ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY
Near Anjugramam Junction, Kanyakumari Main Road, Palkulam, Variyoor P.O - 629401
Kanyakumari Dist, Tamilnadu., E-mail : admin@rcet.org.in, Website : www.rcet.org.in

# ENGINEERING MATHEMATICS -II

***SUBJECT CODE: MA8251***

***(Regulation 2017)***
***Common to all branches of B.E***

## UNIT – I

## MATRICES

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# MATRICES

## 1.1 INTRODUCTION

The definition of matrices and the basic operations related to them originated in a memoir of Cayley in 1958.This grew out of a simple observation on bilinear transformation and theory of Algebraic invariants .Subsequently a number of eminent mathematicians like Sylvester, Hamilton contributed to the development of the theory. Heisenberg used matrices in quantum theory as early as1925 AD. The matrix algebra is a kind of shorthand technique for translating complicated mathematical relationship in to compact forms .It has high degree of applicability in most of the quantitative applied sciences and has become indispensable tool. In Physical, Social and Biological Sciences, application of matrix algebra has become highly wide –spread. In this chapter we shall discuss and solve the Eigen value problem $AX = \lambda X$

## Definition

An arrangement of mn elements in a rectangular form having an ordered set of m rows and n columns is called an m × n matrix A

$$A = \left[\begin{pmatrix} a_{11}\, a_{12}\, a_{13} & \cdots & a_{1n} \\ a_{21} a_{22}\, a_{23} & \ddots & a_{2n} \\ \ldots & \cdots & \ldots . \end{pmatrix}\right]$$

In short $A=[a_{ij}]$ ,$i = 1,2, \ldots \ldots m$

$j = 1,2, \ldots \ldots n$

Here each $a_{ij}$ is called an element of the matrix in, i th row and j th column.

## Special Types of Matrices

Let A be Square Matrix of order n,

1. $A = A^T$,then it is called a symmetric matrix $(i.e)$ $a_{ij} = a_{ji}$ for all $i, j$.
2. If the square matrix $A = -A^T$, then it is called skew –symmetric matrix
$(i.e)$ $a_{ij} = -a_{ji}$ for all $i, j$.
3. An $m \times 1$ matrix is called a column matrix.
4. An $1 \times n$ matrix is called a row matrix.
5. In the square matrix $A=(a_{ij})$ the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the diagonal elements of A and the diagonal elements constitute the principal diagonal of the matrix A .
6. A square matrix is called a diagonal matrix if all the elements except the leading diagonal are zero $(i.e)$ In a diagonal matrix $a_{ij} = 0$ if $i \neq j$.
7. A square matrix $A=(a_{ij})$ is called an upper triangular matrix if all the entries below the leading diagonal are zero.
8. A square matrix $A=(a_{ij})$ is called an lower triangular matrix if all the entries above the leading diagonal are zero.

9. A square matrix A is said to be singular if $|A| = 0$, otherwise it is a non –singular matrix.

## 1.2 CHARACTERISTIC EQUATION

If A is any square matrix of order n, the matrix $A - \lambda I$ where I is the unit matrix and $\lambda$ be scaler of order n can be formed as

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$ is called the characteristic equation of A.

## Working Rule for Characteristic Equation

**Type I:** For $2 \times 2$ matrix

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then the characteristic equation of A is $\lambda^2 - s_1\lambda + s_2 = 0$

Where $s_1$ =Sum of the leading diagonal elements $= a_{11} + a_{22}$

$s_2 = |A|$ =Determinant of a matrix A.

**Type II:** For $3 \times 3$ matrix

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then the characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

Where $s_1$ =Sum of the leading diagonal elements $= a_{11} + a_{22} + a_{33}$

$s_2$ =Sum of minors of leading diagonal elements

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$s_3 = |A|$ = Determinant of a matrix A.

**Example: 1.1 Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$**

**Solution:**

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

$s_1 =$ sum of the main diagonal element

$= 1 + 2 = 3$

$$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$$

Characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

**Example: 1.2 Find the characteristic equation of the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$**

**Solution:**

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

$s_1 =$ sum of the main diagonal element

$= 1 + 4 = 5$

$$s_2 = |A| = \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 4 - 10 = -6$$

Characteristic equation is $\lambda^2 - 5\lambda - 6 = 0$

**Example: 1.3 Find the characteristic equation of the matrix** $\begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = \text{sum of the main diagonal element}$

$= 2 + 2 + 2 = 6$

$s_2 = \text{sum of the minors of the main diagonalelement}$

$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 11$

$s_3 = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2(4 - 0) - 0 + 1(0 - 2)$

$= 8 - 2 = 6$

Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

**Example: 1.4 Find the characteristic equation of the matrix** $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = \text{sum of the main diagonal element}$

$= 2 + 2 + 1 = 5$

$s_2 = \text{sum of the minors of the main diagonalelement}$

$= \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 7$

$s_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(2 - 0) - 1(1 - 0) + 1(0 - 0)$

$= 4 - 1 = 3$

Characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

**Example: 1.5 Find the characteristic polynomial of the matrix** $\begin{pmatrix} \mathbf{0} & \mathbf{-2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$

**Solution:**

The characteristic polynomial is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3$

$s_1 = \text{sum of the main diagonal element}$

$= 0 + 1 + 2 = 3$

$s_2 = \text{sum of the minors of the main diagonalelement}$

$= \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 0$

$$s_3 = |A| = \begin{vmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{vmatrix} = 0 + 2(-2+2) - 2(1+1)$$

$$= -4$$

Characteristic polynomial is $\lambda^3 - 3\lambda^2 + 4$

## 1.3 EIGEN VALUES AND EIGEN VECTORS

### Definition

The values of $\lambda$ obtained from the characteristic equation $|A - \lambda I| = 0$ are called Eigenvalues of 'A'.[or Latent values of A or characteristic values of A]

### Definition

Let A be square matrix of order 3 and $\lambda$ be scaler. The column matrix $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ which satisfies $(A - \lambda I)X = 0$ is called Eigen vector or Latent vector or characteristic vector.

### Linearly Dependent and Independent Eigenvectors

Let A be the matrix whose columns are Eigen vectors of A

* If $|A| = 0$ then the eigenvectors are linearly dependent.

* If $|A| \neq 0$ then the eigenvectors are linearly independent.

**Example: 1.6 Find the Eigen values for the matrix** $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 2 + 3 + 2 = 7$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 = 11$$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6-2) - 2(2-1) + 1(2-3)$$

$$= 8 - 2 - 1 = 5$$

Characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\Rightarrow \lambda = 1\ , \lambda^2 - 6\lambda + 5 = 0$$

$$\Rightarrow \lambda = 1,\ (\lambda - 1)(\lambda - 5) = 0$$

The Eigen values are $\lambda = 1, 1, 5$

**Example: 1.7 Determine the Eigen values for the matrix** $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= -2 + 1 + 0 = -1$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -12 - 3 - 6 = -21$$

$$s_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(0-12) - 2(0-6) - 3(-4+1)$$

$$= 24 + 12 + 9 = 45$$

Characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\Rightarrow \lambda = -3 \;, \qquad \lambda^2 - 2\lambda - 15 = 0$$

$$\Rightarrow \lambda = -3, \;\; (\lambda+3)(\lambda-5) = 0$$

The Eigen values are $\lambda = -3, -3, 5$

## Exercise: 1.1

**Find the Eigen values for the following matrices:**

1. $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ **Ans:** $\lambda = 5; \; -5$

2. $\begin{pmatrix} 2 & -1 \\ -8 & 4 \end{pmatrix}$ **Ans:** $\lambda = 0,6$

3. $\begin{pmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{pmatrix}$ **Ans:** $\lambda = -10, -20, 5$

4. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ **Ans:** $\lambda = 1,2,3$

5. $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$ **Ans:** $\lambda = 1,2,3$

## Eigen values and Eigen vectors for Non – Symmetric matrix

**Example: 1.8 Find the Eigen values and Eigen vectors for the matrix** $\begin{pmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 8 + 7 + 3 = 18$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 40 - 40 + 20 = 0$$

Characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \lambda = 0, \quad (\lambda - 15)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 0,3,15$$

**To find the Eigen vectors:**

Case( i) When $\lambda = 0$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0 \ldots (1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \ldots (2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \ldots (4)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \ldots (5)$$

$$2x_1 - 4x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) When $\lambda = 15$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 6x_2 + 2x_3 = 0 \ldots (7)$$

$$-6x_1 - 8x_2 - 4x_3 = 0 \ldots (8)$$

$$2x_1 - 4x_2 - 12x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

**Example: 1.9 Determine the Eigen values and Eigen vectors of the matrix** $\begin{pmatrix} \mathbf{7} & \mathbf{-2} & \mathbf{0} \\ \mathbf{-2} & \mathbf{6} & \mathbf{-2} \\ \mathbf{0} & \mathbf{-2} & \mathbf{5} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 7 + 6 + 5 = 18$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix} = 26 + 35 + 38 = 99$$

$$s_3 = |A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix} = 182 - 20 + 0 = 162$$

Characteristic equation is $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$$\Rightarrow \lambda = 3, (\lambda^2 - 15\lambda + 54) = 0$$

$$\Rightarrow \lambda = 3, \quad (\lambda - 9)(\lambda - 6) = 0$$

$$\Rightarrow \lambda = 3, 6, 9$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow\begin{pmatrix}7-3 & -2 & 0\\ -2 & 6-3 & -2\\ 0 & -2 & 5-3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$4x_1-2x_2+0x_3=0\ \ldots(1)$$

$$-2x_1+3x_2-2x_3=0\ \ldots(2)$$

$$0x_1-2x_2+2x_3=0\ \ldots(3)$$

From (1) and (2)

$$\frac{x_1}{4-0}=\frac{x_2}{0+8}=\frac{x_3}{12-4}$$

$$\frac{x_1}{4}=\frac{x_2}{8}=\frac{x_3}{8}$$

$$\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

$$X_1=\begin{pmatrix}1\\2\\2\end{pmatrix}$$

Case (ii) When $\lambda=6$ the eigen vector is given by $(A-\lambda I)X=0$ where $X=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$

$$\Rightarrow\begin{pmatrix}7-6 & -2 & 0\\ -2 & 6-6 & -2\\ 0 & -2 & 5-6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_1-2x_2+0x_3=0\ \ldots(4)$$

$$-2x_1+0x_2-2x_3=0\ \ldots(5)$$

$$0x_1-2x_2-x_3=0\ \ldots(6)$$

From (4)and (5)

$$\frac{x_1}{4-0}=\frac{x_2}{0+2}=\frac{x_3}{0-4}$$

$$\frac{x_1}{4}=\frac{x_2}{2}=\frac{x_3}{-4}$$

$$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$$X_2=\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

Case (iii) When $\lambda=9$ the eigen vector is given by $(A-\lambda I)X=0$ where $X=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$

$$\Rightarrow\begin{pmatrix}7-9 & -2 & 0\\ -2 & 6-9 & -2\\ 0 & -2 & 5-9\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$-2x_1-2x_2+0x_3=0\ \ldots(7)$$

$$-2x_1-3x_2-2x_3=0\ \ldots(8)$$

$$0x_1-2x_2-4x_3=0\ \ldots(9)$$

From (7) and (8)

$$\frac{x_1}{4-0}=\frac{x_2}{0-4}=\frac{x_3}{6-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

**Example: 1.10 Determine the Eigen values and Eigen vectors of the matrix** $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 3 + 2 + 5 = 10$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 6 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 10 + 15 + 6 = 31$$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{vmatrix} = 30$$

Characteristic equation is $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 8\lambda + 15) = 0$$

$$\Rightarrow \lambda = 2, \ (\lambda - 5)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 2,3,5$$

