ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY
Near Anjugramam Junction, Kanyakumari Main Road, Palkulam, Variyoor P.O - 629401
Kanyakumari Dist, Tamilnadu., E-mail : admin@rcet.org.in, Website : www.rcet.org.in

DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT: PROBABILITY AND QUEUEING THEORY

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UNIT – III: RANDOM PROCESSES

MA 8402 – PROBABILITY AND QUEUEING THEORY

UNIT – III RANDOM PROCESSES
Markov process and Markov chains

Random process

A random process is an infinite indexed collection of random variables { X(t) : t ε T }, defined over a common probability space. The index parameter t is typically time, but can also be a spatial dimension. Random processes are used to model random experiments that evolve in time such as Daily price of a stock.

If the time(parameter) 't' is fixed, random process is called a random variable.

Continuous-time random process and Discrete state random process.

Consider a random process {X(t), $t \in T$}, where T is the index set or parameter set. The values assumed by X(t) are called the states, and the set of all possible values of the states forms the state space E of the random process.

(a) If the state space E and index set T are both continuous, then the random process is called continuous-time random process.
(b) If the state space E is discrete and the index set T is continuous, then the random process is called discrete state random process

Stationary process

A random process is said to be stationary if its mean ,variance , moments are constants other process are called non stationary

Evolutionary process: A random process is not stationary in any sense is called as evolutionary process

First order stationary process:

A random process is said to be stationary to order one if is first order density function does not change with a shift in time origin.i.e., $f_X(x_1 : t_1) = f_X(x_1, t_1 + \delta)$ for any time t_1 and any real number .i.e., $E[X(t)] = \overline{X}$ = Constant.

second order stationary process:

A random process is said to be stationary to order two if its second-order density functions does not change with a shift in time origin.i.e., $f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2 : t_1 + \delta, t_2 + \delta)$ for all t_1, t_2 and .

n^{th} **order stationary process, when will it become a SSS process?**

A random process $X(t)$ is said to be stationary to order n or n^{th} order stationary if its n^{th} order density function is invariant to a shift of time origin.

$f_X(x_1, x_2, ..., x_n, t_1, t_2, ..., t_n) = f_X(x_1, x_2, ..., x_n, t_1 + \delta, t_2 + \delta, ..., t_n + \delta)$ for all $t_1, t_2, ..., t_n$ & h . n^{th} order stationary process becomes a SSS process when $n \to \infty$

Ergodic Random process.

A random process {X(t)} is called ergodic if all its ensemble averages are equal to appropriate time averages

Autocorrelation

$$R_{XX}(t_1, t_2) = E\{X(t_1) X(t_2)\}$$

Auto covariance

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E\{X(t_1)\} E\{X(t_2)\}$$

I Wide sense stationary process.

A random process {X(t)} is called wide-sense stationary if the following conditions hold:

(i) $E[X(t)] = a\ constant$

(ii) $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = R_{XX}(t_1 - t_2)$ = function of time difference

Auto correlation:

$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = R_{XX}(t_1 - t_2)$

Evolutionary process:

A random process is not stationary in any sense is called as evolutionary process

Problems under Wide sense stationary process (UQ).

1. Prove that first order stationary random process has a constant mean. (DEC 2013)

Let X(t) be a first-order stationary process. Then the first-order probability density function of X(t) satisfies $f_X(x_1 ; t_1) = f_X(x_1 ; t_1 + \varepsilon)$ …..(A) for all t_1 and ε.

Now, consider any two time instants t_1 and t_2, and define the random variable $X_1 = X(t_1)$ and $X_2 = X(t_2)$. By definition, the mean values of X_1 and X_2 are given by

$$E(X_1) = E[X(t_1)] = \int_{x_1=-\infty}^{\infty} x_1 f_X(x_1;t_1)dx_1 \quad \cdots(1)$$

$$E(X_2) = E[X(t_2)] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_2)dx_2 \quad \cdots(2)$$

$$Let\ t_2 = t_1 + \varepsilon \qquad (2) \Rightarrow E[X(t_1+\varepsilon)] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;\, t_1+\varepsilon)dx_2$$

Using (A), $E[X(t_1+\varepsilon)] = \int_{x_2=-\infty}^{\infty} x_2 f_X(x_2;t_1)dx_2 = \int_{x_1=-\infty}^{\infty} x_1 f_X(x_1;t_1)dx_1 = E[X(t_1)]$

which shows first order stationary random process has a constant mean.

