ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY
Near Anjugramam Junction, Kanyakumari Main Road, Palkulam, Variyoor P.O - 629401
Kanyakumari Dist, Tamilnadu., E-mail : admin@rcet.org.in, Website : www.rcet.org.in

DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT: PROBABILITY AND QUEUEING THEORY

SUBJECT CODE : MA8402

REGULATION : 2017

UNIT – IV : QUEUEING MODELS

MA 8402 – PROBABILITY AND QUEUEING THEORY

UNIT IV QUEUEING MODELS

Kendall's notations for queueing model:

$(a / b / c) : (d / e)$ is the Kendall's notation

Where

$a \rightarrow$ Arrival pattern (or)Probability law or distribution

$b \rightarrow$ Service time probability distribution

$c \rightarrow$ Number of servers

$d \rightarrow$ Capacity of the system

$e \rightarrow$ Service discipline.

Various queue disciplines in queueing model:

This is the manner by which customer are selected for service when a queue has formed. The most common queues discipline are

1. FIFO –First In First Out (OR) FCFS – First Come First served
2. LIFO – Last In First Out (OR) LCFS – Last Come First Served
3. SIRO – Selection In Random Order
4. PIR – Priority in Section

Transient state and Steady state of queueing system:

A queueing system is in transient state when its operating characteristics are depend on time. It is in steady state when the characteristics are independent on time.

Customers behavior:

Generally a customer behaves in the following ways

Balking: A customer who refuses to enter queueing system because the queue is too long is said to be balking.

Reneging: A customer who leaves the queue without receiving service because of too much waiting (or due to impatience) is said to have reneged.

Queue with discouragement:

If a customer is discouraged to join the queue expecting a long waiting time or having the impatience in getting the service, the queueing model is said to be the queue with discouragement.

Different types of queueing models:

Model I : (M/M/1);(∞ /FCFS) :Single server & Infinite capacity

Model II :(M/M/c);(∞ /FCFS) : Multiple server & Infinite capacity

Model III : (M/M/1);(k/FCFS) : Single server & Finite capacity

Model IV : (M/M/c);(k/FCFS) : Multiple server & Finite capacity

Notations:

1. P_n = Probability that n number of customer in the system.
2. P_0 = Probability that no customer in the system.
3. $L_s = E(N) =$ The average (Expected) number of jobs in the system.
4. $L_q = E(N-1) =$ The average (Expected) number of jobs in the queue.
5. W_s = Average (Expected) waiting time in the system.
6. W_q = Average (Expected) waiting time in the queue.
7. $\rho =$ The server is busy (or) Traffic intensity (or) Utilization factor.

8. λ' =Effective arrival rate.

Model I : (M/M/1);(∞ /FCFS) : Single server & Infinite capacity:

This model represents a queueing system with single server, Poisson arrival, exponential service time and there is no limit on the system capacity and the customers are served on a first come first served basis.

Characteristics

1. **A supermarket has 2 girls running up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in Poisson fashion at the rate of 10 per hour, find the following:**

- **(i) What is the probability of a customer have to wait for service?**
- **(ii) What is the expected percentage of idle time for each girl?**
- **(iii) What is the expected length of customer's waiting time?**
- **(iv) What is the expected number of idle girls at any time?** **(Ap/May'15)**

Solution:

Model identification: Since there are two girls and infinite capacity.

Hence this problem comes under the model (M/M/c);(∞ /FCFS).

Given data:

Arrival rate $\lambda = 10$ per hr

Service rate $\mu = \frac{1}{4}$ per min (i.e) $\mu = \frac{60}{4} = 15$ per hr.

Number of servers $c = 2$

w.k.t $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\mu c}{c!(\mu c-\lambda)}\left(\frac{\lambda}{\mu}\right)^c \right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{10}{15}\right)^n + \frac{2\times 15}{2!((2\times 15)-10)}\left(\frac{10}{15}\right)^2 \right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(0.6667)^n + \frac{30}{40}(0.6667)^2 \right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(0.6667)^n + 0.3333 \right]^{-1}$$

$$P_0 = \left[(1+0.6667)+0.3333\right]^{-1} = 2^{-1} = 0.5$$

i) The probability that a customer has to wait for the service is

$$P(N_s \geq 2) = \frac{\left(\frac{10}{15}\right)^2}{2!\left(1-\frac{10}{15\times 2}\right)}(0.5) \qquad \because P(N_s \geq c) = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)}P_0$$

$$P(N_s \geq 2) = \frac{(0.6667)^2}{1.3333}(0.5) = \frac{1}{6} = 0.1667$$

ii) Fraction of time that a girl is busy $= \rho = \frac{\lambda}{c\mu} = \frac{10}{2\times 15} = \frac{1}{3}$

$\therefore$ Fraction of time when a girl is idle = 1- Fraction of time that a girl is busy

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ Percentage of idle time of for each girl = $\frac{2}{3}\times 100 = 67\%$

iii) Expected waiting time of customer = $W_s = \frac{L_q}{\lambda} + \frac{1}{\mu}$

Where $L_q = \frac{1}{c.c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left(1-\frac{\lambda}{\mu c}\right)^{-2} P_0$

$$= \frac{1}{(2)2!}(0.6667)^3\left(1-\frac{10}{30}\right)^{-2}(0.5) = 0.083$$

$\therefore W_s = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{0.083}{10} + \frac{1}{15} = 0.0083$ hrs

iv) The expected no. of idle girl:

E(no. of idle girl) = ?

No.of idle girls:	2	1	0
Probability	P_0	P_1	P_2

E(idle time for each girl)=2 P_0+1 P_1+0 P_2

Now, P_0=0.5

w.k.t $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0,\ 0 \le n < c$

$$P_1 = \frac{1}{1!}\left(\frac{10}{15}\right)^1 (0.5) = 0.333 = \frac{1}{3}$$

$$P_2 = \frac{1}{2!}\left(\frac{10}{15}\right)^2 (0.5) = 0.1111 = \frac{1}{9}$$

E(idle no of girl)=2 $P_0 + 1 P_1 + 0 P_2 = 2\times\frac{1}{9} + 1\times\frac{1}{3} + 0 = \frac{5}{9}$

$\therefore$ The expected no. of idle girl = 0.5556

2. **A small mail –order business has one telephone line and a facility for call waiting for two additional customers. Orders arrive at the rate of one per minute and each order requires 2 minutes and 30 seconds to take down the particulars. What is the expected number of calls waiting in the queue? What is the mean waiting time in the queue?**

(Ap/May'15)

Solution:

Model identification:

since there is only one telephone line and the capacity of the system is finite. Hence this problem comes under the model (M/M/1);(k /FCFS).

Given Data:

Arrival rate : $\lambda = 1/$ min

Service rate: $\frac{1}{\mu}=\frac{5}{2} \Rightarrow \mu=\frac{2}{5}$ per min , k=3 $\rho=\frac{\lambda}{\mu}=\frac{1}{2/5}=\frac{5}{2}=2.5$

The expected number of calls waiting in the queue = $L_q = L_s - \frac{\lambda'}{\mu}$

Where $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}, if\ \lambda \neq \mu$ and $\lambda' = \mu\left(1-P_0\right)$

$$L_s = \frac{2.5}{1-2.5} - \frac{4(2.5)^4}{1-(2.5)^4} = 0.3333 + 4.105 = 4.4383$$

$$P_0 = \frac{(1-\rho)}{1-\rho^{k+1}}, \text{ if } \lambda \neq \mu$$

$$P_0 = \frac{1-2.5}{1-(2.5)^4} = \frac{-1.5}{-38.06} = 0.0394$$

$$\lambda' = \frac{2}{5}\left(1-0.0394\right) = 0.3842$$

The expected number of calls waiting in the queue = $L_q = 4.4383 - \frac{0.3842}{0.4} = 3.4778$

The mean waiting time in the queue= $W_q = \frac{L_s}{\lambda'} - \frac{1}{\mu} = \frac{4.4383}{0.3842} - 2.5 = 11.55 - 2.5 = 9.052$

3. Find the system size probabilities for an $M/M/S : FCFS/\infty/\infty$ queueing system under steady state conditions. Also obtain the expression for average number of customers in the system and waiting time of a customer in the system. (Nov/Dec'11), (Ap/May'11)

Solution:

Assume $S = c \Rightarrow \left(M/M/c\right);(\infty/FCFS)$

This model represents a queueing system with poisson arrivels, exponential service time, multiple servers, infinite capacity and FCFS queue service from a single queue.