**To find the Eigen vectors:**

Case( i) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 + 4x_3 = 0 \ldots (1)$$

$$0x_1 + 0x_2 + 6x_3 = 0 \ldots (2)$$

$$0x_1 + 0x_2 + 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{6-0} = \frac{x_2}{0-6} = \frac{x_3}{0-0}$$

$$\frac{x_1}{6} = \frac{x_2}{-6} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + x_2 + 4x_3 = 0 \ldots (4)$$

$$0x_1 - x_2 + 6x_3 = 0 \ldots (5)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{4+6} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{10} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Case (iii) When $\lambda = 5$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 + 4x_3 = 0 \ldots (7)$$

$$0x_1 - 3x_2 + 6x_3 = 0 \ldots (8)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{6+12} = \frac{x_2}{0+12} = \frac{x_3}{6-0}$$

$$\frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

**Example: 1.11 Find the Eigen values and Eigen vectors of the matrix** $\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{-b} & \mathbf{a} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

$s_1$ = sum of the main diagonal element

$= a + a = 2a$

$$s_2 = |A| = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2 + b^2$$

Characteristic equation is $\lambda^2 - 2a\lambda + a^2 + b^2 = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2}$$

$$= \frac{2a \pm \sqrt{-4b^2}}{2}$$

$$= \frac{2a \pm i2b}{2}$$

$$= a \pm ib$$

The Eigen values are $\lambda = a \pm ib$

**To find the Eigen vectors:**

Case (i) When $\lambda = a + ib$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} a - (a + ib) & b \\ -b & a - (a + ib) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-ibx_1 + bx_2 = 0 \ldots (1)$$

$$-bx_1 - ibx_2 = 0 \ldots (2)$$

$$\therefore \ (1) \Rightarrow -ibx_1 = -bx_2$$

$$\frac{x_1}{b} = \frac{x_2}{ib} \Rightarrow \frac{x_1}{1} = \frac{x_2}{i}$$

$$X_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Case (ii) When $\lambda = a - ib$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} a - (a - ib) & b \\ -b & a - (a - ib) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$ibx_1 + bx_2 = 0 \ldots (3)$$

$$-bx_1 + ibx_2 = 0 \ldots (4)$$

$$\therefore (3) \Rightarrow ibx_1 = -bx_2$$

$$\frac{x_1}{-b} = \frac{x_2}{ib} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{i}$$

$$X_2 = \begin{pmatrix} -1 \\ i \end{pmatrix}$$

## Exercise: 1.2

**Find the Eigen values and Eigenvectors of the following matrices:**

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ **Ans:** $\lambda = 1; (1,0,0); \ \lambda = 2; (0,1,1); \lambda = 4; (0,-1,1)$

2. $\begin{pmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{pmatrix}$ **Ans:** $\lambda = -2; (1,-1,0);\ \lambda = 9; (2,2,-1); \lambda = -18; (1,1,4)$

3. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ **Ans:** $\lambda = 1; (2,-1,-4);\ \lambda = 3; (2,1,-2); \lambda = -4; (1,-3,13)$

4. $\begin{pmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{pmatrix}$ **Ans:** $\lambda = 0; (5,1,0);\ \lambda = -1; (2,0,1); \lambda = 2; (0,1,-2)$

5. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **Ans:** $\lambda = 0; (1,-1,0);\ \lambda = 1; (0,0,1); \lambda = 4; (1,1,0)$

## Problems on Symmetric matrices with repeated Eigen values

**Example: 1.12 Determine the Eigen values and Eigen vectors of the matrix** $\begin{pmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 6 + 3 + 3 = 12$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14 = 36$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow \lambda = 2, \ (\lambda - 2)(\lambda - 8) = 0$$

$$\Rightarrow \lambda = 2,2,8$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 8$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \ldots (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \ldots (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \ldots (4)$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (5)$$

$$2x_1 - x_2 + x_3 = 0 \ldots (6)$$

In (1) put $x_1 = 0 \quad \Rightarrow -2x_2 = -2x_3$

$$\Rightarrow \frac{x_2}{1} = \frac{x_3}{1} \ \Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

In (1) put $x_2 = 0 \quad \Rightarrow 4x_1 + 2x_3 = 0$

$$\Rightarrow 4x_1 = -2x_3$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{2} \ \Rightarrow X_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

**Example: 1.13 Find the Eigen values and Eigen vectors of the matrix** $\begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 2 + 3 + 2 = 7$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 = 1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 5$$

Characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 6\lambda + 5) = 0$$

$$\Rightarrow \lambda = 1, \ (\lambda - 1)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 1, 1, 5$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 5$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0 \ldots (1)$$

$$x_1 - 2x_2 + x_3 = 0 \ldots (2)$$

$$x_1 + 2x_2 - 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (4)$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (5)$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (6)$$

In (1) put $x_1 = 0 \quad \Rightarrow 2x_2 = -x_3$

$$\Rightarrow \frac{x_2}{-1} = \frac{x_3}{2} \Rightarrow X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

In (1) put $x_2 = 0 \quad => x_1 = -x_3$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{1} \Rightarrow X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

**Example: 1.14 Find the Eigen values and Eigen vectors of the matrix** $\begin{pmatrix} \mathbf{6} & \mathbf{-6} & \mathbf{5} \\ \mathbf{14} & \mathbf{-13} & \mathbf{10} \\ \mathbf{7} & \mathbf{-6} & \mathbf{4} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$$= 6 - 13 + 4 = -3$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix} = 8 - 11 + 6 = 3$$

$$s_3 = |A| = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix} = -1$$

Characteristic equation is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$$\Rightarrow \lambda = -1, (\lambda^2 + 2\lambda + 1) = 0$$

$$\Rightarrow \lambda = 1, \ (\lambda + 1)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -1, -1, -1$$

**To find the Eigen vectors:**

When $\lambda = -1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6+1 & -6 & 5 \\ 14 & -13+1 & 10 \\ 7 & -6 & 4+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 6x_2 + 5x_3 = 0 \ldots (1)$$

$$14x_1 - 12x_2 + 10x_3 = 0 \ldots (2)$$

$$7x_1 - 6x_2 + 5x_3 = 0 \ldots (3)$$

In (1) put $x_1 = 0 \quad \Rightarrow -6x_2 = -5x_3$

$$\Rightarrow \frac{x_2}{5} = \frac{x_3}{6} \ \Rightarrow X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

In (1) put $x_2 = 0 \quad \Rightarrow 7x_1 = -5x_3$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_3}{7} \ \Rightarrow X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$$

In (1) put $x_3 = 0 \quad \Rightarrow 7x_1 = 6x_2$

$$\Rightarrow \frac{x_1}{6} = \frac{x_2}{7} \ \Rightarrow X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}; X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$

**Example: 1.15 Find the Eigen values and Eigen vectors of the matrix** $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} \end{pmatrix}$

**Solution:**

The characteristic equation s $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 2 + 2 + 2 = 6$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4 + 4 + 4 = 12$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

Characteristic equation is $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 4\lambda + 4) = 0$$

$$\Rightarrow \lambda = 2, \ (\lambda - 2)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 2, 2, 2$$

**To find the Eigen vectors:**

When $\lambda = 2$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-2 & 1 & 0 \\ 0 & 2-2 & 1 \\ 0 & 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + x_2 + 0x_3 = 0 \ldots (1)$$

$$0x_1 + 0x_2 + x_3 = 0 \ldots (2)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

Hence the corresponding Eigen vectors are $X_1 = X_2 = X_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

## Exercise: 1.3

**Find the Eigen values and Eigenvectors of the following matrices:**

1. $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ **Ans:** $\lambda = 1; (2, -1, 0); \ \lambda = 1; (1, 0, -1); \lambda = 5; (1, 1, 1)$

2. $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$ **Ans:** $\lambda = 1; (0, 1, -1); \ \lambda = 1; (1, 0, -1); \lambda = 7; (1, 2, 3)$

3. $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$ **Ans:** $\lambda = 1; (1, 1, -1); \ \lambda = 1; (1, 1, -1); \lambda = 2; (2, 1, 0)$

4. $\begin{pmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ **Ans:** $\lambda = 0; (3, -1, 0); \ \lambda = 0; (3, -1, 0); \lambda = 1; (12, -4, -1)$

5. $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ **Ans:** $\lambda = -3; (3, 0, 1); \ \lambda = -3; (-2, 1, 0); \lambda = 5; (1, 2, -1)$

## 1.4 PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS

**Property: 1(a) The sum of the Eigen values of a matrix is equal to the sum of the elements of the principal (main) diagonal.**

**(or)**

**The sum of the Eigen values of a matrix is equal to the trace of the matrix.**

**1. (b) product of the Eigen values is equal to the determinant of the matrix.**

**Proof:**

Let A be a square matrix of order $n$.

The characteristic equation of A is $|A - \lambda I| = 0$

$(i.e.)\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - \cdots + (-1)S_n = 0$ ... (1)

where $S_1 =$ Sum of the diagonal elements of A.

. . .

. . .

. . .

$S_n =$ determinant of A.

We know the roots of the characteristic equation are called Eigen values of the given matrix.

Solving (1) we get $n$ roots.

Let the $n$ be $\lambda_1, \lambda_2, \ldots \lambda_n$.

i.e., $\lambda_1, \lambda_2, \ldots \lambda_n$. are the Eignvalues of A.

We know already,

$\lambda^n - (\text{Sum of the roots } \lambda^{n-1} + [\text{sum of the product of the roots taken two at a time}]\, \lambda^{n-2} -$

$\ldots + (-1)^n(\text{Product of the roots}) = 0$ ... (2)

Sum of the roots $= S_1 by$ (1)&(2)

$(i.e.)\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = S_1$

$(i.e.)\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

| Sum of the Eigen values = Sum of the main diagonal elements |
|---|

Product of the roots $= S_n$by (1)&(2)

$(i.e.)\lambda_1\lambda_2 \ldots \lambda_n = \det$ of A

| Product of the Eigen values = $\lvert A \rvert$ |
|---|

**Property: 2 A square matrix A and its transpose $A^T$ have the same Eigenvalues.**

**(or)**

**A square matrix A and its transpose $A^T$ have the same characteristic values.**

**Proof:**

Let A be a square matrix of order $n$.

The characteristic equation of A and $A^T$ are

$$|A - \lambda I| = 0 \qquad \ldots\ldots(1)$$

and $$|A^T - \lambda I| = 0 \qquad \ldots\ldots(2)$$

Since, the determinant value is unaltered by the interchange of rows and columns.

We know $|A| = |A^T|$

Hence, (1) and (2) are identical.

∴ The Eigenvalues of A and $A^T$ are the same.

**Property: 3 The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.**

**(or)**

**The Eigen values of a triangular matrix are just the diagonal elements of the matrix.**

**Proof:** Let us consider the triangular matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Characteristic equation of is

$$|A - \lambda I| = 0$$

i.e., $$\begin{vmatrix} a_{11} - \lambda & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

On expansion it gives $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$

i.e., $\lambda = a_{11}, a_{22}, a_{33}$

which are diagonal elements of the matrix A.

**Property: 4 If $\boldsymbol{\lambda}$ is an Eigenvalue of a matrix A, then $\frac{1}{\lambda}, (\lambda \neq 0)$ is the Eignvalue of $A^{-1}$.**

**(or)**

**If $\boldsymbol{\lambda}$ is an Eigenvalue of a matrix A, what can you say about the Eigenvalue of matrix $A^{-1}$. Prove your statement.**

**Proof:**

If X be the Eigenvector corresponding to $\lambda$,

then $AX = \lambda X \qquad \ldots$ (i)

Pre multiplying both sides by $A^{-1}$, we get

$$A^{-1}AX = A^{-1}\lambda X$$

$$(1) \Rightarrow X = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\div \lambda \Rightarrow \quad \frac{1}{\lambda}X = A^{-1}X$$

$$(i.e.) \quad A^{-1}X = \frac{1}{\lambda}X$$

This being of the same form as (i), shows that $\frac{1}{\lambda}$ is an Eigenvalue of the inverse matrix $A^{-1}$.