2. Define a k^{th} order stationary process. When will it become a strict sense stationary proess (APR 2017)

The process is said to be k^{th} order stationary if all its finite dimensional distributions are invariant under translation of time for all $t_1, t_2, \ldots t_n$ and forn=1,2,3……k only and not for all n>k, then the process is called the k^{th} order stationary process. It becomes a strict sense stationary proess when k→∞.

3. Show that the random process $X(t) = A\cos(w_0 t + \theta)$ is wide-sense stationary, if A and w_0 are constants and θ is uniformly distributed random variable in (0,2π) (APR 2017)

Solution:

Since θ is uniformly distributed in $(0, 2\pi)$ p.d.f. is $f(\theta) = \frac{1}{2\pi}\ (0 \le 0 \le 2\pi)$

$$E[X(t)] = \int_0^{2\pi} \frac{1}{2\pi} A\cos(\omega_0 t + \theta)\, d\theta = \frac{A}{2\pi}\left[\sin(\omega_0 t + \theta)\right]_0^{\pi} = \frac{A}{2\pi}\left[\sin(\omega_0 t + 2\pi) - \sin(\omega_0 t)\right]$$

$$= \frac{A}{2\pi}\left[\sin \omega_0 t - \sin \omega_0 t\right] = 0$$

$$R(t_1,t_2)=E\left[X(t_1)X(t_2)\right]=E\left[A\cos(\omega_0 t_1+\theta)A\cos(\omega_0 t_2+\theta)\right]$$

$$=E\left[A^2\cos(\omega_0 t_1+\theta)\cos(\omega_0 t_2+\theta)\right]=\frac{A^2}{2}E\left[\cos\left[\omega_0(t_1+t_2)+2\theta\right]+\cos\omega_0(t_1-t_2)\right]$$

$$=\frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[\cos\left(\omega_0(t_1+t_2)+2\theta\right)+\cos\left(\omega_0(t_1-t_2)\right)\right]d\theta$$

$$=\frac{A^2}{4\pi}\left\{\left[\frac{\sin\left(\omega_0(t_1+t_2)+2\theta\right)}{2}\right]_0^{2\pi}+\cos\left(\omega_0(t_1-t_2)\right)\left[\theta\right]_0^{2\pi}\right\}$$

$$=\frac{A^2}{4\pi}\left\{\frac{\sin\left(\omega_0(t_1+t_2)+4\pi\right)}{2}-\frac{\sin\left(\omega_0(t_1+t_2)\right)}{2}+\cos\left(\omega_0(t_1-t_2)\right)(2\pi)\right\}$$

$$=\frac{A^2}{4\pi}\left[\frac{\sin\omega_0(t_1+t_2)}{2}-\frac{\sin\omega_0(t_1+t_2)}{2}+2\pi\cos\omega_0(t_1-t_2)\right]$$

$$=\frac{A^2}{4\pi}.2\pi\cos\omega_0(t_1-t_2)=\frac{A^2}{2}\cos\omega_0(t_1-t_2)$$ =a function of t_1-t_2

$\therefore$ The process $X(t)$ is W.S.S.

4. Show that the process $X(t)=A\cos\lambda t+B\sin\lambda t$ is wide sense stationary, if E(AB) =0; E(A)=E(B) = 0, & E(A^2) = E(B^2), where A &B are random variables. **(MAY 2013)**

Solution:

Given $X(t)=A\cos\lambda t+B\sin\lambda t$, E(A) = E(B) = E(AB) = 0, E(A^2) = E(B^2) = k(say)

To prove X(t) is wide sense stationary, we have to show

(i) E{X(t)} = constant

(ii) $R_{XX}(t,t+\tau)$ = functionof timedifference = functionof τ

Now, $E\left[X(t)\right]=E\left[A\cos\lambda t+B\sin\lambda t\right]=\cos\lambda t E(A)+\sin\lambda t E(B)=0$ $\therefore$ $E\left[X(t)\right]=$ constant

