, \quad z = 1$$

Example 3.65 Find the invariant points of a function $f(z) = \frac{z^3+7z}{7-6zi}$. [A.U D15/J16 R-13]

Solution:

Given $w = f(z) = \frac{z^3+7z}{7-6zi}$

The invariant points are given by

$$\Rightarrow z = \frac{z^3+7z}{7-6zi}$$

$$\Rightarrow 7 - 6zi = z^2 + 7$$

$$\Rightarrow -6zi = z^2 \Rightarrow z^2 + 6zi = 0 \Rightarrow z(z+6i) = 0$$

$$\Rightarrow z = 0, \; z = -6i$$

PROBLEMS BASED ON BILINEAR TRANSFORMATION

Example: 3.66 Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively. [A.U. A/M 2015 R-13, A.U N/D 2013, N/D 2014]

Solution:

Given $z_1 = 0, \; z_2 = -1, \; z_3 = i,$

$w_1 = i, \; w_2 = 0, \; w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{z}{(z-i)}(1+i)$$

$$\Rightarrow w-i = \frac{z}{(z-i)}(-i+1)$$

$$\Rightarrow w = \frac{z}{(z-i)}(-i+1)+i = \frac{-iz+z+iz+1}{(z-i)} = \frac{z+1}{z-i}$$

Aliter: Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = i,$

$w_1 = i,\ \ w_2 = 0,\ \ \ w_3 = \infty,$

Let the required transformation be

$$w = \frac{az+b}{cz+d} \ldots (1),\ ad-bc \neq 0$$

$$i = \frac{b}{d}$$

$w_1 = \frac{az_1+b}{cz_1+d}$	$w_2 = \frac{az_2+b}{cz_2+d}$	$w_3 = \frac{az_3+b}{cz_3+d}$
$i = \frac{b}{d}$	$0 = \frac{-a+b}{-c+d}$	$\frac{1}{0} = \frac{ai+b}{ci+d}$
$b = di$	$\Rightarrow -a+b = 0$	$\Rightarrow ci+d = 0$
	$\Rightarrow a = b$	$\Rightarrow d = -ci$

$$\therefore a = b = di = c$$

$$\therefore (1) \Rightarrow w = \frac{az+a}{az+\frac{a}{i}} = \frac{z+1}{z+\frac{1}{i}} = \frac{z+1}{z-i}$$

Example: 3.67 Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

[Anna, May 1997] [A.U N/D 2012] [A.U A/M 2017 R-08]

Solution:

Given $z_1 = \infty,\ \ z_2 = i,\ \ z_3 = 0, w_1 = 0,\ \ w_2 = i,\ \ \ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving z_1, and w_3, since $z_1 = \infty, w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z_2-z_3)}{z-z_3}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow w = \frac{-1}{z}$$

Example: 3.68 Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$, show that the transformation maps the interior of the unit circle of the $z-$ plane onto the upper half of the $w-$ plane. [A.U. May 2001] [A.U M/J 2014] [A.U D15/J16 R-13]

Solution:

Given $z_1 = 1,\ z_2 = i,\ z_3 = -1$

$w_1 = 0,\ w_2 = 1,\ w_3 = \infty,$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Omit the factors involving w_3, since , $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \qquad \because \left[\left(\frac{i+1}{i-1}\right)\left(\frac{i+1}{i+1}\right)\right] = \left[\frac{i^2+i+i+1}{i^2-i^2}\right] = \left[\frac{2i}{-2}\right] = -i$$

$$\Rightarrow w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{z-1}{z+1}[-i]$$

$$\Rightarrow w = \frac{(-i)z+i}{(1)z+1}\left[\because w = \frac{az+b}{cz+d}, ad - bc \neq 0 \text{ Form}\right]$$