**Property: 5 If $\boldsymbol{\lambda}$ is an Eigenvalue of an orthogonal matrix, then $\frac{1}{\lambda}$ is an Eigenvalue.**

**Proof:**

Definition: Orthogonal matrix.

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$

i.e., $A^T = A^{-1}$

Let A be an orthogonal matrix.

Given $\lambda$ is an Eignevalue of A.

$\Rightarrow \frac{1}{\lambda}$ is an Eigenvalue of $A^{-1}$

Since, $A^T = A^{-1}$

$\therefore \frac{1}{\lambda}$ is an Eigenvalue of $A^T$

But, the matrices A and $A^T$ have the same Eigenvalues, since the determinants $|A - \lambda I|$ and $|A^T - \lambda I|$ are the same.

Hence, $\frac{1}{\lambda}$ is also an Eigenvalue of A.

**Property: 6 If $\lambda_1, \lambda_2, \ldots \lambda_n$. are the Eignvalues of a matrix A, then $A^m$ has the Eigenvalues $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer)**

**Proof:**

Let $A_i$ be the Eigenvalue of A and $X_i$ the corresponding Eigenvector.

Then $AX_i = \lambda_i X_i \quad \ldots(1)$

We have $A^2X_i = A(AX_i)$

$= A(\lambda_i X_i)$

$= \lambda_i A(X_i)$

$= \lambda_i(\lambda_i X_i)$

$= \lambda_i^2 X_i$

$|||$ 1y $A^3X_i = \lambda_i^3 X_i$

In general, $A^m X_i = \lambda_i^m X_i \quad \ldots.(2)$

Hence, $\lambda_i^m$ is an Eigenvalue of $A^m$.

The corresponding Eigenvector is the same $X_i$.

**Note:** If $\lambda$ is the Eigenvalue of the matrix A then $\lambda^2$ is the Eigenvalue of $A^2$

**Property: 7 The Eigen values of a real symmetric matrix are real numbers.**

**Proof:**

Let $\lambda$ be an Eigenvalue (may be complex) of the real symmetric matrix A. Let the corresponding Eigenvector be X. Let A denote the transpose of A.

We have $AX = \lambda X$

Pre-multiplying this equation by $1 \times n$ matrix $\bar{X}'$, where the bar denoted that all elements of $\bar{X}'$ are the complex conjugate of those of $X'$, we get

$\overline{X'}AX = \lambda\overline{X'}X \quad \ldots.(1)$

Taking the conjugate complex of this we get $X' A\bar{X} = \bar{\lambda}X'\bar{X}$ or

$$X' A \bar{X} = \bar{\lambda} X' \bar{X} \text{ since, } \bar{A} = A \text{ for A is real.}$$

Taking the transpose on both sides, we get

$$(X'A\bar{X})' = (\bar{\lambda} X'\bar{X})' (i.e.,) \overline{X'} A' X = \bar{\lambda} \overline{X'} X$$

$$(i.e.) \overline{X'} A' X = \bar{\lambda} \overline{X'} X \text{ since } A' = A \text{ for A is symmetric.}$$

But, from (1), $\overline{X'} A X = \lambda \overline{X'} X$ Hence $\lambda \overline{X'} X = \bar{\lambda} \overline{X'} X$

Since, $\overline{X'} X$ is an $1 \times 1$ matrix whose only element is a positive value, $\lambda = \bar{\lambda}$ $(i.e.)$ $\lambda$ is real).

**Property: 8 The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal.**

**Proof:**

For a real symmetric matrix A, the Eigen values are real.

Let $X_1, X_2$ be Eigenvectors corresponding to two distinct eigen values $\lambda_1, \lambda_2$ [$\lambda_1, \lambda_2$ are real]

$$AX_1 = \lambda_1 X_1 \qquad \ldots.(1)$$

$$AX_2 = \lambda_2 X_2 \qquad \ldots.(2)$$

Pre multiplying (1) by $X_2'$, we get

$$X_2' AX_1 = X_2' \lambda_1 X_1$$

$$= \lambda_1 X_2' X_1$$

Pre-multiplying (2) by $X_1'$, we get

$$X_1' AX_2 = \lambda_2 X_1' X_2 \qquad \ldots.(3)$$

$$\text{But} (X_2' AX_1)' = (\lambda_1 X_2' X_1)'$$

$$X_1' A X_2 = \lambda_1 X_1' X_2$$

$$(i.e) \quad X_1' A X_2 = \lambda_1 X_1' X_2 \qquad \ldots\ldots (4) \quad [\because A' = A]$$

From (3) and (4)

$$\lambda_1 X_1' X_2 = \lambda_2 X_1' X_2$$

(i.e.,)$(\lambda_1 - \lambda_2) X_1' X_2 = 0$

$\lambda_1 \neq \lambda_2, X_1' X_2 = 0$

$\therefore X_1, X_2$ are orthogonal.

**Property: 9 The similar matrices have same Eigen values.**

**Proof:**

Let A, B be two similar matrices.

Then, there exists an non-singular matrix P such that $B = P^{-1} AP$

$$B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - P^{-1} \lambda IP$$

$$= P^{-1}(A - \lambda I)P$$

$$|B - \lambda I| = |P^{-1}| \, |A - \lambda I| \, |P|$$

$$= |A - \lambda I| \, |P^{-1}P|$$

$= |A - \lambda I|\,|I|$

$= |A - \lambda I|$

Therefore, A, B have the same characteristic polynomial and hence characteristic roots.

∴ They have same Eigen values.

**Property: 10 If a real symmetric matrix of order 2 has equal Eigen values, then the matrix is a scalar matrix.**

**Proof :**

**Rule 1 :** A real symmetric matrix of order $n$ can always be diagonalised.

**Rule 2 :** If any diagonalized matrix with their diagonal elements are equal, then the matrix is a scalar matrix.

Given A real symmetric matrix 'A' of order 2 has equal Eigen values.

By Rule: 1 A can always be diagonalized, let $\lambda_1$ and $\lambda_2$ be their Eigenvalues then

we get the diagonlized matrix $= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Given$\lambda_1 = \lambda_2$

Therefore, we get $= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

By Rule: 2 The given matrix is a scalar matrix.

**Property: 11 The Eigen vector X of a matrix A is not unique.**

**Proof :**

Let λ be the Eigenvalue of A, then the corresponding Eigenvector X such that $A X = \lambda X$ .

Multiply both sides by non-zero K,

$K (AX) = K (\lambda X)$

$\Rightarrow A (KX) = \lambda (KX)$

$(i.e.)$an Eigenvector is determined by a multiplicative scalar.

$(i.e.)$ Eigenvector is not unique.

**Property: 12 $\lambda_1, \lambda_2, \ldots \lambda_n$ be distinct Eigenvalues of an $n \times n$ matrix, then the corresponding Eigenvectors $X_1, X_2, \ldots X_n$ form a linearly independent set.**

**Proof:**

Let $\lambda_1, \lambda_2, \ldots \lambda_m (m \le n)$ be the distinct Eigen values of a square matrix A of order $n$.

Let $X_1, X_2, \ldots X_m$ be their corresponding Eigenvectors we have to prove $\sum_{i=1}^{m} \alpha_i X_i = 0$ implies each $\alpha_i = 0, i = 1,2, \ldots, m$

Multiplying $\sum_{i=1}^{m} \alpha_i X_i = 0$ by $(A - \lambda_1 I)$, we get

$$(A - \lambda_1 I)\alpha_1 X_1 = \alpha_1 (AX_1 - \lambda_1 X_1) = \alpha_1(0) = 0$$

When $\sum_{i=1}^{m} \alpha_i X_i = 0$ Multiplied by

$$(A - \lambda_2 I)(A - \lambda_2 I) \ldots (A - \lambda_{i-1} I)(A - \lambda_i I)\ (A - \lambda_{i+1} I) \ldots\ (A - \lambda_m I)$$

We get, $\alpha_i(\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \ldots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \ldots (\lambda_i - \lambda_m) = 0$

Since, λ's are distinct, $\alpha_i = 0$

Since, $i$ is arbitrary, each $\alpha_i = 0, i = 1, 2, \dots, m$

$\sum_{i=1}^{m} \alpha_i \mathrm{X_i} = 0$ implies each $\alpha_i = 0, i = 1,2, \dots, m$

Hence, $\mathrm{X_1}, \mathrm{X_2}, \dots \mathrm{X_m}$ are linearly independent.

**Property: 13 If two or more Eigen values are equal it may or may not be possible to get linearly independent Eigenvectors corresponding to the equal roots.**

**Property: 14 Two Eigenvectors $\mathbf{X_1}$ and $\mathbf{X_2}$ are called orthogonal vectors if $X_1^T \mathbf{X_2} = 0$**

**Property: 15 If A and B are $n \times n$ matrices and B is a non singular matrix, then A and $B^{-1}$ AB have same eigenvalues.**

**Proof:**

Characteristic polynomial of $B^{-1}$ AB

$= |\mathrm{B^{-1}\,AB} - \lambda \mathrm{I}| = |\mathrm{B^{-1}\,AB} - \mathrm{B^{-1}}(\lambda \mathrm{I})\mathrm{B}|$

$= |\mathrm{B^{-1}}\,(\mathrm{A} - \lambda \mathrm{I})\mathrm{B}| = |\mathrm{B^{-1}}||\mathrm{A} - \lambda \mathrm{I}||\mathrm{B}|$

$= |\mathrm{B^{-1}}|\,|\mathrm{B}|\,|\mathrm{A} - \lambda \mathrm{I}| = |\mathrm{B^{-1}B}||\mathrm{A} - \lambda \mathrm{I}|$

= Characterisstisc polynomial of A

Hence, A and $\mathrm{B^{-1}}$ AB have same Eigenvalues.

## Problems based on properties

**Example: 1.16 Find the sum and product of the Eigen values of the matrix** $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

**Solution:**

Sum of the Eigen values =Sum of the main diagonal elements

$= (-2) + (1) + (0)$

$= -1$

Product of the Eigen values = $\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$= -2(0 - 12) - 2(0 - 6) - 3(-4 + 1)$

$= 24 + 12 + 9 = 45$

**Example: 1.17 Find the sum and product of the Eigen values of the matrix A=** $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

**Solution:**

Sum of the Eigen values $=$ Sum of its diagonal elements $= 1 + 2 + 1 = 4$

Product of Eigen values $= |C| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(2 - 1) - 2(-1 - 1) + 3(-1 - 2)$

$= 1(1) - 2(-2) + 3(-3)$

$= 1 + 4 - 9 = -4$

**Example: 1.18 The product of two Eigen values of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third Eigenvalue.**

**Solution:**

Let Eigen values of the matrix A be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2 = 16$

We know that, $\lambda_1\lambda_2\lambda_3 = |A|$

[Product of the Eigen values is equal to the determinant of the matrix]

$$\therefore \lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + (-6+2) + 2(2-6)$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8$$

$$\Rightarrow \lambda_1\lambda_2\lambda_3 = 32$$

$$\Rightarrow 16\,\lambda_3 = 32$$

$$\Rightarrow \lambda_3 = \frac{32}{16} = 2$$

**Example: 1.19 Two of the Eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are 2 and 8. Find the third Eigen value.**

**Solution:**

We know that, Sum of the Eigen values = Sum of its diagonal elements

$$= 6 + 3 + 3 = 12$$

Given $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

We get, $\lambda_1 + \lambda_2 + \lambda_3 = 12$

$$2 + 8 + \lambda_3 = 12$$

$$\lambda_3 = 12 - 10$$

$$\lambda_3 = 2$$

$\therefore$ The third Eigenvalue $= 2$

**Example: 1.20 If 3 and 15 are the two Eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ find $|A|$, without expanding the determinant.**

**Solution:**

Given $\lambda_1 = 3, \lambda_2 = 15, \lambda_3 = ?$

We know that, Sum of the Eigen values = Sum of the main diagonal elements

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$$

$$3 + 15 + \lambda_3 = 18$$

$$\Rightarrow \lambda_3 = 0$$

We know that, Product of the Eigen values $= |A|$

$$\Rightarrow (3)(15)(0) = |A|$$

$$\Rightarrow |A| = (3)(15)(0)$$

$$\Rightarrow |A| = 0$$

**Example: 1.21 If 2, 2, 3 are the Eigen values of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ find the Eigen values of $A^T$.**

**Solution:**

By Property "A square matrix A and its transpose $A^T$ have the same Eigen values".