$R_{XX}(t,t+\tau)=E\left\{X(t)\,X(t+\tau)\right\}$

$$=E\left\{\left[A\cos\lambda t+B\sin\lambda t\right]\left[A\cos(\lambda t+\lambda\tau)+B\sin(\lambda t+\lambda\tau)\right]\right\}$$

$$=E\left[\begin{array}{l}A^2\cos\lambda t\cos(\lambda t+\lambda\tau)+AB\cos\lambda t\sin(\lambda t+\lambda\tau)+AB\sin\lambda t\cos(\lambda t+\lambda\tau)\\ \qquad +B^2\sin\lambda t\sin(\lambda t+\lambda\tau)\end{array}\right]$$

$$=\cos\lambda t\cos(\lambda t+\lambda\tau)E(A^2)+\cos\lambda t\sin(\lambda t+\lambda\tau)E(AB)+\sin\lambda t\cos(\lambda t+\lambda\tau)E(AB)+\sin\lambda t\sin(\lambda t+\lambda\tau)E(B^2)$$

$$=\cos\lambda t\cos(\lambda t+\lambda\tau)k+\sin\lambda t\sin(\lambda t+\lambda\tau)k$$

$$\left[\because E(AB)=0\ \&\ E(A^2)=E(B^2)=k(say)\right]$$

$$=k\left[\cos\lambda t\cos(\lambda t+\lambda\tau)+\sin\lambda t\sin(\lambda t+\lambda\tau)\right]$$

$= k\cos(\lambda t + \lambda\tau - \lambda t) = k\cos\lambda\tau = a$ function of time difference. $\therefore$ X(t) is wide sense stationary.

5. Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ is wide sense stationary, if E(AB) =0; E(A)=E(B) = 0, & $E(A^2) = E(B^2)$, where A &B are random variables. (MAY 2013)

Solution:

Given $X(t) = A\cos\lambda t + B\sin\lambda t$, E(A) = E(B) = E(AB) = 0, $E(A^2) = E(B^2) = k$(say)

To prove X(t) is wide sense stationary, we have to show

(i) E{X(t)} = constant

(ii) $R_{XX}(t,t+\tau)$ = functionof timedifference = functionof τ

Now, $E[X(t)] = E[A\cos\lambda t + B\sin\lambda t] = \cos\lambda t E(A) + \sin\lambda t E(B) = 0 \therefore E[X(t)] =$ constant

$R_{XX}(t,t+\tau) = E\{X(t)\, X(t+\tau)\}$

$= E\{[A\cos\lambda t + B\sin\lambda t][A\cos(\lambda t+\lambda\tau) + B\sin(\lambda t+\lambda\tau)]\}$

$$= E\left[\begin{array}{l} A^2\cos\lambda t\cos(\lambda t+\lambda\tau) + AB\cos\lambda t\sin(\lambda t+\lambda\tau) + AB\sin\lambda t\cos(\lambda t+\lambda\tau) \\ \qquad + B^2\sin\lambda t\sin(\lambda t+\lambda\tau) \end{array}\right]$$

$$= \cos\lambda t\cos(\lambda t+\lambda\tau)E(A^2) + \cos\lambda t\sin(\lambda t+\lambda\tau)E(AB) + \sin\lambda t\cos(\lambda t+\lambda\tau)E(AB) + \sin\lambda t\sin(\lambda t+\lambda\tau)E(B^2)$$

$= \cos\lambda t\cos(\lambda t+\lambda\tau)k + \sin\lambda t\sin(\lambda t+\lambda\tau)k$

$\left[\because E(AB) = 0 \;\&\; E(A^2) = E(B^2) = k(say)\right]$

$= k[\cos\lambda t\cos(\lambda t+\lambda\tau) + \sin\lambda t\sin(\lambda t+\lambda\tau)]$

$= k\cos(\lambda t + \lambda\tau - \lambda t) = k\cos\lambda\tau = a$ function of time difference. $\therefore$ X(t) is wide sense stationary.