If $n < c$, then only n of the c servers will be busy and others are idle and hence mean service rate will be $n\mu$.

If $n \geq c$, all c servers will be busy and hence the mean service rate is $c\mu$

$$\therefore \mu_n = \begin{cases} n\mu, & 0 \leq n < c \\ c\mu, & n \geq c \end{cases} \text{ and } \lambda = \lambda_n\ \forall n$$

By birth and death process

$$P_n = \frac{\lambda_0\lambda_1\lambda_2\lambda_3....\lambda_{n-1}}{\mu_1\mu_2\mu_3\mu_4....\mu_n}P_0$$

Case i: When $0 \leq n < c$

$$P_n = \frac{\lambda\lambda\lambda\lambda\cdots\lambda\ (n\text{ times})}{1\mu\, 2\mu 3\mu\, 4\mu....\text{n }\mu}P_0$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)P_0$$

Case ii: when $n \geq c$

$$P_n = \frac{\lambda\lambda\lambda\lambda......(n\text{ times})}{\mu_1\mu_2....\mu_c....\mu_{c+1}....\mu_n}P_0$$

$$= \frac{\lambda^n}{\{1\mu\,2\mu\,3\mu...(c-1)\mu\}\{c\mu\,c\mu...(c-(c-1))\text{ times}\}}P_0$$

$$= \frac{\lambda^n}{(c-1)!\mu^{c-1}c^{n-c+1}\mu^{n-c+1}}P_0$$

$$= \frac{\lambda^n}{(c-1)!c^{n-c+1}\mu^{n-c+1+c-1}}P_0$$

$$= \frac{\lambda^n}{c(c-1)!c^{n-c}\mu^n}P_0$$

$$P_n = \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad n \ge c \qquad \because n! = n(n-1)!$$

$$P_n = \begin{cases} \dfrac{1}{c!c^{n-c}}\left(\dfrac{\lambda}{\mu}\right)^n P_0 & , n \ge c \\ \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n P_0 & , 0 \le n < c \end{cases}$$

To find P_0

Since $\sum_{n=0}^{\infty} P_n = 1$

$$\Rightarrow \sum_{n=0}^{c-1} P_n + \sum_{n=c}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=c}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow P_0\left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n\right] = 1 \text{---------} (1)$$

Consider

$$\Rightarrow \sum_{n=c}^{\infty} \frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n = \frac{1}{c!c^{-c}}\sum_{n=c}^{\infty}\left(\frac{\lambda}{\mu c}\right)^n$$

$$= \frac{1}{c!c^{-c}}\left[\left(\frac{\lambda}{\mu c}\right)^c + \left(\frac{\lambda}{\mu c}\right)^{c+1} + \left(\frac{\lambda}{\mu c}\right)^{c+2} + \dots\right]$$

$$= \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left[1 + \left(\frac{\lambda}{\mu c}\right)^1 + \left(\frac{\lambda}{\mu c}\right)^2 + \left(\frac{\lambda}{\mu c}\right)^3 + \dots\right]$$

$$= \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left[1 - \left(\frac{\lambda}{\mu c}\right)\right]^{-1}$$

	$$= \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^{c}\left[\frac{\mu c-\lambda}{\mu c}\right]^{-1}$$ $$\Rightarrow \sum_{n=c}^{\infty}\frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^{n} = \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^{c}\left[\frac{\mu c}{\mu c-\lambda}\right]$$ (1) $$\Rightarrow P_0 = \left[\sum_{n=0}^{c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n} + \sum_{n=c}^{\infty}\frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^{n}\right]^{-1}$$ $$\Rightarrow P_0 = \left[\sum_{n=0}^{c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n} + \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^{c}\left(\frac{\mu c}{\mu c-\lambda}\right)\right]^{-1}$$ To find the average number of customer in the system: (L_s) $$L_s = L_q + \frac{\lambda}{\mu}$$ Where L_q = Average number of customer in the queue $$L_q = \sum_{n=c}^{\infty}(n-c)P_n$$ $$L_q = \sum_{n=c}^{\infty}(n-c)\frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^{n}P_0$$ $$= \frac{1}{c!c^{-c}}P_0\sum_{n=c}^{\infty}(n-c)\left(\frac{\lambda}{\mu c}\right)^{n}$$ $$= \frac{1}{c!c^{-c}}P_0\left[0+(c+1-c)\left(\frac{\lambda}{\mu c}\right)^{c+1}+(c+2-c)\left(\frac{\lambda}{\mu c}\right)^{c+2}+(c+3-c)\left(\frac{\lambda}{\mu c}\right)^{c+3}+\dots\right]$$ $$= \frac{1}{c!c^{-c}}P_0\left(\frac{\lambda}{\mu c}\right)^{c+1}\left[1+(2)\left(\frac{\lambda}{\mu c}\right)^{1}+(3)\left(\frac{\lambda}{\mu c}\right)^{2}+\dots\right]$$ $$= \frac{1}{c!c^{-c}}P_0\left(\frac{\lambda}{\mu c}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2}$$ $$L_q = \frac{1}{c\,c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2}P_0$$ $$\therefore L_s = L_q + \frac{\lambda}{\mu} = \frac{1}{c\,c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2}P_0 + \frac{\lambda}{\mu}$$
4.	**In a cinema theatre people arrive to purchase tickets at the average rate of 6 per minute and it takes 7.5 seconds on the average to purchase a ticket. If a person arrives just 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket.** **i) Can he expect to be seated for the start of the picture?** **ii) What is the probability that he will be seated when the film starts?** **iii) How early must he arrive in order to be 99% sure of being seated for the start of the picture? (Nov/Dec'14)** **Solution:**

Model identification:
since there is only one counter and the arrival of persons is infinite, capacity of the system is infinie. Hence this problem comes under the model (M/M/1);(∞ /FCFS).

Given Data:

Arrival rate $\lambda = 6$ per min

Service rate $\mu = \frac{1}{7.5}$ per sec $\Rightarrow \mu = \frac{60}{7.5} = 8$ per min

i) Expected total time required to purchase the ticket and to reach the seat

=waiting time in the system + time to reach the seat = $W_s + 1.5$.

Where $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} = 0.5$

Expected total time required to purchase the ticket and to reach the seat =0.5+1.5=2 min

ii) P(he will be the seated for the start of the picture)

=P(Total time < 2 min)

$$= P\left(W_s < \frac{1}{2}\right) = 1 - P\left(W_s > \frac{1}{2}\right) = 1 - e^{-(\mu-\lambda)t} = 1 - e^{-(8-6)\frac{1}{2}} = 0.63$$

iii) Suppose t minutes be the time of arrival so that he is seated 99%, then

$P(W \le t) = 0.99$

$\Rightarrow 1 - P(W > t) = 0.99$

$\Rightarrow P(W > t) = 1 - 0.99 = 0.01$

$e^{-(\mu-\lambda)t} = 0.01 \Rightarrow e^{-2t} = 0.01 \Rightarrow -2t = \log(0.01) \Rightarrow t = 2.3$

This is the waiting time in the system. that is to purchase ticket.

He takes 1.5 minutes to reach the seat after purchasing ticket.

$\therefore$ total time = 2.3+1.5=3.8

Hence he must arrive atleast 3.8 minutes earlier so as to be 99% sure of seeing the start of the film.

5. **There are three typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour,**

(a) What fraction of the time all the typists will be busy?

(b) What is the average number of letters waiting to be typed?

(c) What is the average time a letter has to spend for waiting and for being typed.

(d) What is the probability that a letter will take longer than 20 min waiting to be typed and being typed? Assume that arrival and service rates follow poisson distribution.

(May-June'13), (May/June'12), (Nov/Dec'11), (Nov/Dec'10)

Solution:

This is an (M / M / c) : (∞ / FCFS) model.