Hence, Eigen values of $A^T$ are 2, 2, 3

**Example: 1.22 If the Eigen values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ are $2, -2$ then find the Eigen values of $A^T$.**

**Solution:**

Eigen values of $A =$ Eigen values $of$ $A^T$

$\therefore$ Eigen values $of$ $A^T$ are $2, -2$.

**Example: 1.23 Find the Eigen values of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ without using the characteristic equation idea.**

**Solution:**

Given $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ clearly given matrix A is an upper triangular matrix. Then by property, the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Hence, the Eigen values are 2, 2, 2.

**Example: 1.24 Find the Eigen values of $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$**

**Solution:**

Given $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Clearly given matrix A is a lower triangular matrix

Hence, by property the Eigen values of A are 2, 3, 4.

**Example: 1.25 Two of the Eigen values of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the Eigen values of $A^{-1}$.**

**Solution:**

Sum of the Eigen values $=$ Sum of the main diagonal elements

$$= 3 + 5 + 3 = 11$$

Let K be the third Eigen value

$$\therefore 3 + 6 + k = 11$$

$$\Rightarrow 9 + k = 11$$

$$\Rightarrow k = 2$$

$\therefore$ The Eigenvalues of $A^{-1}$ are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

**Example: 1.26 Two Eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the Eigenvalues of $A^{-1}$.**

**Solution:**

Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Let the Eigen values of the matrix A be $\lambda_1, \lambda_2, \lambda_3$

Given condition is $\lambda_2 = \lambda_3 = 1$

We have, Sum of the Eigen values = Sum of the main diagonal elements

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$$

$$\Rightarrow \lambda_1 + 1 + 1 = 7$$

$$\Rightarrow \lambda_1 + 2 = 7$$

$$\Rightarrow \lambda_1 = 5$$

Hence, the Eigen values of A are 1, 1, 5

Eigen values of $A^{-1}$ are $\frac{1}{1}, \frac{1}{1}, \frac{1}{5}$, i.e., $1, 1, \frac{1}{5}$

**Example: 1.27 Find the Eigen values of the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$**

**Solution:**

We know that, the given matrix is a upper triangular matrix.

Therefore, the Eigen values of A are 2, 3, 4

**Example: 1.28 Find the Eigen values of $A^3$ given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$**

**Solution:**

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Clearly given A is an upper triangular matrix

Hence, the Eigen values are 1, 2, 3

$(i.e.)$ The Eigen values of the given matrix A are 1, 2, 3.

By the property, the Eigen values of the matrix $A^3$ are $1^3, 2^3, 3^3$. i.e., 1, 8, 27.

**Example: 1.29 If 1 & 2 are the Eigen values of a $2 \times 2$ matrix A, what are the Eigenvalues of $A^3$ and $A^{-1}$?**

**Solution:**

We know that, if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigenvalues of A, then $\lambda_1^m, \lambda_2^m \ldots \lambda_n^m$ are the Eigenvalues of $A^m$.

Given 1 and 2 are the Eigen values of A.

$\therefore 1^2$ and $2^2$ i.e., 1 and 4 are the Eigen values of $A^2$ ; $1$ *and* $\frac{1}{2}$ are the Eigenvalues of $A^{-1}$.

**Example: 1.30 If $\alpha$ and $\beta$ are the Eigen values of $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$, form the matrix whose Eigenvalues are $\alpha^3$ and $\beta^3$.**

**Note:** If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigenvalues of A, then $\lambda_1^k, \lambda_2^k, \lambda_3^k \ldots \lambda_n^k$ are the Eigenvalues of $A^k$ for any positive integer.

**Solution:**

Let $A = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$, Let $\alpha, \beta$ be the Eigen values of A

$\Rightarrow A^2 = A.A$

$$= \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 10 & -8 \\ -8 & 26 \end{pmatrix}$$

$\Rightarrow A^3 = A^2 A$

$$= \begin{pmatrix} 10 & -8 \\ -8 & 26 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 38 & -50 \\ -50 & 138 \end{pmatrix}$$

Hence, the required matrix is $\begin{pmatrix} 38 & -50 \\ -50 & 138 \end{pmatrix}$

**Example: 1.31 Form the matrix whose Eigen values are $\alpha - 5, \beta - 5, \gamma - 5$ where $\alpha, \beta, \gamma$ are the Eigenvalues of $A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$**

**Solution:**

**Note:** The matrix $A - KI$ has the Eigen values $\lambda_1 - k, \lambda_2 - k, \ldots \lambda_n - k$.

Hence, the required matrix is $A - 5I = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix}$$

**Example: 1.32 Two Eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and they are $\frac{1}{5}$ times to the third. Find them.**

**Solution:**

Let the third Eigen value be $\lambda_3$.

We know that, $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2 = 7$

Given $\lambda_1 = \lambda_2 \ldots (2)$ $\lambda_1 = \frac{1}{5}\lambda_3 \ldots (3)$ $\lambda_2 = \frac{1}{5}\lambda_3 \ldots (4)$

$(1) \Rightarrow \quad 2\lambda_1 + \lambda_3 = 7$ by (2)

$\frac{2}{5}\lambda_3 + \lambda_3 = 7$ by (3)

$\Rightarrow \quad \frac{7}{5}\lambda_3 = 7$ $\boxed{\Rightarrow \lambda_3 = 5}$

(3) $\Rightarrow \quad \lambda_1 = \frac{1}{5}(5) = 1$ $\boxed{\Rightarrow \lambda_3 = 5}$

(2) $\Rightarrow \quad \boxed{\lambda_2 = 1}$ $[\because \lambda_1 = \lambda_2]$

Hence, the Eigen values are 1, 1, 5.

**Example: 1.33 If 2, 3 are the Eigen values of $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, find the value of a.**

**Solution:**

$$\text{Let A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$$

Let $\lambda_1, \lambda_2, \lambda_3$ by the Eigenvalues of A.

Given $\lambda_1 = 2, \lambda_2 = 3$

We know that, $\lambda_1 + \lambda_2 + \lambda_3 =$ Sum to the main diagonal elements

$$\Rightarrow 2 + 3 + \lambda_3 = 2 + 2 + 2$$

$$\Rightarrow 5 + \lambda_3 = 6$$

$$\Rightarrow \lambda_3 = 1$$

We know that, $\lambda_1\, \lambda_2\, \lambda_3 = |A|$

$$\Rightarrow (2)(3)(1) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix}$$

$$\Rightarrow 6 = 2(4-0) - 0(0-0) + 1(0-2a)$$

$$\Rightarrow 6 = 8 - 2a$$

$$\Rightarrow 2a = 8 - 6 = 2$$

$$\therefore a = 1$$

**Example: 1.34 If the Eigen values of A of order $3 \times 3$ are 2, 3 and 1, then find the Eigen values of adjoint of A.**

**Solution:**

Given the Eigen values of A are 2, 3, 1

The Eigen values of $A^{-1}$ are $\frac{1}{2}, \frac{1}{3}, 1,$

Formula: $\text{adj A} = |A|\, A^{-1}$

$|A| =$ Product of the Eigenvalues of $A = (2)(3)(1) = 6$

$\therefore \text{adj A} = 6\, A^{-1}$

$\therefore$ The Eivenvalues of adj A are $6\left(\frac{1}{2}\right), 6\left(\frac{1}{3}\right), 6\,(1)$, i.e., $3, 2, 6$

**Example: 1.35 If $2, -1, -3$ are the Eigenvalues of the matrix A, then find the Eigenvalues of the matrix $A^2 - 2I$.**

**Solution:**

Given the Eigen values of A are $2, -1, -3$

The Eigen values of $A^2 - 2I$ are $2, -1, 7$

Since $2^2 - 2(1) = 2, (-1)^2 - 2(1) = -1, (-3)^2 - 2(1) = 7$

**Example: 1.36 What are the Eigen values of the matrix $A + 3I$ if the Eigenvalues of the matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ are 6 and $-1$? why ?**

**Solution:**

The Eigen values of A are 6 and $-1$

The Eigen values of $A + 3I$ are $6 + 3$and $-1 + 3$

i.e., The Eigen values of $A + 3I$ are 9 and 2

**Reason:**

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are Eigenvalues of A, then

$\lambda_1 + k, \lambda_2 + k, \lambda_n + k$ are the Eigenvalues of $A + kI$, then

**Example: 1.37 Find the Eigen values of $3A + 2I$, where $A = \begin{bmatrix} 5 & 4 \\ 0 & 2 \end{bmatrix}$**

**Solution:**

The Eigen values of A are 5 and 2.

The Eigen values of $3A + 2I$ are $3(5) + 2$ and $3(2) + 2$

$(i.e.)$ The Eigen values of 3A $+2I$ are 17 and 8

## Exercise: 1.4

1. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & 3 \end{bmatrix}$, then find the sum and product of all Eigenvalues of A.

   **Ans:** Sum $= 4$; Product $= -13$

2. Find the product of the Eigen values of

   (a) $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ **Ans:** Product $= -6$

   (b) $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$ **Ans:** Product $= -1$

3. If the Eigen values of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are $-2, 3, 6$, then find the Eigenvalues of $A^T$.

   **Ans:** $2, 3, 6,$

4. Find the Eigen values of $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ **Ans:** 1, 2, 3

5. Find the Eigen values of the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ **Ans:** $1, \frac{1}{2}, \frac{1}{3},$

6. Find the Eigen values of $A^2$ if $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$ **Ans:** 4, 9, 16

7. Obtain the Eigen values of $A^3$ where $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ **Ans:** 1, 64

**1.5 DIAGONALISATION OF A MATRIX BY ORTHOGONAL TRANSFORMATION**

## Orthogonal matrix

## Definition

A matrix ‘A’ is said to be orthogonal if $AA^T = A^T A = I$

**Example: 1.38 Show that the following matrix is orthogonal** $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

**Solution:**

Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\Rightarrow A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

∴ A is orthogonal.

## Modal Matrix

Modal matrix is a matrix in which each column specifies the eigenvectors of a matrix .It is denoted by N.

A square matrix A with linearity independent Eigen vectors can be diagonalized by a similarly transformation, $D = N^{-1}AN$, where N is the modal matrix .The diagonal matrix D has as its diagonal elements, the Eigen values of A.

## Normalized vector

Eigen vector $X_r$ is said to be normalized if each element of $X_r$ is being divided by the square root of the sum of the squares of all the elements of $X_r$.i.e.,the normalized vector is $\frac{X}{|X|}$

$X_r = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; Normalized vector of $X_r = \begin{bmatrix} x_1/\sqrt{x_1^2+x_2^2+x_3^2} \\ x_2/\sqrt{x_1^2+x_2^2+x_3^2} \\ x_3/\sqrt{x_1^2+x_2^2+x_3^2} \end{bmatrix}$

**Working rule for diagonalization of a square matrix A using orthogonal reduction:**

i) Find all the Eigen values of the symmetric matrix A.

ii) Find the Eigen vectors corresponding to each Eigen value.

iii) Find the normalized modal matrix N having normalized Eigen vectors as its column vectors.

iv) Find the diagonal matrix $D = N^T AN$.The diagonal matrix D has Eigen values of A as its diagonal elements.