To find z:

$$\Rightarrow wz + w = -iz + i$$

$$\Rightarrow wz + iz = -w + i$$

$$\Rightarrow z[w + i] = -w + i$$

$$\Rightarrow z = \frac{(w-i)}{w+i}$$

To prove: $|z| < 1$ maps $v > 0$

$$\Rightarrow |z| < 1$$

$$\Rightarrow \left|\frac{-(w-i)}{w+i}\right| < 1$$

$$\Rightarrow \left|\frac{w-i}{w+i}\right| < 1$$

$$\Rightarrow |w - i| < |w + i|$$

$$\Rightarrow |u + iv - i| < |u + iv + i|$$

$$\Rightarrow |u + i(v - 1)| < |u + i(v + i)|$$

$$\Rightarrow u^2 + (v-1)^2 < u^2 + (v+1)^2$$

$$\Rightarrow (v-1)^2 < (v+1)^2$$

$$\Rightarrow \ v^2 - 2v + 1 < v^2 + 2v + 1$$

$$\Rightarrow \ -4v < 0$$

$$\Rightarrow v > 0$$

Example: 3.69 Determine the bilinear transformation that maps the points $-1, 0, 1$, in the z plane onto the points $0, i, 3i$ in the w plane. [Anna, May 1999]

Solution:

Given $z_1 = -1, \ z_2 = 0, \ z_3 = 1,$

$w_1 = 0, \ w_2 = i, \ w_3 = 3i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{[z-(-1)][0-1]}{(z-1)[0-(-1)]}$$

$$\Rightarrow \frac{w(-2i)}{(w-3i)(i)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\Rightarrow \frac{-2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow \frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow 2wz - 2w = wz + w - 3zi - 3i$$

$$\Rightarrow 2wz - 2w - wz - w = -3i(z+1)$$

$$\Rightarrow w[2z - 2 - z - 1] = -3i(z+1)$$

$$\Rightarrow w[z-3] = -3i(z+1)$$

$$\Rightarrow w = -3i\frac{(z+1)}{(z-3)}$$

Note: Either image or object or both are infinity should not apply the following Aliter method.

Aliter:

Given $z_1 = -1, \ z_2 = 0, \ z_3 = 1,$

$w_1 = 0, \ w_2 = i, \ w_3 = 3i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Let $A = \frac{w_2-w_3}{w_2-w_1} = \frac{i-3i}{i-0} = \frac{-2i}{i} = -2$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{0-1}{0+1} = -1$$

$$\Rightarrow a = Aw_1 - Bw_3 = 0 + 3i = 3i$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-1)(3i)(-1) - 0 = 3i$$

$$\Rightarrow c = A - B = (-2) - (-1) = -1$$

$$\Rightarrow d = Bz_1 - Az_3 = (-1)(-1) - (-2)(1) = 3$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(3i)+z(3i)}{(-1)z+3}$$

Example: 3.70 Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, 1, -i$ respectively. [Anna, May 2002]

Solution:

Given $z_1 = -1, \quad z_2 = 0, \quad z_3 = 2,$

$w_1 = 0, \quad w_2 = i, \quad w_3 = -i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{Let } A = \frac{w_2-w_3}{w_2-w_1} = \frac{i+i}{i-0} = \frac{2i}{i} = 2$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{0-2}{0+2} = -1$$

$$\Rightarrow a = Aw_1 - Bw_3 = (2)(0) - (-1)(-1) = -i$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-1)(-i)(-2) - (2)(0)(2) = -2i$$

$$\Rightarrow c = A - B = 2 - (-1) = 3$$

$$\Rightarrow d = Bz_1 - Az_3 = (-1)(-1) - (2)(2) = -2$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i)z+(-2i)}{3z+(-2)}$$

Example: 3.71 Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. Hence find the fixed points. [A.U, May 2001] [A.U April 2016 R-15 U.D]

Solution:

Given $z_1 = 1, \quad z_2 = i, \quad z_3 = -1,$

$w_1 = i, \quad w_2 = 0, \quad w_3 = -i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{0+i}{0-i} = -1$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{i+1}{i-1} = -i$$

$$\Rightarrow a = Aw_1 - Bw_3 = (-1)(i) - (-i)(-i) = -i + 1$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-i)(-i)(1) - (-1)(i)(-1) = -1 - i$$

$$\Rightarrow c = A - B = (-1) - (-i) = -1 + i$$

$$\Rightarrow d = Bz_1 - Az_3 = (-i)(1) - (-1)(-1) = -i - 1$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(-1-i)}{(-1+i)z+(-i-1)} = \frac{iz+1}{(-i)z+1}$$

Example: 3.72 Find the bilinear transformation which maps $z = 0$ onto $w = -i$ and has -1 and 1 as the invariant points. Also show that under this transformation the upper half of the z plane maps onto the interior of the unit circle in the w plane. [A.U A/M 2017 R-13]

Solution:

Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = 1,$

$w_1 = -i,\ \ w_2 = -1,\ \ w_3 = 1,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{-1-1}{-1+i} = \frac{-2}{-1+i} = 1+i$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{-1-1}{-1-0} = 2$$

$$\Rightarrow a = Aw_1 - Bw_3 = (1+i)(-i) - 2(1) = -i + 1 - 2 = -i - 1$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (2)(1)(0) - (1+i)(-i)(1) = i - 1$$

$$\Rightarrow c = A - B = (1+i) - 2 = i - 1$$

$$\Rightarrow d = Bz_1 - Az_3 = (2)(0) - (1+i)(1) = -(1+i)$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(i-1)}{(i-1)z+(-1-i)} = \frac{z+(-i)}{(-i)z+1}$$

We know that, $z = \frac{-dw+b}{cw-a} = \frac{-w-i}{-iw-1} = \frac{w+i}{1+wi}$

$$z = \frac{u+iv+i}{1(u+iv)i}$$

$$= \frac{u+iv+i}{1+iu-v} = \frac{u+iv+i}{(1-v)+iu}$$

$$= \left[\frac{u+iv+i}{(1-v)+iu}\right]\left[\frac{1-v-iu}{(1-v)-iu}\right]$$

$$= \frac{u-uv-iu^2+iv-iv^2+uv+i-iv+u}{(1-v)^2+u^2}$$

$$x + iy = \frac{2u+i[-u^2-v^2+1]}{(1-v)^2+u^2}$$

$$\Rightarrow \quad y \quad = \frac{1-u^2-v^2}{(1-v)^2+u^2}$$

Upper half of the z −plane

$$\Rightarrow y \geq 0$$

$$\Rightarrow \frac{1-u^2-v^2}{(1-v)^2+u^2} \geq 0$$

$$\Rightarrow 1 - u^2 - v^2 \geq 0$$

$$\Rightarrow 1 \geq u^2 + v^2$$

$$\Rightarrow u^2 + v^2 \leq 1$$

Therefore the upper half of the z −plane maps onto the interior of the unit circles in the w-plane.

Example: 3.73 Find the Bilinear transformation that maps the points $1+i, -i, 2-i$ of the z −plane into the points $0, 1, i$ of the w-plane. [A.U M/J 2007, N/D 2007]

Solution:

$$\text{Given}\quad \begin{array}{r|c} z_1 = 1+i & w_1 = 0 \\ z_2 = -i & w_2 = 1 \\ z_3 = 2-i & w_3 = i \end{array}$$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{1-i}{1-0} = 1-i = \frac{1-i}{1+2i}(1+2i) = \frac{3+i}{1+2i}$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{-i-2+i}{-i-1-i} = \frac{-2}{-1-2i} = \frac{2}{1+2i}$$

$$\Rightarrow a = Aw_1 - Bw_3 = \left(\frac{3+i}{1+2i}\right)(0) - \left(\frac{2}{1+2i}\right)(i) = \frac{-2i}{1+2i}$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = \left(\frac{2}{1+2i}\right)(i)(1+i) - 0 = \frac{-2+2i}{1+2i}$$

$$\Rightarrow c = A - B = \frac{3+i}{1+2i} - \frac{2}{1+2i} = \frac{1+i}{1+2i}$$

$$\Rightarrow d = Bz_1 - Az_3 = \left(\frac{2}{1+2i}\right)(1+i) - \left(\frac{3+i}{1+2i}\right)(2-i) = \frac{-5+3i}{1+2i}$$