$\lambda = 15$ / hr. & $\mu = 6$ / hr, c = 3, $\therefore \frac{\lambda}{\mu} = 2.5$ & $\rho = \frac{\lambda}{c\mu} = \frac{2.5}{3} = 0.833$

(a) All the typists will be busy if there are at least 3 customers (letters) in the system

$p(n \ge 3) = p(3) + p(4) + p(5) + \ldots = 1 - [p_0 + p_1 + p_2]$

$$\Rightarrow P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!c^{-c}}\left(\frac{\lambda}{\mu c}\right)^c\left(\frac{\mu c}{\mu c - \lambda}\right)\right]^{-1} = 0.0449$$

$$P_1 = \frac{\lambda}{1!\,\mu} P_0 = 2.5P_0, \quad P_2 = \frac{\lambda^2}{2!\,\mu^2} P_0 = \frac{1}{2}(2.5)^2 P_0$$

	$P(n\geq 3) = 1 - [1 + 2.5 + (2.5^2/2)].(.0449) = 1-0.2974625 = 0.7025375 = 0.7025$ (b) Waiting to be typed (queue) (c) $L_q = \frac{1}{c\,c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left[1-\frac{\lambda}{\mu c}\right]^{-2} P_0$ $L_q = \frac{1}{3\times 6}(2.5)^4 P_0 = 3.5078$ (c) $W_s = \frac{1}{\lambda}\left[L_q + \frac{\lambda}{\mu}\right] = \frac{1}{15}[3.5078 + 2.5] = 0.4005$ hr. $d)\ P(W>t) = e^{-\mu t}\left\{1+\frac{\left(\frac{\lambda}{\mu}\right)^s\left[1-e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right]P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)\left(s-1-\frac{\lambda}{\mu}\right)}\right\}$ $d)\ P(W>t) = e^{-6\times\frac{1}{3}}\left\{1+\frac{(2.5)^3\left[1-e^{(-2)(-0.5)}\right](0.0449)}{6\left(1-\frac{2.5}{3}\right)(-0.5)}\right\} = 0.4616$
6.	**A TV repairman finds that the time spend on his job has an exponential distribution with mean 30 minutes. If he repair sets in the order in which they come in and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 hour day.** **a) Find the repairman's expected idle time on each day?** **b) How many jobs are ahead of average set just brought? (Nov/Dec'13) asst(May/June'12)** **Solution:** Model identification: since there is only one repair man and the capacity of the system is infinity. Hence this problem comes under the model (M/M/1);(∞/FCFS). Given Data: Arrival rate $\lambda = 10$ per 8hr day Service rate $\frac{1}{\mu} = 30 \Rightarrow \mu = \frac{1}{30}$ per min $\Rightarrow (i.e)\ \mu = 8\times 2 = 16$ per 8hr day i) The repairman's idle time = $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{10}{16} = \frac{6}{16} = \frac{3}{8}$/day The Expected idle time = $8\times\frac{3}{8} = 3\ hrs$ ii) The average number of jobs in the system = $L_s = \frac{\lambda}{\mu-\lambda} = \frac{10}{16-10} = 1.667$ 2 jobs. **Another method:**

Arrival rate $\lambda = \frac{10}{8} = \frac{5}{4}$ per hr

Service rate $\mu = \frac{1}{30}$ per min

$(i.e)\ \mu = \frac{60}{30} = 2\ /\ hr$

i) The repairman's idle time $= 1 - \frac{\lambda}{\mu} = 1 - \frac{10}{16} = \frac{6}{16} = \frac{3}{8}$

The Expected idle time each day $= 8 \times \frac{3}{8} = 3\ hrs$

ii) Number of the jobs ahead of the average set brought in = The average number of jobs in the system

$$= L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{5}{4}}{2 - \frac{5}{4}} = 1.667 \quad 2 \text{ jobs.}$$

7. **A group of engineers has two terminals available to aid their calculations. The average computing job requires 20 minutes of terminal time and each engineer requires some computation one in half an hour. Assume that these are distributed according to an exponential distribution. If there are 6 engineers in the group, find**

a) the expected number of engineers waiting to use the terminals in the computing center.

b) the total time lost per day.

Solution:

Since there are 2 terminals, also since there are 6 engineers in the group, the capacity of the system is finite. Hence this problem comes under the model (M/M/c);(k/FCFS)

Given data:

Arrival rate $\lambda = \frac{1}{1/2} = 2$ per hr

Service rate $\mu = \frac{1}{20}$ per min (i.e) $\mu = \frac{60}{20} = 3$ per hr.

Number of servers $c = 2$

Capacity $= k = 6$

Expected number of engineers waiting to use in the computing center $= L_s$

$$L_s = L_q + \frac{\lambda'}{\mu}$$

Where $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c + \sum_{n=c}^{k} \left(\frac{\lambda}{\mu c} \right)^{n-c} \right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!} \left(\frac{2}{3} \right)^n + \frac{1}{2!} \left(\frac{2}{3} \right)^2 + \sum_{n=2}^{6} \left(\frac{2}{3 \times 2} \right)^{n-2} \right]^{-1}$$

	$= \left[1 + \frac{2}{3} + \frac{1}{2} \times \left(\frac{2}{3}\right)^2 \left\{1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4\right\}\right]^{-1} = 0.5003$ And $\rho = \frac{\lambda}{\mu c} = \frac{1}{3}$ $L_q = \left(\frac{\lambda}{\mu}\right)^c \frac{\rho}{c!(1-\rho)^2}\left\{1 - \rho^{k-c} - (k-c)(1-\rho)\rho^{k-c}\right\} P_0$ $L_q = \left(\frac{2}{3}\right)^2 \frac{\frac{1}{3}}{2!\left(1-\frac{1}{3}\right)^2}\left\{1 - \left(\frac{1}{3}\right)^{6-2} - (6-2)\left(1-\frac{1}{3}\right)\left(\frac{1}{3}\right)^{6-2}\right\}(0.5003) = 0.0796$ Effective arrival rate $\lambda' = \mu\left[c - \sum_{n=0}^{c-1}(c-n)P_n\right] = 3\left[2 - \sum_{n=0}^{1}(2-n)P_n\right]$ $\lambda' = 3\left[2 - (2P_0 + P_1)\right]$ where $P_1 = \left(\frac{\lambda}{\mu}\right) P_0$ $\lambda' = 3\left[2 - \left(2 \times 0.5003 + \frac{2}{3} \times 0.5003\right)\right] = 1.9976$ Expected number of engineers waiting to use in the computing center = L_s $L_s = L_q + \frac{\lambda'}{\mu} = 0.0796 + \frac{1.9976}{3} = 0.7455$
8.	**Consider a single server queueing system with Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours and the maximum permissible number of calling units in the system is two. Find the steady state probability distribution of the number of calling units in the system and the expected number of calling units in the system.** **Solution:** <u>Model identification</u>: since there is only one server and the maximum number of calling source is 2, capacity of the system is finite. Hence this problem comes under the model (M/M/1);(k/FCFS). <u>Given Data:</u> Arrival rate $\lambda = 3$per hr Service rate $\mu = \frac{1}{0.25} = \frac{1}{1/4}$ per hr (i.e) $\mu = 4$ per hr Capacity of the system $k = 2$ $\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$ i) Steady state Probability distribution of the number of calling units is $P_n, n \geq 0$ $(i.e)$ To find P_0, P_1, P_2 $P_n = \frac{(1-\rho)\rho^n}{1-\rho^{k+1}} = \frac{(1-0.75)(0.75)^n}{1-0.75^3} = (0.4324)(0.75)^n$ $P_0 = (0.4324)(0.75)^0 = 0.4324$ $P_1 = (0.4324)(0.75)^1 = 0.3243$

$P_2 = (0.4324)(0.75)^2 = 0.2432$

ii) The average number of calling units in the system is

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} \quad \text{since } \lambda \neq \mu \qquad \text{(or) } L_s = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$L_s = \frac{0.75}{1-0.75} - \frac{(2+1)(0.75)^3}{1-(0.75)^3} = 0.81$$

Another method to find L_s:

$$L_s = \sum_{n=0}^{k} nP_n = \sum_{n=0}^{2} n(0.43)(0.75)^n = 0.43\left[0+0.75+2(0.75)^2\right] = 0.81$$

9. **A bank has two tellers working on savings accounts. The first teller handles withdrawals only while the second teller handles deposits only. It has been found that the service time distribution for the deposits and withdrawals both is exponential with mean service time 3 minutes per customer. Depositors are found to arrive in a Poisson fashion throughout the day with mean arrival rate 16 per hour. Withdrawers also arrive in a Poisson fashion with mean arrival rate of 14 per hour.**

i) What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits?

ii) What would be the effect if this could only be accomplished by increasing the service time to 3.5 minutes?