**Example: 1.39 Diagonalize the matrix** $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{1} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 2 + 1 + 1 = 4$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3 + 1 + 1 = -1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -4$$

Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 3\lambda - 4) = 0$$

$$\Rightarrow \lambda = 1, \ (\lambda + 1)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = -1, 1, 4$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 1$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 0 \ldots (1)$$

$$x_1 + 0x_2 - 2x_3 = 0 \ldots (2)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-2-0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_1 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

Case( ii) When $\lambda = -1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \dots (4)$$

$$x_1 + 2x_2 - 2x_3 = 0 \dots (5)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \dots (6)$$

From (4) and (5)

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 - x_3 = 0 \dots (7)$$

$$x_1 - 3x_2 - 2x_3 = 0 \dots (8)$$

$$-x_1 - 2x_2 - 3x_3 = 0 \dots (9)$$

From (7) and (8)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

To check $X_1, X_2$ & $X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

**Example: 1.40 Diagonalize the matrix** $\begin{pmatrix} \mathbf{10} & \mathbf{-2} & \mathbf{-5} \\ \mathbf{-2} & \mathbf{2} & \mathbf{3} \\ \mathbf{-5} & \mathbf{-3} & \mathbf{5} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 10 + 2 + 5 = 17$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 3 \\ -3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 1 + 25 + 16 = 42$$

$$s_3 = |A| = \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & -3 & 5 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 17\lambda + 42) = 0$$

$$\Rightarrow \lambda = 0, 3, 14$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 0$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-0 & -2 & -5 \\ -2 & 2-0 & 3 \\ -5 & -3 & 5-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$10x_1 - 2x_2 - 5x_3 = 0 \dots (1)$$

$$-2x_1 + 2x_2 + 3x_3 = 0 \dots (2)$$

$$-5x_1 + 3x_2 + 5x_3 = 0 \dots (3)$$

From (1) and (2)

$$\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

$$X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-3 & -2 & -5 \\ -2 & 2-3 & 3 \\ -5 & -3 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 2x_2 - 5x_3 = 0 \dots (4)$$

$$-2x_1 - x_2 + 3x_3 = 0 \dots (5)$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \dots (6)$$

From (4) and (5)

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 14$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-14 & -2 & -5 \\ -2 & 2-14 & 3 \\ -5 & -3 & 5-14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x_1 - 2x_2 - 5x_3 = 0 \dots (7)$$

$$-2x_1 - 12x_2 + 3x_3 = 0 \dots (8)$$

$$-5x_1 + 3x_2 - 9x_3 = 0 \dots (9)$$

From (7) and (8)

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$\frac{x_1}{-6} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1{}^T X_2 = (1 \quad -5 \quad 4)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 5 + 4 = 0$$

$$X_2{}^T X_3 = (1 \quad 1 \quad 1)\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 1 + 2 = 0$$

$$X_3{}^T X_1 = (-3 \quad 1 \quad 2)\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = -3 - 5 + 8 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{42}} \\ \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & -3 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

**Example: 1.41 Diagonalize the matrix** $\begin{pmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 6 + 3 + 3 = 12$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14 = 36$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow \lambda = 2,2,8$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 8$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \ldots (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \ldots (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \ldots (4)$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (5)$$

$$2x_1 - x_2 + x_3 = 0 \ldots (6)$$

Put $x_1 = 0 \quad \Rightarrow -2x_2 = -2x_3$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new vector orthogonal to both $X_1$ and $X_2$

$(i.e)\ X_1^T X_3 = 0\ \&\ X_2^T X_3 = 0$

$$\begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\ \&\ \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$2a - b + c = 0 \ldots (7)$$

$$a + b + c = 0 \ldots (8)$$

From (7) and (8)

$$\frac{a}{-1-1} = \frac{b}{0-2} = \frac{c}{2}$$

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$$\frac{a}{-1} = \frac{b}{-1} = \frac{c}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T AN$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

**Example: 1.42 Diagonalize the matrix** $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 3 + 3 + 3 = 9$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 + 8 + 8 = 24$$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 16$$

Characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 8\lambda + 16) = 0$$

$$\Rightarrow \lambda = 1, 4, 4$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-1 & 1 & 1 \\ 1 & 3-1 & -1 \\ 1 & -1 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 + x_2 + x_3 = 0 \ldots (1)$$

$$x_1 + 2x_2 - x_3 = 0 \ldots (2)$$

$$x_1 - x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-4 & 1 & 1 \\ 1 & 3-4 & -1 \\ 1 & -1 & 3-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + x_2 + x_3 = 0 \ldots (4)$$

$$x_1 - x_2 - x_3 = 0 \ldots (5)$$

$$x_1 - x_2 - x_3 = 0 \ldots (6)$$

put $x_1 = 0 \Rightarrow x_2 = -x_3$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new vector orthogonal to both $X_1$ and $X_2$

$(i.e)$ $X_1^T X_3 = 0$ & $X_2^T X_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \ \& \ \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-a + b + c = 0 \ldots (7)$$

$$0a - b + c = 0 \ldots (8)$$

From (7) and (8)

$$\frac{a}{1+1} = \frac{b}{0+1} = \frac{c}{1+0}$$

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T AN$

$$= \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

**Example: 1.43 Reduce the matrix A=$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ to the diagonal form, find $A^3$.**

**Solution:**

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

$s_1 =$ sum of the main diagonal element

= 3+3=6

$$s_2 = |A| = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8$$

Characteristic equation is $\lambda^2 - 6\lambda + 8 = 0$

$$\lambda = 2,4$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-2 & -1 \\ -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_1 - x_2 = 0$$

$$-x_1 + x_2 = 0$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Case (ii)When $\lambda = 4$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-4 & -1 \\ -1 & 3-4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

$$-x_1 - x_2 = 0$$

$$-x_1 - x_2 = 0$$

$$\Rightarrow -x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

$$X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

To check $X_1$& $X_2$ are orthogonal

$$X_1{}^T X_2 = (1 \quad 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$X_2{}^T X_1 = (1 \quad -1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

To find $A^3$: $A^3 = ND^3N^T$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & 64 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 36 & -28 \\ -28 & 36 \end{pmatrix}$$

**Example: 1.44 Find the 3×3 square symmetric matrix A having Eigen values 2,3,6 and corresponding Eigen vectors $(1 \quad 0 \quad -1)^T, (1 \quad 1 \quad 1)^T,$ and $(1 \quad -2 \quad 1)^T$.**

**Solution:**

$$\text{Given } \lambda = 2,3,6; \quad X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}; \quad N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$D = N^T A N \Rightarrow A = NDN^T$$

$$DN^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} & 0 & \frac{-2}{\sqrt{2}} \\ \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} \\ \frac{6}{\sqrt{6}} & \frac{-12}{\sqrt{6}} & \frac{6}{\sqrt{6}} \end{pmatrix}$$

$$A = NDN^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} & 0 & \frac{-2}{\sqrt{2}} \\ \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} & \frac{3}{\sqrt{3}} \\ \frac{6}{\sqrt{6}} & \frac{-12}{\sqrt{6}} & \frac{6}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 1+1+1 & 0+1-2 & -1+1+1 \\ 0+1-2 & 0+1+4 & 0+1-2 \\ -1+1+1 & 1+0-2 & 1+1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

**Example: 1.45 Reduce the matrix A =** $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{pmatrix}$ **to the diagonal form, find $A^3$.**

**Solution:**

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 1 + 5 + 1 = 7$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0$$

$$s_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = -36$$

Characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$

$$\Rightarrow \lambda = 3\ ;\ (\lambda^2 - 4\lambda - 12) = 0$$

$$\Rightarrow \lambda = -2,6,3$$

**To find the Eigen vectors:**

Case (i) When $\lambda = -2$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 + 3x_3 = 0 \ldots (1)$$

$$x_1 + 7x_2 + x_3 = 0 \ldots (2)$$

$$3x_1 + x_2 + 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 + 3x_3 = 0 \ldots (4)$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (5)$$

$$3x_1 + x_2 - 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 6$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-5x_1 + x_2 + 3x_3 = 0 \ldots (7)$$

$$x_1 - x_2 + x_3 = 0 \ldots (8)$$

$$3x_1 + x_2 - 5x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1{}^T X_2 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0$$

$$X_2{}^T X_3 = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 - 2 + 1 = 0$$

$$X_3{}^T X_1 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\Rightarrow D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\Rightarrow D^3 = \begin{pmatrix} -8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 216 \end{pmatrix}$$

$A^3 = ND^3N^T$

$$= \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} -8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 216 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 41 & 63 & 49 \\ 63 & 153 & 63 \\ 49 & 63 & 41 \end{pmatrix}$$

## Exercise: 1.5

**1. Diagonalise the following using orthogonal transformation**

**a.** $A = \begin{pmatrix} 5 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$ **Ans:** $D(2,3,6)$

**b.** $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **Ans:** $D(1,3,6)$

**c.** $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **Ans:** $D(8,2,2)$

**d.** $A = \begin{pmatrix} 1 & -1 & -12 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ **Ans:** $D(2,2,-1)$

**2.** a Reduce $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ to diagonal form by an orthogonal transformation and also find $A^4$ .

**Ans:** $A^4 = \begin{pmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{pmatrix}$

b. Reduce the matrix $A=\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$ to the diagonal form, find $A^3$.

**Ans:** $A^3 = \begin{pmatrix} 104 & 0 & 112 \\ 0 & 216 & 0 \\ 112 & 0 & 104 \end{pmatrix}$

c. Reduce the matrix A=$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{pmatrix}$ to the diagonal form, find $A^4$.

**Ans:** $A^4 = \begin{pmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & 160 & 81 \end{pmatrix}$

3. Find the 3x3 square symmetric matrix A having Eigen values -2,1, 1 and corresponding Eigen vectors$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ .

**Ans:** $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

## 1.6 REDUCTION OF QUADRATIC FORM TO CANON ICAL FORM BY ORTHOGONAL TRANSFORMATION

### Linear Form

The expression of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_1, a_2 \ldots a_n$ are constants is called a linear form in the variables $x_1, x_2 \ldots x_n$. This linear form can also be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{i=1}^{n} a_ix_i$

### Bilinear Form

Any expression which is linear and homogeneous in each of the sets of variables $\{x_1, x_2 \ldots \ldots \ldots \ldots . . x_n\}, \{y_1, y_2, \ldots \ldots . y_n\}$ is called a bilinear form in these variables.

The general bilinear form of the two sets of variables $\{x_1, x_2, x_3\} and \ \{y_1, y_2, y_3\}$ can be written as

$$f(x,y) = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{13}x_1y_3 + a_{21}x_2y_1 + a_{22}x_2y_2 + a_{23}x_2y_3 + a_{31}x_3y_1 + a_{32}x_3y_2 + a_{33}x_3y_3 \quad \ldots (1)$$

This bilinear form can written as $f(x,y) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_iy_j$

Equation (1) can be written in matrix form as

$$f(x,y) = (x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = X'AY$$

Where $X'$ is the transpose of $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $and\ Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is called the matrix of the bilinear form.

### Quadratic Form

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

The general Quadratic form in three variables $\{x_1, x_2, x_3\}$ is given by

$$f(x_1,x_2,x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_2x_2 + a_{33}x_3^2$$

This Quadratic form can written as $f(x_1,x_2,x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_iy_j$

$$f(x_1,x_2,x_3) = (x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= X'AX$$

Where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and A is called the matrix of the Quadratic form.