6. If the process X(t)=P+Qt, where P and Q are independent random variables with E(P)= p, E(Q) =q, Var(P) = σ_1^2 and Var(Q) = σ_2^2 then find E{(X(t)} and $R(t_1,t_2)$. Is the process {X(t)} stationary? (APRIL 2017)

Solution:

E{X(t)}=E{P+Qt}=E(P)+E(Q)t=p+qt which is not a constant.

$R(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\{(p+qt_1)(p+qt_2)\}$

$= E\{p^2\} + E\{PQ\}(t_1+t_2) + E(Q^2)t_1t_2$

$= \sigma_1^2 + p^2 + pq(t_1+t_2) + (\sigma_2^2 + q^2)t_1t_2$. (where E(PQ)=E(P).E(Q)=pq)

Which is not a functions of $t_1 - t_2$.

Therefore {X(t)} is not stationary.

7. The process {X(t)} whose probabilitydistribution under certain condition is given by

$$P[X(t)=n]=\begin{cases}\dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n=1,2,3,\cdots\\ \dfrac{at}{1+at}, & n=0.\end{cases}$$

Show that {X(t)} is not stationary. **(MAY 2012, DEC 2013)**

Solution:

Given $P[X(t)=n]=\begin{cases}\dfrac{(at)^{n-1}}{(1+at)^{n+1}}, & n=1,2,3,\cdots\\ \dfrac{at}{1+at}, & n=0.\end{cases}$

$$E[X(t)]=\sum_{n=0}^{\infty} nP\{X(t)=n\}=0+1P\{X(t)=1\}+2P\{X(t)=2\}+3P\{X(t)=3\}+\cdots$$

$$=1\left(\frac{1}{(1+at)^2}\right)+2\left(\frac{at}{(1+at)^3}\right)+3\left(\frac{(at)^2}{(1+at)^4}\right)+\cdots$$

$$=\frac{1}{(1+at)^2}\left[1+2\frac{at}{1+at}+3\left(\frac{at}{1+at}\right)^2+\cdots\right]=\frac{1}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-2}$$

$$=\frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}=1=a\text{ constant}$$

$$E[X^2(t)]=\sum_{n=0}^{\infty} n^2P\{X(t)=n\}=\sum_{n=0}^{\infty}\{n(n+1)-n\}P\{X(t)=n\}$$

$$=\sum_{n=0}^{\infty} n(n+1)P\{X(t)=n\}-nP\{X(t)=n\}$$

$$=\sum_{n=0}^{\infty} n(n+1)P\{X(t)=n\}-1 \qquad \left\{\because \sum_{n=0}^{\infty} nP\{X(t)=n\}=1\right\}$$

$$=[0+1.2P\{X(t)=1\}+2.3P\{X(t)=2\}+3.4P\{X(t)=3\}+\cdots]-1$$

$$=\left\{2\left(\frac{1}{(1+at)^2}\right)+6\left(\frac{at}{(1+at)^3}\right)+12\left(\frac{(at)^2}{(1+at)^4}\right)+\cdots\right\}-1$$

$$=\frac{2}{(1+at)^2}\left[1+3\frac{at}{1+at}+4\left(\frac{at}{1+at}\right)^2+\cdots\right]-1=\frac{2}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-3}-1$$

$$=\frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-3}-1=1+at-1=at\text{, not a constant.}$$

So, X(t) is not a stationary process.

8. If a random process $\{X(t)\}$ **is defined by** $\{X(t)\}=\sin(\omega t+Y)$ **where** Y **is uniformly distributed in** $(0,2\pi)$**. Show that** $\{X(t)\}$ **is WSS.**

Solution:

Since y is uniformly distributed in $(0, 2\pi)$,

$$f(y) = \frac{1}{2\pi}, \ 0 < y < 2\pi$$

$$E[X(t)] = \int_0^{2\pi} X(t) f(y) dy$$

$$= \frac{1}{2\pi}\int_0^{2\pi} sin\,(\omega t + y)\, dy$$

$$= \frac{1}{2\pi}\left[-cos(\omega t + y)\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}\left[cos\,(\omega t + 2\pi) - cos\,\omega t\right] = 0$$