Solution:

Case(i) :Given $\frac{1}{\mu} = 3\text{min} \Rightarrow \mu = 20 / hr$

Average waiting time in the queue	Depositors	Withdrawers
When there is a separate channel then for the (i)depositors $\lambda_1 = 16 / hr$ (ii)withdrawers $\lambda_2 = 14$ / hr	$W_q = \frac{\lambda_1}{\mu(\mu-\lambda_1)} = \frac{16}{20(20-16)} = \frac{1}{5} hr \text{ (or) } 12 \text{ min}$	$W_q = \frac{\lambda_2}{\mu(\mu-\lambda_2)} = \frac{14}{20(20-14)} = \frac{7}{60} hr \text{ (or) } 7 \text{ min}$
If both tellers do the service then s=2, $\mu = 20 / hr$ $\lambda = \lambda_1 + \lambda_2 = 30 / hr$	$W_q = \frac{1}{\mu}.\frac{1}{s.s!}.\frac{\left(\frac{\lambda}{\mu}\right)^s.P_0}{\left(1-\frac{\lambda}{\mu s}\right)^2} = \frac{1}{20}.\frac{1}{2.2}.\frac{(1.5)^2}{(1-.75)^2}.\frac{1}{7} = \frac{0.45}{7}$ h or 3.86 min Where $p_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n\,!}\frac{\lambda^n}{\mu^n} + \frac{\lambda^s}{\mu^s\, s\,!}\frac{1}{\left(1-\frac{\lambda}{s\mu}\right)}\right]^{-1}$	

$$= \left[1 + 1.5 + \frac{(1.5)^2}{2 \times 0.25}\right]^{-1} = \frac{1}{7}$$

Hence if both tellers do both types of service, the customers get benefited as their waiting time is considerably reduced.

Case (ii): If both tellers do both types of service with increased service time then

s=2, $\mu = \frac{60}{3.5} = \frac{120}{7} / hr$, $\lambda = \lambda_1 + \lambda_2 = 30 / hr$

$$P_0 = \left[1 + 1.75 + \frac{(1.75)^2}{2\,x\frac{1}{8}}\right]^{-1} = \frac{1}{15}$$

$$W_q = \frac{1}{\mu} \cdot \frac{1}{s.s!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^s . P_0}{\left(1 - \frac{\lambda}{\mu s}\right)^2} = \frac{7}{120} \cdot \frac{1}{2.2} \cdot \frac{(1.75)^2}{\left(1 - \frac{7}{8}\right)^2} \cdot \frac{1}{15} = \frac{2.86}{15} \text{ h or } 11.44 \text{ min}$$

So, if this arrangement is adopted, withdrawers stand to lose as their waiting time is increased considerably and depositors get slightly benefited.

10. **A one person barber shop has 6 chairs to accommodate people waiting for a haircut. Assume that customers who arrive when all the 6 chairs are full leave without entering the barber shop. Customers arrive at the rate of 3 per hour and spend an average of 15 minutes in the barber's chair. Compute**

1) the probability that a customer can go to the barber's chair without waiting.

2) the average waiting time in the queue and in the system.

3) the average number of customers in the system and in the queue.

4) the probability that there are seven customers in the system. **(May/June 2016)**

Solution:

Model identification:

since there is only one barber and the capacity of the system is finite. Hence this problem comes under the model (M/M/1);(k /FCFS).

Given Data:

Arrival rate $\lambda = 3/hr$

Service rate $\mu = \frac{1}{15} \times 60 = 4 / hr$ per min , k=6+1=7 $\rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$

1) The probability that a customer can go to the barber's chair without waiting

$$= P_0 = \frac{(1-\rho)}{1-\rho^{k+1}}, \text{ if } \lambda \neq \mu$$

$$P_0 = \frac{1 - \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^8} = 0.2778$$

	2) The average waiting time in the queue $W_S = \frac{L_S}{\lambda'}$ Where $L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}, if\ \lambda \neq \mu$ and $\lambda' = \mu(1-P_0)$ $\lambda' = 4(1-0.2778) = 2.89$ $L_s = \frac{0.75}{1-0.75} - \frac{8(0.75)^8}{1-(0.75)^8} = 3 - 0.89 = 2.11$ The average waiting time in the System $W_S = \frac{2.11}{2.89} = 0.73$ The average waiting time in the System $W_q = \frac{L_S}{\lambda'} - \frac{1}{\mu} = 0.73 - \frac{1}{3} = 0.417$ 3) The average number of customers in the system= $L_s = 2.11 \cong 2$ The average number of customers in the queue = $L_q = L_s - \frac{\lambda'}{\mu} = 2.11 - \frac{2.89}{4} = 1.387$ 4) The probability that there are seven customers in the system $P_n = \rho^n \left(\frac{(1-\rho)}{1-\rho^{k+1}}\right) = \rho^n P_0$ if $\lambda \neq \mu$ $P_7 = (0.75)^7 (0.2778) = 0.0371$
11.	**Derive $L_s, L_q, W_s \& W_q$ for queues with impatience customer, where the arrival rate is inversely propotional to the number of customers in the system.** **Solution:** **Take** $\lambda_n = b\ \lambda_n = \frac{\lambda}{n+1}$, $n = 0, 1, 2, \ldots$ **And** $\mu_n = \mu$, $n = 1, 2, 3, \ldots$ $P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}}$ $= \frac{1}{1 + \frac{\left(\frac{\lambda}{1}\right)}{\mu} + \frac{\left(\frac{\lambda}{1}\right)\left(\frac{\lambda}{2}\right)}{\mu\mu} + \frac{\left(\frac{\lambda}{1}\right)\left(\frac{\lambda}{2}\right)\left(\frac{\lambda}{3}\right)}{\mu\mu\mu} + \ldots}$

$$= \frac{1}{1 + \frac{\left(\frac{\lambda}{\mu}\right)}{1!} + \frac{\left(\frac{\lambda}{\mu}\right)^2}{2!} + \frac{\left(\frac{\lambda}{\mu}\right)^3}{3!} + \ldots}$$

$$= \frac{1}{e^{\left(\frac{\lambda}{\mu}\right)}} \quad where,\ \rho = \frac{\lambda}{\mu} < 1$$

$$P_0 = e^{-\rho}\ where,\ \rho = \frac{\lambda}{\mu} < 1$$

Now,

$$P_n = P_0 \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n},\ n = 1, 2, 3, \ldots.$$

$$P_n = P_0 \rho^n \left(1 \times \frac{1}{2} \times \frac{1}{3} \times \ldots \times \frac{1}{n}\right) = e^{-\rho} \frac{\rho^n}{n!},\ n = 0, 1, 2, 3, \ldots$$

$$\therefore L_s = \sum_{n=1}^{\infty} n P_n = \rho = \frac{\lambda}{\mu}$$

$$\therefore L_q = \sum_{n=1}^{\infty} (n-1) P_n = \frac{\lambda}{\mu} + e^{\frac{\lambda}{\mu}} - 1$$

By Little's formula,

$$W_s = \frac{L_s}{\lambda} = \frac{\frac{\lambda}{\mu}}{\mu} = \frac{\lambda}{\mu^2}$$

$$W_s = \frac{L_q}{\lambda} = \frac{\frac{\lambda}{\mu} + e^{\frac{\lambda}{\mu}} - 1}{\lambda} = \frac{\rho + e^{\rho} - 1}{\lambda}$$

12. Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min. between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.

(a) Find the average number of persons waiting in the system.

(b) What is the probability that a person arriving at the booth will have to wait in the queue?

(c) What is the probability that it will take him more than 10 min. altogether to wait for the phone and complete his call?

(d) Estimate the fraction of the day that the phone will be in use.

This is an infinite queueing model with single server

Arrivalrate $\lambda = \frac{1}{12}$ / min. , Service rate $\mu = \frac{1}{4}$ / min.