We know that, $w = \frac{az+b}{cz+d}, ad-bc \neq 0$

$$\Rightarrow \; w = \frac{\left(\frac{-2i}{1+2i}\right)z + \left(\frac{2i-2}{1+2i}\right)}{\left(\frac{1+i}{1+2i}\right)z + \left(\frac{3i-5}{1+2i}\right)}$$

$$\Rightarrow w = \frac{(-2i)z + (2i-2)}{(1+i)z + (3i-5)}$$

Verification:

(i) If $z = 1+i$,then

$$w = \frac{(-2i)(1+i)+(2i-2)}{(1+i)(1+i)+(3i-5)}$$

$$= \frac{-2i+2+2i-2}{(1+i)(1+i)+(3i-5)} \quad = 0$$

(ii) If $z = -i$, then

$$w = \frac{(-2i)(-i)+(2i-2)}{(1+i)(-i)+(3i-5)}$$

$$= \frac{-2+2i-2}{-i+1+3i-5} = \frac{2i-4}{2i-4} = 1$$

(iii)If $z = -i$, then

$$w = \frac{(-2i)(2-i)+(2i-2)}{(1+i)(2-i)+(3i-5)} = \frac{-4i-2+2i-2}{2-i+2i+1+3i-5}$$

$$= \frac{-2i-4}{4i-2} = \frac{-i-2}{2i-1} \times \frac{2i+1}{2i+1}$$

$$= \frac{2-i-4i-2}{-4-1} = \frac{-5i}{-5} = i$$

Exercise: 3.6

1. Find the fixed points of the following mappings

   (i) $w = \frac{2z-5}{z+4}$ Ans. $z = -1 \pm 2i$

   (ii) $w = \frac{z-2}{z+3}$ Ans. $z = -1 \pm i$

   (iii) $w = \frac{1}{z-2i}$ Ans. $z = i$

   (iv) $w = \frac{5z+4}{z+5}$ Ans. $z = \pm 2$

2. Define bilinear transformation.
3. Find the most general bilinear transformation that maps the upper half of the z −plane onto the interior of the unit circle in the w-plane.
4. Find the bilinear transformation for the following

   (1) $-i, 0, i$; $-1, i, 1$ **Ans:** $w = -i\left(\frac{z-1}{z+1}\right)$

   (2) $1, -1, \infty$; $1+i, 1-i, 1$ **Ans:** $w = \frac{z+i}{z}$

   (3) $0, 1, \infty$; $i, 1-i$ **Ans:** $w = \frac{z+i}{1+zi}$

   (4) $1, i, -1$; $2, i, -2$ **Ans:** $w = -\left(\frac{6z-2i}{iz-3}\right)$

   (5) $0, 1, \infty$; $-5, -1, 3$ **Ans:** $w = \frac{3z-5}{z+1}$

   (6) $\infty, i, 0$; $0, -i, \infty$ **Ans:** $w = \frac{1}{z}$

   (7) $-i, 0, i$; $\infty, -1, 0$ **Ans:** $w = \frac{z-1}{z+1}$

   (8) $0, 1, \infty$; $i, -1, -i$ **Ans:** $w = -i\left(\frac{z+i}{z-i}\right)$

   (9) $0, 1, -1$; $-1, 0, \infty$ **Ans:** $w = \frac{z-1}{z+1}$