$$R_{XX}(t_1, t_2) = E\left[sin(\omega t_1 + y) sin(\omega t_2 + y)\right]$$

$$= E\left[\frac{cos(\omega(t_1 - t_2)) - cos(\omega(t_1 + t_2) + 2y)}{2}\right]$$

$$= \frac{1}{2}E\left[cos(\omega(t_1 - t_2))\right] - \frac{1}{2}E\left[cos(\omega(t_1 + t_2) + 2y)\right]$$

$$= \frac{1}{2}cos(\omega(t_1 - t_2)) - \frac{1}{2}\int_0^{2\pi} cos(\omega(t_1 + t_2) + 2y)\frac{1}{2\pi}dy$$

$$= \frac{1}{2}cos(\omega(t_1 - t_2)) - \frac{1}{4\pi}\left[\frac{sin(\omega(t_1 + t_2) + 2y)}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}cos(\omega(t_1 - t_2)) - \frac{1}{8\pi}\left[sin(\omega(t_1 + t_2) + 2\pi) - sin\omega(t_1 + t_2)\right]$$

$= \frac{1}{2}cos\,(\omega\,(t_1 - t_2))$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

9. Verify whether the sine wave random process $X(t) = Y \sin \omega t$, Y is uniformly distributed in the interval $(-1,1)$ is WSS or not

Solution:

Since y is uniformly distributed in $(-1,1)$,

$$f(y) = \frac{1}{2}, \ -1 < y < 1$$

$$E[X(t)] = \int_{-1}^{1} X(t) f(y) dy$$

$$= \int_{-1}^{1} y\, sin\omega t\, \frac{1}{2} dy$$

$$= \frac{\sin\omega t}{2}\int_{-1}^{1} y\,dy$$

$$= \frac{\sin\omega t}{2}(0) = 0$$

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$

$$= E\left[y^2 \sin\omega t_1 \sin\omega t_2\right]$$

$$= E\left[y^2\,\frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{2}\right]$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{2}E\left(y^2\right)$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{2}\int_{-1}^{1} y^2 f(y)\,dy$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{2}\,\frac{1}{2}\int_{-1}^{1} y^2\,dy$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{4}\left(\frac{y^3}{3}\right)_{-1}^{1}$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{4}\left(\frac{2}{3}\right)$$

$$= \frac{\cos\omega(t_1-t_2)-\cos w(t_1+t_2)}{6}$$

$R_{XX}(t_1,t_2) \neq$ a function of time difference alone. Hence it is not a WSS Process.

10. If $X(t) = Y\cos t + Z\sin t$ for all t & where Y & Z are independent binary random variables. Each of which assumes the values – 1 & 2 with probabilities $\frac{2}{3}$ & $\frac{1}{3}$ respectively, prove that $\{X(t)\}$ is WSS.

Solution:

Given

$Y = y$	:	-1	2
$P[Y = y]$	:	$\frac{2}{3}$	$\frac{1}{3}$

$$E(Y) = E(Z) = -1\times\frac{2}{3} + 2\times\frac{1}{3} = 0$$

$$E\left(Y^2\right) = E\left(Z^2\right) = (-1)^2\times\frac{2}{3} + (2)^2\times\frac{1}{3}$$

$$E\left(Y^2\right) = E\left(Z^2\right) = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$$

Since Y & Z are independent

$E[YZ] = E[Y]E[Z] = 0$ ----(1)

Hence $E\left[X(t)\right] = E\left[y\cos t + z\sin t\right]$

$= E[y]\cos t + E[z]\sin t$

$E\left[X(t)\right] = 0 =$ is a constant. $\left[\because E(y) = E(z) = 0\right]$

$R_{XX}(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$

$= E\left[(y\cos t_1 + z\sin t_1)(y\cos t_2 + z\sin t_2)\right]$

$= E\left[y^2\cos t_1\cos t_2 + yz\cos t_1\sin t_2 + zy\sin t_1\cos t_2 + z^2\sin t_1\sin t_2\right]$

$= E(y^2)\cos t_1\cos t_2 + E[yz]\cos t_1\sin t_2 + E[zy]\sin t_1\cos t_2 + E\left[z^2\right]\sin t_1\sin t_2$

$= E(y^2)\cos t_1\cos t_2 + E(z^2)\sin t_1\sin t_2$

$= 2\left[\cos t_1\cos t_2 + \sin t_1\sin t_2\right]$ $\left[\because E(y^2) = E(z^2) = 2\right]$

$= 2\cos(t_1 - t_2) =$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

11. Check whether the two random process given by $X(t) = A\cos\omega t + B\sin\omega t$ & $Y(t) = B\cos\omega t - A\sin\omega t$. Show that $X(t)$ & $Y(t)$ are jointly WSS if A & B are uncorrelated random variables with zero mean and equal variance random variables are jointly WSS.