**Note:** To write the matrix of a quadratic form as

$$A = \begin{pmatrix} \text{coeff. of} x^2 & 1/2\text{coeff. of} xy & 1/2\text{coeff. of} xz \\ 1/2\text{coeff. of} xy & \text{coeff. of} y^2 & 1/2\text{coeff. of} yz \\ 1/2\text{coeff. of } xz & 1/2\text{coeff. of} yz & \text{coeff. of } z^2 \end{pmatrix}$$

**Example: 1.46 Write down the Quadratic form in to matrix form**

**(i)$2x^2 + 3y^2 + 6xy$**

**Solution:**

$$A = \begin{pmatrix} \text{coeff. of } x^2 & 1/2\text{coeff. of } xy \\ 1/2\text{coeff. of } xy & \text{coeff. of} y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$$

**(ii) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$**

**Solution:**

$$A = \begin{pmatrix} \text{coeff. of} x^2 & 1/2\text{coeff. of} xy & 1/2\text{coeff. of} xz \\ 1/2\text{coeff. of} xy & \text{coeff. of} y^2 & 1/2\text{coeff. of} yz \\ 1/2\text{coeff. of } xz & 1/2\text{coeff. of} yz & \text{coeff. of } z^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 4 \\ -1 & 5 & -1/2 \\ 4 & -1/2 & -6 \end{pmatrix}$$

**(iii) $12\,x_1^2 + 4x_2^2 + 5x_3^2 - 4x_2x_3 + 6x_1x_3 - 8x_1x_2$**

**Solution:**

$$A = \begin{pmatrix} \text{coeff. of } x_1^2 & 1/2\text{coeff. of } x_1x_2 & 1/2\text{coeff. of } x_1x_3 \\ 1/2\text{coeff. of } x_1x_2 & \text{coeff. of } x_2^2 & 1/2\text{coeff. of } x_2x_3 \\ 1/2\text{coeff. of } x_1x_3 & 1/2\text{coeff. of } x_2x_3 & \text{coeff. of } x_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & -4 & 3 \\ -4 & 4 & -2 \\ 3 & -2 & 5 \end{pmatrix}$$

**Example: 1.47 Write down the matrix form in to Quadratic form**

**(i)$\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & -2 & 5 \end{pmatrix}$**

**Solution:**

Quadratic form is $2\,x_1^2 - 2x_2^2 + 6x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

**(ii)** $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix}$

**Solution:**

Quadratic form is $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3$.

## NATURE OF QUADRATIC FORM DETERMINED BY PRINCIPAL MINORS

Let A be a square matrix of order n say $A = \left[\begin{pmatrix} a_{11}\, a_{12}\, a_{13} & \cdots & a_{1n} \\ a_{21} a_{22}\; a_{23} & \ddots & a_{2n} \\ \ldots & \cdots & \ldots . \end{pmatrix}\right]$

The principal sub determinants of A are defined as below.

$$s_1 = a_{11}$$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$s_n = |A|$$

The quadratic form $Q = X^T AX$ is said to be

1. Positive definite: If $s_{1,} s_{2,} s_{3 \ldots\ldots\ldots} s_n > 0$
2. Positive semidefinite: If $s_{1,} s_{2,} s_{3 \ldots\ldots\ldots} s_n \geq 0$ and atleast one $s_i = 0$
3. Negative definite: If $s_{1,} s_{3,} s_{5 \ldots\ldots\ldots} < 0$ and $s_{2,} s_{4,} s_{6 \ldots\ldots\ldots} > 0$
4. Negative semidefinite: If $s_{1,} s_{3,} s_{5 \ldots\ldots\ldots} < 0$ and $s_{2,} s_{4,} s_{6 \ldots\ldots\ldots} > 0$ and atleast one $s_i = 0$
5. Indefinite: In all other cases

**Example: 1.48 Determine the nature of the Quadratic form $\mathbf{12x_1^2 + 3x_2^2 + 12x_3^2 + 2x_1x_2}$**

**Solution:**

$$A = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

$s_1 = a_{11} = 12 > 0$

$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 12 & 1 \\ 1 & 3 \end{vmatrix} = 35 > 0$

$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{vmatrix} = 430 > 0$ , Postive definite

**Example: 1.49 Determine the nature of the Quadratic form $\mathbf{x_1^2 + 2x_2^2}$**

**Solution:**

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$s_1 = a_{11} = 1 > 0$

$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 2 > 0$

$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$ , Positive semidefinite

**Example: 1.50 Determine the nature of the Quadratic form $\mathbf{x^2 - y^2 + 4z^2 + 4xy + 2yz + 6zx}$**

**Solution:**

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{pmatrix}$

$s_1 = a_{11} = 1 > 0$

$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -5 < 0$

$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 0$ , Indefinite

**Example: 1.51 Determine the nature of the Quadratic form $\mathbf{xy + yz + zx}$**

**Solution:**

Let A = $\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$

$s_1 = a_{11} = 0$

$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -1/4 < 0$

$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{vmatrix} = \frac{1}{4} > 0$ , Indefinite

## RANK, INDEX AND SIGNATURE OF A REAL QUADRATIC FORMS

Let $Q = X^T AX$ be quadratic form and the corresponding canonical form is $d_1{y_1}^2 + d_2{y_2}^2 + \cdots + d_n{y_n}^2$.

The **rank** of the matrix A is number of non –zero Eigen values of A. If the rank of A is ‘r’, the canonical form of Q will contain only “r” terms .Some terms in the canonical form may be positive or zero or negative.

The number of positive terms in the canonical form is called the **index**(p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form .i.e,$p - (r - p) = 2p - r$ is called the signature of the quadratic form and usually denoted by s. Thus $s = 2p - r$.

**Example: 1.52 Reduce the Quadratic form$\mathbf{x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 + 6x_2x_3}$ to canonical form through an orthogonal transformation .Find the nature rank, index, signature and also find the non zero set of values which makes this Quadratic form as zero.**

**Solution:**

Given $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 1 + 2 + 1 = 4$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 + 1 + 1 = 3$$

$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda = 0$

$$\Rightarrow \lambda = 0 \,; (\lambda^2 - 4\lambda + 3) = 0$$

$$\Rightarrow \lambda = 0,1,3$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 0$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 + 0x_3 = 0 \ldots (1)$$

$$-x_1 + 2x_2 + x_3 = 0 \ldots (2)$$

$$0x_1 + x_2 + x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - x_2 + 0x_3 = 0 \ldots (4)$$

$$-x_1 - x_2 + x_3 = 0 \ldots (5)$$

$$0x_1 + x_2 - 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - x_2 + 0x_3 = 0 \ldots (7)$$

$$-x_1 + x_2 + x_3 = 0 \ldots (8)$$

$$0x_1 + x_2 + 0x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -1 + 2 - 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T AN$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Canonical form $= Y^T DY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T DY = (y_1, y_2, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0{y_1}^2 + {y_2}^2 + 3{y_3}^2$$

Rank = 2

Index = 2

Signature = 2 – 0 = 2

Nature is positive semi definite.

**To find non zero set of values:**

Consider the transformation X = NY

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{3}} - \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{3}} + \frac{2y_2}{\sqrt{6}} + 0y_3$$

$$x_3 = \frac{-y_1}{\sqrt{3}} + \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{2}}$$

Put $y_2 = 0$ & $y_3 = 0$

$x_1 = \frac{y_1}{\sqrt{3}};\ x_2 = \frac{y_1}{\sqrt{3}};\ x_3 = \frac{-y_1}{\sqrt{3}}$

Put $y_1 = \sqrt{3}$

$x_1 = 1; x_2 = 1;\ x_3 = -1$ which makes the Quadratic equation zero.

**Example: 1.53 Reduce the Quadratic form $x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$ to canonical form through an orthogonal transformation .Find the nature rank,index,signature and also find the non zero set of values which makes this Quadratic form as zero**

**Solution:**

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$
$$= 1 + 1 + 1 = 3$$
$$s_2 = \text{sum of the minors of the main diagonalelement}$$
$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1 + 1 + 0 = 2$$
$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$$\Rightarrow \lambda = 0\,; (\lambda^2 - 3\lambda + 2) = 0$$
$$\Rightarrow \lambda = 0,1,2$$

**To find the Eigen vectors:**

Case (i) When $\lambda = 0$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$x_1 - x_2 + 0x_3 = 0 \ldots (1)$$
$$-x_1 + x_2 + 0x_3 = 0 \ldots (2)$$
$$0x_1 + 0x_2 + x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{1-0} = \frac{x_2}{0+1} = \frac{x_3}{0}$$
$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$
$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 1-1 & 0 \\ 0 & 0 & 1-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$0x_1 - x_2 + 0x_3 = 0 \ldots (4)$$

$$-x_1 + 0x_2 + 0x_3 = 0 \ldots (5)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Case (iii) When $\lambda = 2$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-2 & -1 & 0 \\ -1 & 1-2 & 0 \\ 0 & 0 & 1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 - x_2 + 0x_3 = 0 \ldots (7)$$

$$-x_1 - x_2 + 0x_3 = 0 \ldots (8)$$

$$0x_1 + 0x_2 - x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = (1 \quad 1 \quad 0) \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0 + 0 + 0 = 0$$

$$X_2^T X_3 = (0 \quad 0 \quad -1) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 + 0 + 0 = 0$$

$$X_3^T X_1 = (1 \quad -1 \quad 0) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 - 1 + 0 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Thus the diagonal matrix $D = N^T AN$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Canonical form $= Y^T DY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T DY = (y_1, y_2, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0{y_1}^2 + {y_2}^2 + 2{y_3}^2$$

Rank = 2

Index = 2

Signature = 2 - 0 = 2

Nature is positive semi definite.

**To find non zero set of values:**

Consider the transformation X = NY

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{2}} + 0 + \frac{y_3}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + 0 - \frac{y_3}{\sqrt{2}}$$

$$x_3 = 0 - y_2 - 0$$

Put $y_2 = 0$ & $y_3 = 0$

$x_1 = \frac{y_1}{\sqrt{2}};\ x_2 = \frac{y_1}{\sqrt{2}};\ x_3 = 0$

Put $y_1 = \sqrt{2}$

$x_1 = 1; x_2 = 1;\ x_3 = 0$ which makes the Quadratic equation zero.

**Example: 1.54 Reduce the Quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form through an orthogonal transformation .Find the nature rank, index, signature**

**Solution:**

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 2 + 1 + 1 = 4$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3 + 1 + 1 = -1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -4$$

Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\lambda = -1, 1, 4$$

**To find the Eigen vectors:**

Case (i) When $\lambda = -1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \ldots (1)$$

$$x_1 + 2x_2 - 2x_3 = 0 \ldots (2)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 0 \ldots (4)$$

$$x_1 + 0x_2 - 2x_3 = 0 \ldots (5)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-2+0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (7)$$

$$x_1 - 3x_2 - 2x_3 = 0 \ldots (8)$$

$$-x_1 - 2x_2 - 3x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = (0 \quad 1 \quad 1) \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_2^T X_3 = (2 \quad -1 \quad 1) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

$$X_3^T X_1 = (1 \quad 1 \quad -1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^TAN$

$$= \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Canonical form $= Y^T DY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T DY = (y_1, y_2, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -{y_1}^2 + {y_2}^2 + 4{y_3}^2$$

Rank = 3

Index = 2

Signature = 2 – 1 = 1

Nature is indefinite.