(a) $L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} = 0.5$ customer

(b) $P(n > 0) = 1 - P(n = 0) = 1 - P$ (no customer in the system) $= 1 - p_0$

	$= 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$ (c) $P(W_s > 10) = e^{-(\mu - \lambda)10} = e^{-5/3} = 0.1889$ (d) P(phone will be idle) = P(n = 0) = $P_o = 1 - \frac{\lambda}{\mu} = \frac{2}{3}$ $\therefore$ P (Phone will be in use) $= 1 - \frac{2}{3} = \frac{1}{3}$
13.	**Customers arrive at a one man barber shop according to a poisson process with a mean inter arrival time of 20 min. Customers spend an average of 15 min. in the barber chair, then** **(a) What is the probability that a customer need not wait for a hair cut ?** **(b) What is the expected no. of customers in the barber shop and in the queue?** **(c) How much time can a customer expect to spend in the barber shop?** **(d) Find the average time that the customers spend in the queue.** **(e) What is the probability that there will be 6 or more customers waiting for service?** This is an infinite queueing model with single server $\lambda = \frac{1}{20}$/ min. and $\mu = \frac{1}{15}$/ min. (a) $p(n = 0) = 1 - \frac{\lambda}{\mu} = 1 - \frac{\frac{1}{20}}{\frac{1}{15}} = \frac{1}{4}$, (b) $L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/20}{1/15 - 1/20} = 3$, $L_q = L_s - \frac{\lambda}{\mu} = 3 - \frac{3}{4} = \frac{9}{4} = 2.2$ (c) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{15} - \frac{1}{20}} = 60 \text{ min.}$ (d) $W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\frac{1}{20}}{\frac{1}{15}\left(\frac{1}{15} - \frac{1}{20}\right)} = 45$ min. (e) $p(n \geq 6) = \left(\frac{\lambda}{\mu}\right)^6 = \left(\frac{1/20}{1/15}\right)^6 = 0.1779$
14	**A two person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all 5 chairs are full, leave without entering shop . Customers arrive at average rate of 4 / hr and spend average of 12 min. in the barber chair. Compute** $p_0, p_1, p_7, \text{ and } L_q$ This is an (M/ M/ 2) : (7 / FIFO) $\lambda = 4/\text{hr}, \ \mu = \frac{1}{12}/\text{min.} = 5/\text{hr}, \quad s = 2, \ k = 7, \ \frac{\lambda}{\mu} = \frac{4}{5} = 0.8, \quad \rho = \frac{\lambda}{s\mu} = \frac{4}{2(5)} = 0.4$

<table>
<tr><td></td><td>

(a) $\frac{1}{p_0} = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\frac{\lambda^n}{\mu^n} + \frac{\lambda^s}{\mu^s s!}\sum_{n=s}^{k}\left(\frac{\lambda}{s\mu}\right)^{n-s}\right] = \left[\sum_{n=0}^{1}\frac{1}{n!}\frac{\lambda^n}{\mu^n} + \frac{\lambda^2}{\mu^2 2!}\sum_{n=2}^{7}\left(\frac{\lambda}{s\mu}\right)^{n-2}\right]$

$= \left[1+(0.8)+\frac{1}{2}(0.8)^2\left((0.4)^0+(0.4)^1+(0.4)^2+(0.4)^3+(0.4)^4+(0.4)^5\right)\right] = 2.311488$

$\therefore p_0 = \frac{1}{2.311488} = 0.4289$

(b) $p_1 = \frac{1}{1!}\left(\frac{\lambda}{\mu}\right)^1 p_0 = 0.3431$

(c) $p_7 = \frac{1}{s^{n-s}\, s!}\left(\frac{\lambda}{\mu}\right)^7 p_0 = \frac{1}{2^5\, 2!}(0.8)^7(0.4289) = 0.0014$

(d) $L_q = p_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!\,(1-\rho)^2}\left[1-\rho^{k-s}-(k-s)(1-\rho)\rho^{k-s}\right] = 0.1462$

</td></tr>
<tr><td>15</td><td>

Derive p_0, L_s, L_q, W_s, W_q for $(M/M/1):(\infty/FIFO)$ queueing model.

By birth and death process $p_n = \frac{\lambda_0\lambda_1\lambda_2\ldots\ldots\lambda_{n-1}}{\mu_0\mu_1\mu_2\ldots.\mu_{n-1}} p_0 = \frac{\lambda\,\lambda\,\lambda\ldots\ldots n \text{ times}}{\mu\,\mu\,\mu\ldots\ldots n \text{ times}} p_0$

$\lambda_n = \lambda \quad \forall\ n \quad \& \quad \mu_n = \mu \quad \forall\ n$

$= \frac{\lambda^n}{\mu^n} p_0 = \left(\frac{\lambda}{\mu}\right)^n p_0$

$p_0 + p_1 + p_2 + \cdots = 1$

$\Rightarrow p_0 + \frac{\lambda}{\mu}p_0 + \left(\frac{\lambda}{\mu}\right)^2 p_0 + \left(\frac{\lambda}{\mu}\right)^3 p_0 + \cdots = 1$

$\Rightarrow p_0\left(1+\left(\frac{\lambda}{\mu}\right)+\left(\frac{\lambda}{\mu}\right)^2+\left(\frac{\lambda}{\mu}\right)^3+\cdots\right) = 1$

$\Rightarrow p_0\left(1-\frac{\lambda}{\mu}\right)^{-1} = 1 \Rightarrow p_0 = 1-\frac{\lambda}{\mu}$

To find L_s**:**

$L_s = E(N) = \sum n\, p_n = 0.p_0 + 1.p_1 + 2.p_2 + \ldots\ldots = 1.\frac{\lambda}{\mu}p_0 + 2.\left(\frac{\lambda}{\mu}\right)^2 p_0 + 3\left(\frac{\lambda}{\mu}\right)^3 p_0 + \ldots\ldots$

$= p_0 .\frac{\lambda}{\mu}\left(1+2.\frac{\lambda}{\mu}+3.\left(\frac{\lambda}{\mu}\right)^2+\ldots\ldots\right) = p_0 .\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)^{-2} = \left(1-\frac{\lambda}{\mu}\right).\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)^{-2} = \frac{\lambda}{\mu-\lambda}$

To find L_q, W_s, W_q

Using Litttle's formula

$L_q = L_s - \frac{\lambda}{\mu} = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)}$, $W_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$

</td></tr>
</table>

16	**A car service station has 2 bays offering service simultaneously. Because of space constraints, only 4 cars are accepted for servicing. The arrival pattern follows poison distribution with 12cars per day. The service time in both the bays is exponentially distributed with μ=8carsperdayperbay .Find the average number of cars in the service station, the average number of cars waiting for service and the average time a car spends in the system.** This is a multiple server model with finite capacity. Arrivalrateλ=12/day and Service rate μ=8/day ,S=2,K=4 $p_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{k}\left(\frac{\lambda}{\mu}\right)^{n-s} \right]^{-1} = \left[1 + \frac{1.5}{1} + \frac{1}{2}(1.5)^2\left[1+(0.75)+(0.75)^2\right]\right]^{-1}$ $P_0 = 0.1960$, $\rho = \frac{\lambda}{s\mu}$, $L_q = P_0\left(\frac{\lambda}{\mu}\right)^s \frac{\rho}{s!(1-\rho)^2}\left[1-\rho^{k-s}-(k-s)(1-\rho)\rho^{k-s}\right]$ $L_q = 0.1960(1.5)^2\left(\frac{0.75}{2(0.25)^2}\right)\left[1-(0.75)^2-2(0.25)(0.75)^2\right] = 0.4134$ car. $L_s = L_q + s\sum_{n=0}^{s-1}(s-n)p_n = 0.4134 + 2\sum_{n=0}^{1}(2-n)p_n = 2.4134 - 2(2p_0+p_1) = 1.73$ cars. $W_s = \frac{L_s}{\lambda'}$ where $\lambda' = \mu\left[s - \sum_{n=0}^{s-1}(s-n)\,p_n\right] = 8\left[2-(2p_0+p_1)\right] = 10.512$ $W_s = \frac{1.73}{10.512} = 0.1646$ day Average time that a car has to spend in the system = 0.16
17	**Patients arrive at a clinic according to Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with a mean rate of 20 per hour.** **(a) Find the effective arrival rate at the clinic.** **(b)What is the probability that an arriving patient will not wait?** **(c)What is the expected waiting time until a patient is discharged from the clinic?** Arrivalrate $\lambda = 30/$ hr., Servicerate $\mu = 20/$ hr., $M/M/1:K/FIFO$ model. $\lambda \neq \mu$, $p_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{k+1}} = \frac{1-3/2}{1-(3/2)^{16}} = 0.00076$ (a) The effective arrival rate is $\lambda' = \mu(1-p_0) = 20(1-0.00076) = 19.98$hr. (b) $P(\text{a patient will not wait}) = p(n=0) = p_0 = 0.00076$