**Example: 1.55 Reduce the Quadratic form $x^2 + y^2 + z^2 2xy - 2yz - 2zx$ to canonical form through an orthogonal transformation .Find the nature rank, index, signature.**

**Solution:**

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 1 + 1 + 1 = 3$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$$

$$s_3 = |A| = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4$$

Characteristic equation is $\lambda^3 - 3\lambda^2 + 4 = 0$

$$\lambda = -1,2,2$$

**To find the Eigen vectors:**

Case (i) When $\lambda = -1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1+1 & -1 & -1 \\ -1 & 1+1 & -1 \\ -1 & -1 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 - x_2 - x_3 = 0 \ldots (1)$$

$$-x_1 + 2x_2 - x_3 = 0 \ldots (2)$$

$$-x_1 - x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-2 & -1 & -1 \\ -1 & 1-2 & -1 \\ -1 & -1 & 1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 - x_2 - x_3 = 0 \ldots (4)$$

$$-x_1 - x_2 - x_3 = 0 \ldots (5)$$

$$-x_1 - x_2 - x_3 = 0 \ldots (6)$$

Put $x_1 = 0 \Rightarrow -x_2 = x_3$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new vector orthogonal to both $X_1$ and $X_2$

$(i.e)$ $X_1^T X_3 = 0$ & $X_2^T X_3 = 0$

$$(1 \quad 1 \quad 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \ \& \ (0 \quad -1 \quad 1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$a + b + c = 0 \ldots (7)$$

$$0a - b + c = 0 \ldots (8)$$

From (7) and (8)

$$\frac{a}{1+1} = \frac{b}{-1} = -\frac{c}{-1+0}$$

$$\frac{a}{2} = \frac{b}{-1} = \frac{c}{-1}$$

$$X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ; $X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T AN$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Canonical form = $Y^T DY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T DY = (y_1, y_2, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -{y_1}^2 + 2{y_2}^2 + 2{y_3}^2$$

Rank = 3

Index = 2

Signature = 2 – 1 = 1

Nature is indefinite.

## Exercise: 1.6

1. Reduce the Quadratic form $2x_1^2+5x_2^2+3x_3^2+4x_1x_2$ to canonical form through an orthogonal transformation .Find the nature, rank, index, signature.

**Ans:** $N=\begin{pmatrix}\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\\ 0 & 1 & 0\end{pmatrix}$ $C.F={y_1}^2+3{y_2}^2+6{y_3}^2$ ; R = 3, I = 3, S = 3, N: positive definite.

2. Reduce the Quadratic form $3x_1^2-3x_2^2-5x_3^2-2x_1x_2-6x_2x_3-6x_3x_1$ to canonical form through an orthogonal transformation .Find the nature, rank, index, signature.

**Ans:** $N=\begin{pmatrix}\frac{-3}{10} & 0 & \frac{1}{\sqrt{14}}\\ 0 & \frac{3}{\sqrt{35}} & \frac{2}{\sqrt{14}}\\ \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{14}}\end{pmatrix}$ $C.F=4{y_1}^2-{y_2}^2-8{y_3}^2$ ; R = 3, I = 1,S = - 1; N: Indefinite.

3. Reduce the Quadratic form $3x_1^2+2x_2^2+3x_3^2-2x_1x_2-2x_2x_3$ to canonical form through an orthogonal transformation .Find the nature, rank, index, signature

**Ans:** $N=\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\end{pmatrix}$ $C.F={y_1}^2+3{y_2}^2+4{y_3}^2$ ; R = 3,I = 3,S = 3, N: positive definite.

4. Reduce the Quadratic form $10x_1^2+2x_2^2+5x_3^2-4x_1x_2+6x_2x_3-10x_3x_1$ to canonical form through an orthogonal transformation .Find the nature, rank, index, signature.

**Ans:** $N=\begin{pmatrix}\frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}}\\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}}\\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}}\end{pmatrix}$ $C.F=3{y_2}^2+14{y_3}^2$ ; R = 2, I = 2, S = 2, N: positive semidefinite.

5. Reduce the Quadratic form $5x_1^2+26x_2^2+10x_3^2+6x_1x_2+4x_2x_3+14x_3x_1$ to canonical form through an orthogonal transformation .Find the nature, rank, index, signature.

**Ans:** $N=\begin{pmatrix}\frac{-16}{\sqrt{378}} & \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{27}}\\ \frac{1}{\sqrt{378}} & \frac{-1}{\sqrt{14}} & \frac{5}{\sqrt{27}}\\ \frac{11}{\sqrt{378}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{27}}\end{pmatrix}$ $C.F=14{y_2}^2+27{y_3}^2$ ; R = 2, I = 2, S = 2, N: positive semidefinite.

## 1.7 CAYLEY HAMILTON THEOREM

### Cayley Hamilton Theorem

**Statement:** Every square matrix satisfies its own characteristic equation.

### Uses of Cayley Hamilton Theorem:

To calculate (i) the positive integral power of A and

(ii) the inverse of a non-singular square matrix A.

## Problems based on Cayley – Hamilton theorem

**Example: 1.56 Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation.**

**Solution:**

Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_1 = 0$$

Where $S_1 =$ sum of the main diagonal elements.

$= 1 + 1 = 2$

$S_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 + 4 = 5$

∴ The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

**To prove:** $A^2 - 2A + 5I = O$

$$A^2 = AA = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

Therefore, the given matrix satisfies its own characteristic equation.

**Example: 1.57 Verify Cayley – Hamilton theorem find $A^4$ and $A^{-1}$ when $A = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$**

**Solution:**

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - S_1\lambda^2 + S_2\lambda = 0$ where

$S_1 =$ sum of its leading diagonal elements $= 2 + 2 + 2 = 6$

$S_2 =$ sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4 - 1) + (4 - 2) + (4 - 1) = 3 + 2 + 3 = 8$$

$$S_3 = |A| = 2(4 - 1) + 1(-2 + 1) + 2(1 - 2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6 - 1 - 2 = 3$$

∴ The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$i.e., \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By Cayley-Hamilton theorem

[Every square matrix satisfies its own characteristic equation]

$(i.e.)\ A^3 - 6A^2 + 8A - 3I = 0 \qquad \dots(1)$

**Verification:**

$$A^2 = A\times A = \begin{bmatrix} -2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix} -2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} = \begin{bmatrix} 7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix}$$

$$A^3 = A\times A^2 = \begin{bmatrix} -2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix} 7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix} = \begin{bmatrix} 29 & -28 & 38\\ -22 & 23 & -28\\ 22 & -22 & 29\end{bmatrix}$$

$$\therefore\quad A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38\\ -22 & 23 & -28\\ 22 & -22 & 29\end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix}$$

$$+8\begin{bmatrix} -2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38\\ -22 & 23 & -28\\ 22 & -22 & 29\end{bmatrix} - \begin{bmatrix} 42 & -36 & 54\\ -30 & 36 & -36\\ 30 & -30 & 42\end{bmatrix} + \begin{bmatrix} 16 & -8 & 16\\ -8 & 16 & -8\\ 8 & -8 & 16\end{bmatrix} - \begin{bmatrix} 3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = O$$

**To find $A^4$:**

$(1) \Rightarrow\ A^3 - 6A^2 - 8A + 3I \quad \ldots(2)$

Multiply A on both sides, we get

$$A^4 = 6A^3 - 8A^2 + 3A = 6[6A^2 - 8A + 3I\,] - 8A^2 + 3A \text{ by } (2)$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$A^4 = 28A^2 - 45A + 18I \qquad \ldots(3)$$

$$(1) \Rightarrow\quad A^4 = 28\begin{bmatrix} 7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 196 & -168 & 252\\ -140 & 168 & -168\\ 140 & -140 & 196\end{bmatrix} - \begin{bmatrix} 90 & -45 & 90\\ -45 & 90 & -45\\ 45 & -45 & 90\end{bmatrix} + \begin{bmatrix} 18 & 0 & 0\\ 0 & 18 & 0\\ 0 & 0 & 18\end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162\\ -95 & 96 & -123\\ 95 & -95 & 124\end{bmatrix}$$

**To find $A^{-1}$:**

$(1)\times\ A^{-1} \Rightarrow A^2 - 6A + 8I - 3\ A^{-1} = O$

$3\ A^{-1} = A^2 - 6A + 8I$

$$3\ A^{-1} = \begin{bmatrix} 7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9\\ -5 & 6 & -6\\ 5 & -5 & 7\end{bmatrix} + \begin{bmatrix} -12 & 6 & -12\\ 6 & -12 & 6\\ -6 & 6 & -12\end{bmatrix} + \begin{bmatrix} 8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{bmatrix}$$

$$3\ A^{-1} = \begin{bmatrix} 3 & 0 & -3\\ 1 & 2 & 0\\ -1 & 1 & 3\end{bmatrix} \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3\\ 1 & 2 & 0\\ -1 & 1 & 3\end{bmatrix}$$

**Example: 1.58 Find $\mathbf{A^{-1}}$ if $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley- Hamilton theorem.**

**Solution:**

The characteristic equation of A is $|A - \lambda I| = 0$

$(i.e.)\ \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{sum of its leading diagonal elements}$

$= 1 + 2 + (-1) = 2$

$S_2 = \text{sum of the minors of its leading diagonal elements}$

$$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$= (-2 + 1) + (-1 - 8) + (2 + 3)$

$= (-1) + (-9) + 5 = -5$

$$S_3 = |A| = \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(-2 + 1) + 1(-3 + 2) + (3 - 4)$

$= 1(-1) + 1(-1) + 4(-1)$

$= -1 - 1 - 4 = -6$

∴ The Characteristic equation is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley Hamilton Theorem we get

[Every square matrix satisfies its own characteristic equation]

$\therefore A^3 - 2A^2 - 5A + 6I\ = O$ ... (1)

**To find $\mathbf{A^{-1}}$**

$(1) \times\ A^{-1} \Rightarrow A^2 - 2A - 5I + 6\,A^{-1} = O$

$A^2 - 2A - 5I + 6\,A^{-1} = O$

$6\,A^{-1} = -A^2 + 2A + 5I$

$A^{-1} = \frac{1}{6}[-A^2 + 2A + 5I]$ ... (2)

$A^2 = A \times A$

$$= \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2) ⇒ $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

**Example: 1.59 If $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, then find $A^n$ interms of A and I.**

**Solution:**

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$S_1 =$ sum of its leading diagonal elements

$= 1 + 2 = 3$

$S_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

Therefore, the characteristic equation of A is

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 2, \lambda = 1$$

Hence, the Eigen values of A are 1, 2.

**To find $A^n$**

When $\lambda^n$ is divided by $\lambda^2 - 3\lambda + 2$

Let the quotient be $Q(\lambda)$ and remainder be $a\lambda + b$.

$\lambda^n = (\lambda^2 - 3\lambda + 2)Q(\lambda) + a\lambda + b$ ... (1)

| when $\lambda = 1$ | when $\lambda = 2$ |
|---|---|
| $1^n = a + b$ | $2^n = 2a + b$ |

$2a + b = 2^n$ ..... (2)

$a + b = 2^n$ ..... (3)

Solving (2) and (3), we get

(2) – (3) ⇒ $a = 2^n - 1^n$

(2) – 2 × (3) ⇒ $b = -2^n + 2(1^n)$

i.e., $a = 2^n - 1^n$

$b = 2(1^n) - 2^n$

Since, $A^2 - 3A + 2I = O$ by Cayley-Hamilton Theorem

(1) ⇒ $A^n = aA + bI$

$A^n = (2^n - 1^n)A + [2(1^n) - 2^n]I$

**Example: 1.60 Use Cayley – Hamilton theorem to find the value of the matrix given by**

**(i)$f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$**

**(ii)** $\mathbf{A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I}$ **if the matrix** $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

**Solution:**

The characteristic equation of A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \text{ where}$$

$S_1$ = sum of the main diagonal elements

$= 2 + 1 + 2 = 5$

$S_2$ = sum of the minors of main diagonal elements

$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= (2 - 0) + (4 - 1) + (2 - 0) = 2 + 3 + 2 = 7$

$= (-1) + (-9) + 5 = -5$

$S_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$

$= 2(2 - 0) - 1(0 - 0) + 1(0 - 1) = 4 - 1 = 3$

Therefore, the characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By C – H theorem, we get

$$A^3 - 5A^2 + 7A - 3I = 0 \; \dots (1)$$

Let i) $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

ii) $g(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

(i)

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ & A^8 - 5A^7 + 7A^6 - 3A^5 \\ \cline{2-2} & (-) \qquad A^4 - 5A^3 + 8A^2 - 2A \\ & A^4 - 5A^3 + 7A^2 - 3A \\ \cline{2-2} & (-) \qquad A^2 + A + 1\,I \end{array}$$

$f(A) = (A^3 - 5A^2 + 7A - 3I)(A^2 + A) + A^2 + A + I$

$= O + A^2 + A + I$ by (1)

$= A^2 + A + I \qquad \dots (2)$

Now, $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$\therefore A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

(ii)

$$\begin{array}{r|l} & \mathrm{A}^5 + 8\mathrm{A} + 35\mathrm{I} \\ \hline \mathrm{A}^3 - 5\mathrm{A}^2 + 7\mathrm{A} - 3\mathrm{I} & \mathrm{A}^8 - 5\mathrm{A}^7 + 7\mathrm{A}^6 - 3\mathrm{A}^5 + \mathrm{A}^4 - 5\mathrm{A}^3 + 8\mathrm{A}^2 - 2\mathrm{A} + 1\,\mathrm{I} \\ & \mathrm{A}^8 - 5\mathrm{A}^7 + 7\mathrm{A}^6 - 3\mathrm{A}^5 \\ \hline & (-) \qquad 8\mathrm{A}^4 - 5\mathrm{A}^3 + 8\mathrm{A}^2 - 2\mathrm{A} \\ & \qquad\qquad 8\mathrm{A}^4 - 40\mathrm{A}^3 + 56\mathrm{A}^2 - 24\mathrm{A} \\ & (-) \qquad 35\mathrm{A}^3 - \mathrm{A}^2 + 22\mathrm{A} + 1\,\mathrm{I} \\ & \qquad\qquad 35\mathrm{A}^3 - 175\,\mathrm{A}^2 + 245\mathrm{A} - 105\mathrm{I} \\ & (-) \qquad 127\,\mathrm{A}^2 - 223\mathrm{A} + 106\,\mathrm{I} \end{array}$$

$$\mathrm{g}(A) = (\mathrm{A}^3 - 5\mathrm{A}^2 + 7\mathrm{A} - 3\mathrm{I})(\mathrm{A}^4 + 8\mathrm{A} + 35\mathrm{I}) + 127\mathrm{A}^2 - 223\mathrm{A} + 106\mathrm{I}$$

$$= O + 127\mathrm{A}^2 - 223\mathrm{A} + 106\,\mathrm{I}$$

$$= 127\mathrm{A}^2 - 223\mathrm{A} + 106\,\mathrm{I}$$

$$= 127\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 223\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 106\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{g}(A) = \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}$$

**Example: 1.61 Use Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$ to express as a linear polynomial in “A” is $\mathbf{A^5 - 4A^4 - 7A^3 + 11A^2 - A + 10\, I}$**

**Solution:**

Given $\mathrm{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation of A is $|\mathrm{A} - \lambda \mathrm{I}| = 0$

i.e., $\lambda^2 - \mathrm{S}_1\lambda + \mathrm{S}_2 = 0$ where

$\mathrm{S}_1 =$ sum of the main diagonal elements

$= 1 + 3 = 4$

$\mathrm{S}_2 = |\mathrm{A}| = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

$\therefore$ The characteristic equation A is $\lambda^2 - 4\lambda - 5 = 0$

By Cayley Hamilton Theorem we get

$\mathrm{A}^2 - 4\mathrm{A} + 5\,\mathrm{I} = O$ ... (1)

**To find:** (i) $\mathrm{A}^5 - 4\mathrm{A}^4 - 7\mathrm{A}^3 + 11\mathrm{A}^2 - \mathrm{A} + 10\,\mathrm{I}$

$$\begin{array}{r|l} & \mathrm{A}^3 - 2\mathrm{A} + 3\mathrm{I} \\ \hline \mathrm{A}^2 - 4\mathrm{A} + 5\,\mathrm{I} = O & \end{array}$$

$$\begin{array}{r} A^5 - 4A^4 - 7A^3 - 11A^2 - A - 10I \\ A^5 - 4A^4 - 7A^3 \qquad\qquad\qquad \\ \hline -2A^3 - 11A^2 - A \\ -2A^3 - 8A^2 - 10A \\ \hline 3A^2 - 11A - 10I \\ 3A^2 - 12A - 15I \\ \hline A + 5I \end{array}$$

$\therefore A^5 - 4A^4 - 7A^3 + 11A^2 - A + 10\,I = (A^2 + 4A - 5I)(A^3 - 2A + 3I) + A + 5I$

$= 0 + A + 5I = A + 5I$ by (1)

which is a linear polynomial in A.

**Example: 1.62 Using Cayley Hamilton theorem find $A^{-1}$ when $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$**

**Solution:**

The Characteristic equation of A is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of the main diagonal elements $= 1 + 1 + 1 = 3$

$S_2 =$ Sum of the minors of the main diagonal elements.

$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$

$= (1 - 1) + (1 - 3) + (1 - 0)$

$= 0 - 2 + 1 = -1$

$S_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$= 1(1 - 1) - 0(2 + 1) + 3(-2 - 1)$

$= 0 - 0 + 3(-3) = -9$

$\therefore$ The characteristic equation A is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley - Hamilton Theorem every square matrix satisfies its own Characteristic equation

$\therefore A^3 - 3A^2 - A + 9\,I = O$

$A^{-1} = \frac{-1}{9}[A^2 - 3A - I]$ ... (1)

$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 + 0 + 3 & 0 + 0 - 3 & 3 + 0 + 3 \\ 2 + 2 - 1 & 0 + 1 + 1 & 6 - 1 - 1 \\ 1 - 2 + 1 & 0 - 1 - 1 & 3 + 1 + 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$

$$-3A = \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix}$$

$$(1) \Rightarrow A^{-1} = \frac{-1}{9}\left[\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 0 & -9 \\ -6 & -3 & -3 \\ -3 & 3 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

**Example: 1.63 Verify Cayley- Hamilton for the matrix A=** $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

**Solution :**

Given A = $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The characteristic equation A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda S_3 = 0 \cdots (1) \quad \text{where}$$

$S_1 =$ Sum of the main diagonal elements

$$= 1 + 2 + 1 = 4$$

$S_2 =$ Sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$= (2 - 6) + (1 - 7) + (2 - 12)$$

$$= -4 - 6 - 10 = -20$$

$$S_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 1(2 - 6) - 3(4 - 3) + 7(8 - 2)$$

$$= -4 - 3(1) + 7(6)$$

$$= -4 - 3 + 42 = 35$$

$$\therefore (1) \Rightarrow \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

By Cayley –Hamilton theorem

$$(2) \Rightarrow A^3 - 4A^2 - 20A - 351 = O$$

To find $A^2$ $and$ $A^3$:

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 12 + 7 & 3 + 6 + 14 & 7 + 9 + 7 \\ 4 + 8 + 3 & 12 + 4 + 6 & 28 + 6 + 3 \\ 1 + 8 + 1 & 3 + 4 + 2 & 7 + 6 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28+ & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$A^3 - 4A^2 - 20A - 35\,I$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + \begin{bmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{bmatrix} + \begin{bmatrix} -20 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{bmatrix} + \begin{bmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ The given matrix A satisfies its own characteristic equation.

Hence, cayley Hamilton theorem is verified.

## Exercise: 1.8

1.Verify Cayley – Hamilton Theorem and find its inverse.

$(a) \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ **Ans**: $A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

$(b) \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **Ans**: $A^{-1} = \frac{1}{20}\begin{bmatrix} 7 & -2 & -3 \\ 1 & 4 & 1 \\ -2 & 2 & 8 \end{bmatrix}$

$(C) \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ **Ans**: $A^{-1} \begin{bmatrix} 1 & 0 & 1 \\ -1 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

$(d) \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ **Ans**: $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$

$(e) \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$ **Ans**: $A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$

$(f) \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$ **Ans**: $A^{-1} = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$(g)\begin{bmatrix}1 & 3 & 7\\4 & 2 & 3\\1 & 2 & 1\end{bmatrix}$ **Ans**: $A^{-1}=\frac{1}{35}\begin{bmatrix}-4 & 11 & -5\\-1 & -6 & 25\\6 & 1 & -10\end{bmatrix}$

2. If $A=\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$, then prove that $A^3-3A^2-9A-5I=0$ Hence, find $A^4$.

$$\textbf{Ans}: A^4=\begin{bmatrix}209 & 208 & 208\\208 & 209 & 208\\208 & 208 & 209\end{bmatrix}$$

3. Find $A^n$ using Cayley –Hamilton theorem, taking $A=\begin{bmatrix}7 & 2\\3 & 6\end{bmatrix}$ also find $A^3$.

$$\textbf{Ans}: A^n=\left[\frac{9^n-4^n}{5}\right]\begin{bmatrix}7 & 2\\3 & 6\end{bmatrix}+\left[\frac{9.4^n-4.9^n}{5}\right]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \text{ and } A^3=\begin{bmatrix}463 & 266\\399 & 330\end{bmatrix}$$

4. Calculate $A^4$ for the matrix $A=\begin{bmatrix}2 & 1 & 2\\0 & 2 & 3\\0 & 0 & 5\end{bmatrix}$ **Ans**: $\begin{bmatrix}16 & 32 & 577\\180 & 16 & 609\\0 & 0 & 625\end{bmatrix}$

5. Verify Cayley –Hamilton theorem for the matrix (i) $A=\begin{bmatrix}3 & -1\\-1 & 5\end{bmatrix}$ (ii) $A=\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$

6. Using cayley –Hamilton theorem, compute $A^3$ for $A=\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$ **Ans**: $\begin{bmatrix}41 & 84\\42 & 83\end{bmatrix}$

7. Given that $A=\begin{bmatrix}1 & 2 & 1\\0 & 1 & -1\\3 & -1 & 1\end{bmatrix}$, Express $A^6-5A^5+8A^4-2A^3-9A^2+35A+6I$

as a linear polynomial in A, using cayley Hamilton theorem. **Ans**: $4A+42I$

8. Obtain the matrix $A^6-25A^2+122A$ where $A=\begin{bmatrix}0 & 0 & 2\\2 & 1 & 0\\-1 & -1 & 3\end{bmatrix}$ **Ans**: $\begin{bmatrix}-34 & 0 & -20\\-20 & -54 & 0\\10 & 10 & -74\end{bmatrix}$

9. Given that $A=\begin{bmatrix}1 & 0 & 3\\2 & 1 & -1\\1 & -1 & 1\end{bmatrix}$, compute the value of

$(A^6-5A^5+8A^4-2A^3-9A^2+31A-36I)$, using Cayley – Hamilton theorem. **Ans**: $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

10. Find $A^n$, using Cayley Hamilton theorem, when $A=\begin{bmatrix}5 & 3\\1 & 3\end{bmatrix}$, Hence find $A^4$.

$$\textbf{Ans}: A^n=\left[\frac{6^n-2^n}{4}\right]\begin{bmatrix}5 & 3\\1 & 3\end{bmatrix}+\left[\frac{(3)2^n-6^n}{2}\right]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}976 & 960\\320 & 336\end{bmatrix}$$

11. Find $A^n$, using Cayley Hamilton theorem, when $A=\begin{bmatrix}7 & 3\\2 & 6\end{bmatrix}$, Also find $A^3$.

$$\textbf{Ans}: A^n=\left[\frac{9^n-4^n}{5}\right]\begin{bmatrix}7 & 3\\2 & 6\end{bmatrix}+\left[\frac{9.4^n-4.9^n}{5}\right]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}463 & 399\\266 & 330\end{bmatrix}$$