	(c) $L_s = \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} = (-3) - \frac{16\left(\frac{3}{2}\right)^{16}}{1-\left(\frac{3}{2}\right)^{16}} = 13\text{patients}$ $W_s = \frac{L_s}{\lambda'} = \frac{13}{19.98} = 0.65\text{hr or } 39\text{min.}$
18	A Telephone exchange has two long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes. (1) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day? (2) If the subscribers will wait and are serviced in turn, what is the expected waiting time? Solution: Model identification: Since there are two operators, infinite capacity. Hence this problem comes under the model (M/M/c);(∞ /FCFS). Given data: Arrival rate $\lambda = 15$ / hr Service rate $\mu = \frac{1}{5}$ per min (i.e) $\mu = \frac{60}{5} = 12$ per hr Number of servers $c = 2$ w.k.t $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\mu c}{c!(\mu c - \lambda)}\left(\frac{\lambda}{\mu}\right)^c\right]^{-1}$ $P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{15}{12}\right)^n + \frac{2\times 12}{2!((2\times 12)-15)}\left(\frac{15}{12}\right)^2\right]^{-1}$ $= \left[\sum_{n=0}^{1} \frac{1}{n!}(1.25)^n + \frac{24}{18}(1.25)^2\right]^{-1}$ $= \left[\sum_{n=0}^{1} \frac{1}{n!}(1.25)^n + 2.078\right]^{-1}$ $P_0 = [(1+1.25)+2.078]^{-1} = 4.328^{-1} = 0.2311$ i) the probability that a subscriber will have to wait for his service is $P(N_s \geq 2) = \frac{\left(\frac{15}{12}\right)^c}{2!\left(1-\frac{15}{24}\right)}(0.2311)$ $\quad\quad P(N_s \geq c) = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)} P_0$ $P(N_s \geq 2) = \frac{(1.25)^2}{0.75}(0.2311) = 0.4814$

	ii) If the subscriber will wait and are serviced in turn then the expected waiting time $W_s = \frac{L_q}{\lambda} + \frac{1}{\mu}$ Where $L_q = \frac{1}{c.c!}\left(\frac{\lambda}{\mu}\right)^{c+1}\left(1-\frac{\lambda}{\mu c}\right)^{-2} P_0$ $= \frac{1}{(2)2!}(1.25)^3\left(1-\frac{15}{24}\right)^{-2} (0.2311)$ $= \frac{1}{4}(1.25)^3\left(1-\frac{15}{24}\right)^{-2} (0.2311)$ $=1.953\times7.111\times0.2311 = 3.209$ $\therefore W_s = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{3.209}{15} + \frac{1}{12} = 0.1368$ hr
19	**People arrive at a theatre ticket booth in Poisson distributed arrival rate of 25 per hour. Average service time is 2 minutes following exponential distribution. Calculate (1) the mean number in waiting line (2) the mean waiting time (3) the utilization factor.** **Soln.** $\lambda = 25$ *per hour*, $\mu = \frac{1}{2}\times 60 = 30$ *per hour*, $\rho = \frac{\lambda}{\mu} = \frac{25}{30} = 0.833$ (1) *Length of queue* $L_q = \frac{\rho^2}{1-\rho} = 4\,(appr.)$ (2) *Mean waiting time* $= \frac{L_q}{\lambda} = 9.6$ min (3) *Utilisation factor* $= \rho = 0.833$
20	Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 5 trains, find the probability that the yard is empty and the average number of trains in the system, given that the inter arrival time and service time are following exponential distribution. **Solution:** <u>Model identification</u>: since there is only one server and the maximum number of calling source is 2, capacity of the system is finite. Hence this problem comes under the model (M/M/1);(k/FCFS). <u>Given Data:</u> Arrival rate $\lambda = \frac{1}{15}$ per min Service rate $\mu = \frac{1}{33}$ per min Capacity of the system $k = 5$ $\rho = \frac{\lambda}{\mu} = \frac{33}{15} = 2.2$

	i) Probability that the yard is empty $= P_0 = \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \dfrac{1-\rho}{1-\rho^{k+1}} = \dfrac{1-2.2}{1-2.2^6} = 0.01068$ ii) Average number of trains in the system: $L_s = \dfrac{\rho}{1-\rho} - \dfrac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}$ since $\lambda \neq \mu$ (or) $L_s = \dfrac{\lambda}{\mu-\lambda} - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$ $L_s = \dfrac{2.2}{1-2.2} - \dfrac{(5+1)(2.2)^6}{1-(2.2)^6} = (-1.8333) - \left(\dfrac{680.28}{-112.37}\right) = 4.221$ **Another method to find L_s:** $L_s = \sum_{n=0}^{5} nP_n = P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5$ $L_s = \rho P_0 + 2\rho^2 P_0 + 3\rho^3 P_0 + 4\rho^4 P_0 + 5\rho^5 P_0$ $L_s = P_0\left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5\right]$ $L_s = P_0\left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5\right]$ $= 0.01168\left[2.2 + 2\times 2.2^2 + 3\times 2.2^3 + 4\times 2.2^4 + 5\times 2.2^5\right]$ $L_s = 4.221$
21	**Self service system is followed in a super market at a metropolis. The customer arrivals occur according to a Poisson distribution with mean 40 per hour. Service time per customer is exponentially distributed with mean 6 minutes. Find the expected number of customers in the system and what is the percentage of time that the facility is idle?** **Solution:** Model identification: It is self service model and the customer himself is treated as server. The number of server is unlimited . Hence this problem comes under the model M/M/ ∞ queues. Given Data: Arrival rate $\lambda = 40$ per hr Service rate $\mu = \dfrac{1}{6}$ per min $(i.e)\ \mu = \dfrac{60}{6} = 10/hr$ i) The average (Expected) number of customer in the system $L_s = \dfrac{\lambda}{\mu} = \dfrac{40}{10} = 4$ customer ii) P(the facility is idle) = $P_0 = e^{-\frac{\lambda}{\mu}} = e^{-4} = 0.0183$ Percentage of idle facility = 0.0183x100=1.83%

<table>
<tr><td>22</td><td>A telephone exchange receives one call every 4 minutes and connects one call every 3 minutes. If the rate of arrival follows Poisson distribution and the service time follows exponential distribution find the average waiting time for a call in the queue and in the system.
<u>Model identification</u>:
since there are one server and the capacity of the system is infinity. Hence this problem comes under the model (M/M/1);(∞ /FCFS).
<u>Given Data:</u>
Arrival rate $\lambda = \frac{1}{4}$ per min
Service rate $\mu = \frac{1}{3}$ per min
i) The average waiting time for a call in the System
$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{3} - \frac{1}{4}} = 12 \text{ min}$
ii) The average waiting time for a call in the queue
$W_q = W_s - \frac{1}{\mu} = 12 - 3 = 9 \text{ min}$</td></tr>
<tr><td>23</td><td>A petrol pump has two pumps. The service times follow the exponential distribution with mean 4 minutes and cars arrive for service in a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service and what is the probability that the pumps remain idle?
Solution:
Model identification: Since the petrol bunk has 2 pumps, infinite capacity.
Hence this problem comes under the model (M/M/c);(∞ /FCFS).
Given data:
Arrival rate $\lambda = 10$ per hr
Service rate $\mu = \frac{1}{4}$ per min (i.e) $\mu = \frac{60}{4} = 15$ per hr.
Number of servers $c = 2$
w.k.t $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \frac{\mu c}{c!(\mu c - \lambda)} \left(\frac{\lambda}{\mu} \right)^c \right]^{-1}$
$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!} \left(\frac{10}{15} \right)^n + \frac{2 \times 15}{2!((2 \times 15) - 10)} \left(\frac{10}{15} \right)^2 \right]^{-1}$
$= \left[\sum_{n=0}^{1} \frac{1}{n!} (0.6667)^n + \frac{30}{40} (0.6667)^2 \right]^{-1}$
$= \left[\sum_{n=0}^{1} \frac{1}{n!} (0.6667)^n + 0.3333 \right]^{-1}$
$P_0 = [(1 + 0.6667) + 0.3333]^{-1} = 2^{-1} = 0.5$
i) The probability that a customer has to wait for the service is</td></tr>
</table>

	$P(N_s \geq 2) = \dfrac{\left(\frac{10}{15}\right)^c}{2!\left(1-\frac{10}{15\times 2}\right)}(0.5)$ $\qquad P(N_s \geq c) = \dfrac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)} P_0$ $P(N_s \geq 2) = \dfrac{(0.6667)^2}{1.3333}(0.5) = \dfrac{1}{6} = 0.1667$ ii) Probability of time that a pump is busy $= \rho = \dfrac{\lambda}{c\mu} = \dfrac{10}{2\times 15} = \dfrac{1}{3}$ $\therefore$ Probability of time when a pump is idle = 1- Probability of time that a pump is busy $= 1 - \dfrac{1}{3} = \dfrac{2}{3}$ $\therefore$ Percentage of idle time of for each pump $= \dfrac{2}{3} \times 100 = 67\%$
24	**A shipping company has a single unloading dock with ships arriving in a Poisson fashion at an average rate of 3 per day. The unloading time distribution for a ship with n unloading crews is found to be exponent with average unloading time $\frac{1}{2n}$ days. The company has a large labour supply without regular working hours, and to avoid long waiting times, the company has a policy of using as many unloading crews as there are ships waiting in line or being unloaded. Find the average number of unloading crews working at any time and the probability that more than 4 crews will be needed.** **Solution:** Arrival rate $\lambda = 3$ ships/day Service rate $\mu_n = n\mu = \dfrac{1}{1/2n} = 2n \Rightarrow \mu = 2$ ships/day for one unloading crew. i) Expected number of unloading crews is equal to the number of ship in the system $L_s = \dfrac{\lambda}{\mu} = \dfrac{3}{2} = 1.5$ ii) $P_n = e^{-\frac{\lambda}{\mu}} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n \quad n = 0,1,2,3,..$ $P_n = e^{-1.5}\dfrac{(1.5)^n}{n!},\ n = 0,1,2,3,...$ P(more than 4 crews will be needed) = P(atleast 5 ships in the system) $= P(N \geq 5) = 1 - \sum_{n=0}^{4} P_n = 1 - \sum_{n=0}^{4} e^{-1.5}\dfrac{(1.5)^n}{n!}$ $= 1 - e^{-1.5}\left[1 + 1.5 + \dfrac{(1.5)^2}{2!} + \dfrac{(1.5)^3}{3!} + \dfrac{(1.5)^4}{4!}\right]$ $= 1 - 0.2231\left[1 + 1.5 + 1.125 + 0.5625 + 0.2109\right]$ $= 1 - 0.98129 = 0.0187$
25	**Explain Markovian Birth - Death process and obtain the expressions for steady state probabilities.** Markovian Birth - Death process:

Let n be the size of the customer at time t.

Let $P_n(t)$ be the probability of n customers in the system at t.

Let λ_n = average arrival rate when n customers are in the system.

μ_n = average service rate when n customers are in the system

Now,

$P_n(t+\Delta t)$ is the probability of n customers at time $t+\Delta t$

The presence of n customers in the system at time $t+\Delta t$ can happen in any one of the following four mutually exclusive ways:

i) Presence of n customers at time t and no arrival or departure during Δt time.

ii) Presence of n-1 customers at time t and one arrival and departure during Δt time.

iii) Presence of n+1 customers at time t and no arrival and one departure during Δt time.

iv) Presence of n customers at time t and one arrival and one departure during Δt time.

$$\therefore P_n(t+\Delta t) = P_n(t)(1-\lambda_n\Delta t)(1-\mu_n\Delta t) + P_{n-1}(t)\lambda_{n-1}\Delta t(1-\mu_{n-1}\Delta t)$$
$$+ P_{n+1}(t)(1-\lambda_{n+1}\Delta t)\mu_{n+1}\Delta t + P_n(t)\,\lambda_n\Delta t\,\mu_n\Delta t$$

$i.e., P_n(t+\Delta t) = P_n(t) - (\lambda_n + \mu_n)P_n(t)\Delta t + \lambda_{n-1}P_{n-1}(t)\Delta t + \mu_{n+1}P_{n+1}(t)$, on omitting terms containing $(\Delta t)^2$ which is negligibly small.

$$\therefore \frac{P_n(t+\Delta t)}{P_n(t)} = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) \quad ----(1)$$

Taking limits on both sides of (1) as $\Delta t \to 0$, we have

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) \quad ----(2)$$

Equation (2) does not hold good for n=0, as $P_{n-1}(t)$ does not exists.

Hence we derive the differential equation satisfied by $P_0(t)$ independently.

Proceeding as before,

$$P_0(t+\Delta t) = P_0(t)(1-\lambda_0\Delta t) + P_1(t)(1-\lambda_1\Delta t)\mu_1\Delta t$$

[by the possibilities (i) and (iii) given above and as no departure is possible when n=0]

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda_0 P_0(t) + P_1(t)\mu_1 \quad ----(3)$$

Taking limits on both sides of (3) as $\Delta t \to 0$, we have

$$P_0'(t) = -\lambda_0 P_0(t) + P_1(t)\mu_1 \quad ----(4)$$

Now in steady state, $P_n(t)$ and $P_0(t)$ are independent of time and hence $P_n'(t)$ and $P_0'(t)$ become zero.

Hence the differential equation (2) and (4) reduce to the difference equations

$$\lambda_{n-1}P_{n-1} - (\lambda_n + \mu_n)P_n + \mu_{n+1}P_{n+1} = 0 \quad ------(5) \; and$$

$$\lambda_0 P_0 + \mu_1 P_1 = 0 \qquad -----(6)$$

To find steady state probabilities

From eqn (6) derived above, we have

$$P_1 = \frac{\lambda_0}{\mu_1}P_0 \qquad -----(7)$$

Putting n=1 in (5)

$$\mu_2 P_2 = (\lambda_1 + \mu_1)P_1 - \lambda_0 P_0$$

$$= (\lambda_1 + \mu_1)\frac{\lambda_0}{\mu_1} P_0 - \lambda_0 P_0$$

$$= \frac{\lambda_0\lambda_1}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2} P_0$$

Successively putting n=1,2,3,.... in (5) and proceeding similarly, we can get

$$P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} P_0 \ etc.$$

Finally $P_n = \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n} P_0 \quad n = 1, 2, 3, \ldots$

Since the number of customers in the system can be 0 or 1 or 2 or 3 etc., which events are mutually exclusive and exhaustive, we have $\sum_{n=0}^{\infty} P_n = 1$

i.e., $P_0 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n} P_0 = 1$

$$P_0 = \frac{1}{\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}}$$

26 **An airport has a single runway. Airplanes have been found to arrive at the rate of 15 per hour. It is estimated that each landing takes 3 minutes. Assuming a Poisson process for arrivals and an exponential distribution for landing times. Find the expected number of airplanes waiting to land, expected waiting time. What is the probability that the waiting time will be more than 5 minutes? (NOV/DEC 2017)**

Solution:

Model identification:

since there is only one single runway and the capacity of the system is infinity. Hence this problem comes under the model (M/M/1);(∞ /FCFS).

Given Data:

Arrival rate $\lambda = 15 / hr \Rightarrow \lambda = \frac{15}{60} = \frac{1}{4} = 0.25$ min

Service rate $\frac{1}{\mu} = 3\text{min} \Rightarrow \mu = \frac{1}{3} = 0.3333\text{min}$

i) The expected number of airplanes waiting to land is

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{1/16}{\frac{1}{3}\left(\frac{1}{3} - \frac{1}{4}\right)} = \frac{9}{4} = 2.25$$

ii) Expected waiting time of the airplanes is

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1/4}{\frac{1}{3}\left(\frac{1}{3} - \frac{1}{4}\right)} = 9\,\text{min}$$

iii) Probability that the waiting time of a customer in the queue exceeds t is

$$P(W_q > t) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$$

$$P(W_q > 5) = \frac{1/4}{1/3} e^{\left(\frac{1}{3}-\frac{1}{4}\right)} = \frac{3}{4} e^{\left(\frac{1}{12}\right)} = 1.226$$

27 **Let there be an automobile inspection situation with three inspection stalls. Assume that cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls up to it. The station can accommodate almost four cars waiting at one time. The arrival pattern is Poisson with a mean of one car every minute during the peak hours. The service time is exponential with a mean of 6 minutes. Find the average number of customers in the system during the peak hours, the average waiting time and the average number per hour that cannot enter the station because of full capacity. (NOV/DEC 2017)**

Solution:

Since there are 3 inspection stalls, also since there are 4 cars in the group, the capacity of the system is finite. Hence this problem comes under the model (M/M/c);(k/FCFS)

Given data:

Arrival rate $\lambda = 1$ per min

Service rate $\mu = \frac{1}{6}$ per min

Number of servers $c = 3$

Capacity = $k = 4+3=7$

Expected number of engineers waiting to use in the computing center = L_s

$$L_s = L_q + \frac{\lambda'}{\mu}$$

Where $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \sum_{n=c}^{k}\left(\frac{\lambda}{\mu c}\right)^{n-c}\right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{2} \frac{1}{n!}(6)^n + \frac{1}{3!}(6)^2 \sum_{n=3}^{7}(2)^{n-3}\right]^{-1}$$

$$= \left[(1+6+18)+6(1+2+4+8+16)\right]^{-1}$$

$$= (25+186)^{-1} = 0.00474$$

$$\boxed{P_0 = 0.00474}$$

And $\rho = \frac{\lambda}{\mu c} = 2$

Expected number of cars in the queue

$$L_q = \left(\frac{\lambda}{\mu}\right)^c \frac{\rho}{c!(1-\rho)^2}\left\{1-\rho^{k-c}-(k-c)(1-\rho)\rho^{k-c}\right\}P_0$$

$$= (6)^3 \frac{2}{3!(1-2)^2}\left\{1-(2)^{7-3}-(7-3)(1-2)(2)^{7-3}\right\}(0.00474)$$

$$= 72(1-16+64)(0.00474) = 1.67$$

$$\boxed{L_q = 1.67}$$

Effective arrival rate $\lambda' = \mu\left[c - \sum_{n=0}^{c-1}(c-n)P_n\right] = \frac{1}{6}\left[3 - \sum_{n=0}^{2}(3-\text{n})P_n\right]$

$\lambda' = \frac{1}{6}\left[3-\left(3P_0 + 2P_1 + P_3\right)\right]$ where $P_1 = \left(\frac{\lambda}{\mu}\right)P_0 ; P_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0 ; P_2 = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 P_0$

$P_1 = 6(0.00474); P_2 = \frac{1}{2}(6)^2 (0.00474); P_3 = \frac{1}{6}(6)^3 (0.00474)$

$P_1 = 0.0284;\ P_2 = 0.0853;\ P_3 = 0.1706$

$\lambda' = \frac{1}{6}\left[3-\left(3(0.0284)+2(0.0853)+0.1706\right)\right] = 0.4289$

$\boxed{\lambda' = 0.4289}$

Expected number of cars in the system = L_s

$L_s = L_q + \frac{\lambda'}{\mu} = 1.67 + (0.4289)(6) = 4.24$

$\boxed{L_s = 4.24 \text{ cars}}$

Expected waiting time in the system $W_s = \frac{L_q}{\lambda'} + \frac{1}{\mu} = \frac{1.67}{0.4289} + 6 = 9.8$ min

Expected no. of cars per hr that cannot enter the station is

$P_7 \times 60 = \left(\frac{1}{3!3^{7-3}}(6)^7 (0.00474\right) \times 60 = 163.8 \qquad \because P_n = \left\{\frac{1}{c!c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0,\ n < c \le k\right.$

28 **Show that for a single service station, Poisson arrivals and exponential service time, the probability that exactly *n* calling units in the queueing system is $P_n = (1-e)\,e^n,\ n \ge 0$ where *e* is the traffic intensity.**

Also, find the expected number of units in the system. (NOV/DEC 2017)

Solution:

Let n be the size of the customer at time t.

Let $P_n(t)$ be the probability of n customers in the system at t.

Let λ_n = average arrival rate when n customers are in the system.

μ_n = average service rate when n customers are in the system

Now,

$P_n(t+\Delta t)$ is the probability of n customers at time $t+\Delta t$

The presence of n customers in the system at time $t+\Delta t$ can happen in any one of the following four mutually exclusive ways:

i) presence of n customers at time t and no arrival or departure during Δt time.

ii) presence of n-1 customers at time t and one arrival and departure during Δt time.

iii) presence of n+1 customers at time t and no arrival and one departure during Δt time.

iv) presence of n customers at time t and one arrival and one departure during Δt time .

$$\therefore P_n(t+\Delta t) = P_n(t)(1-\lambda_n\Delta t)(1-\mu_n\Delta t) + P_{n-1}(t)\lambda_{n-1}\Delta t(1-\mu_{n-1}\Delta t)$$
$$+ P_{n+1}(t)(1-\lambda_{n+1}\Delta t)\mu_{n+1}\Delta t + P_n(t)\ \lambda_n\Delta t\ \mu_n\Delta t$$

i.e., $P_n(t+\Delta t) = P_n(t) - (\lambda_n + \mu_n)\, P_n(t)\Delta t + \lambda_{n-1}P_{n-1}(t)\Delta t + \mu_{n+1}P_{n+1}(t)$, on omitting terms containing $(\Delta t)^2$ which is negligibly small.

$$\therefore \frac{P_n(t+\Delta t)}{P_n(t)} = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) ----(1)$$

Taking limits on both sides of (1) as $\Delta t \to 0$, we have

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) ----(2)$$

Equation (2) does not hold good for n=0, as $P_{n-1}(t)$ does not exists.

Hence we derive the differential equation satisfied by $P_0(t)$ independently.

Proceeding as before,

$$P_0(t+\Delta t) = P_0(t)(1-\lambda_0\Delta t) + P_1(t)(1-\lambda_1\Delta t)\mu_1\Delta t$$

[by the possibilities (i) and (iii) given above and as no departure is possible when n=0]

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda_0 P_0(t) + P_1(t)\mu_1 ----(3)$$

Taking limits on both sides of (3) as $\Delta t \to 0$, we have

$$P_0'(t) = -\lambda_0 P_0(t) + P_1(t)\mu_1 ----(4)$$

Now in steady state, $P_n(t)$ and $P_0(t)$ are independent of time and hence $P_n'(t)$ and $P_0'(t)$ become zero.

Hence the differential equation (2) and (4) reduce to the difference equations

$$\lambda_{n-1}P_{n-1} - (\lambda_n + \mu_n)\, P_n + \mu_{n+1}P_{n+1} = 0 ------(5) \text{ and}$$

$$\lambda_0 P_0 + \mu_1 P_1 = 0 \qquad -----(6)$$

To find steady state probabilities

From eqn (6) derived above, we have

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \qquad -----(7)$$

Putting n=1 in (5)

$$\mu_2 P_2 = (\lambda_1 + \mu_1)\, P_1 - \lambda_0 P_0$$

$$= (\lambda_1 + \mu_1)\frac{\lambda_0}{\mu_1} P_0 - \lambda_0 P_0$$

$$= \frac{\lambda_0\lambda_1}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2} P_0 \quad -----(8)$$

Successively putting n=1,2,3,.... in (5) and proceeding similarly, we can get

$$P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} P_0 \; etc.$$

Finally $P_n = \dfrac{\lambda_0\lambda_1\lambda_2....\lambda_{n-1}}{\mu_1\mu_2\mu_3....\mu_n} P_0 \quad n = 1,2,3,...... ------(9)$

Since the number of customers in the system can be 0 or 1 or 2 or 3 etc., which events are mutually exclusive and exhaustive, we have $\sum_{n=0}^{\infty} P_n = 1$

i.e., $P_0 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n} P_0 = 1$

$$P_0 = \frac{1}{\sum_{n=1} \frac{\lambda_0\lambda_1\lambda_2\ldots\lambda_{n-1}}{\mu_1\mu_2\mu_3\ldots\mu_n}} \quad ------(10)$$

Now,

Let N denote the no. of customer in the queueing system

And N is a discrete random variable, which can take the value 0,1,2,3,….

Such that $P(N = n) = P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$, from equation (9) and (10), we have

$$P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n} = 1-\frac{\lambda}{\mu}$$

$$\boxed{P_0 = 1-\frac{\lambda}{\mu}}$$

$$\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)}$$

The expected number of units in the system

$$L_s = \sum nP_n$$

$$= \sum n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum n\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-2}$$

$$\boxed{L_s = \frac{\lambda}{\mu-\lambda}}$$