Solution:

Given $E(A) = E(B) = 0$

$Var(A) = Var(B) = \sigma^2$

$\therefore E(A^2) = E(B^2) = \sigma^2$

As $A \& B$ uncorrelated are $E[AB] = E[A]E[B] = 0$.

$E\left[X(t)\right] = E\left[A\cos\omega t + B\sin\omega t\right]$

$= E(A)\cos\omega t + E(B)\sin\omega t = 0$

$E\left[X(t)\right] = 0 =$ is a constant.

$R_{XX}(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$

$= E\left[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)\right]$

$= E\left[A^2\cos\omega t_1\cos\omega t_2 + AB\cos\omega t_1\sin\omega t_2 + BA\sin\omega t_1\cos\omega t_2 + B^2\sin\omega t_1\sin\omega t_2\right]$

$= \cos\omega t_1\cos\omega t_2\, E\left[A^2\right] + \cos\omega t_1\sin\omega t_2\, E[AB] + \sin\omega t_1\cos\omega t_2\, E[BA] + \sin\omega t_1\sin\omega t_2\, E\left[B^2\right]$

$= \sigma^2\left[\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2\right]$

$= \sigma^2\cos\omega(t_1 - t_2)$ $\left[\because E(A^2) = E(B^2) = \sigma^2 \;\&\; E(AB) = E(BA) = 0\right]$

$R_{XX}(t_1, t_2) =$ is a function of time difference.

$E\left[Y(t)\right] = E\left[B\cos\omega t - A\sin\omega t\right]$

$= E(B)\cos\omega t - E(A)\sin\omega t = 0$

$$R_{YY}(t_1,t_2) = E\left[\left(B\cos\omega t_1 - A\sin\omega t_1\right)\left(B\cos\omega t_2 - A\sin\omega t_2\right)\right]$$
$$= E\left[B^2\cos\omega t_1\cos\omega t_2 - BA\cos\omega t_1\sin\omega t_2 - AB\sin\omega t_1\cos\omega t_2 + A^2\sin\omega t_1\sin\omega t_2\right]$$
$$= E\left(B^2\right)\cos\omega t_1\cos\omega t_2 - E\left(BA\right)\cos\omega t_1\sin\omega t_2 - E\left(AB\right)\sin\omega t_1\cos\omega t_2 + E\left(A^2\right)\sin\omega t_1\sin\omega t_2$$
$$= \sigma^2\cos\omega\left(t_1 - t_2\right) \qquad [\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \&\ E\left(AB\right) = E\left(BA\right) = 0]$$

$R_{YY}(t_1,t_2)$ is a function of time difference.

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$
$$= E\left[\left(A\cos\omega t_1 + B\sin\omega t_1\right)\left(B\cos\omega t_2 + A\sin\omega t_2\right)\right]$$
$$= E\left[AB\cos\omega t_1\cos\omega t_2 - A^2\cos\omega t_1\sin\omega t_2 + B^2\sin\omega t_1\cos\omega t_2 - BA\sin\omega t_1\sin\omega t_2\right]$$
$$= \sigma^2\left[\sin\omega t_1\cos\omega t_2 - \cos\omega t_1\sin\omega t_2\right]$$
$$= \sigma^2\sin\omega\left(t_1 - t_2\right) \qquad [\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \&\ E\left(AB\right) = E\left(BA\right) = 0]$$

$R_{XY}(t_1,t_2) =$ is a function of time difference.

Since $\{X(t)\}$ & $\{Y(t)\}$ are individually WSS & also $R_{XY}(t_1,t_2)$ is a function of time difference.

$\therefore$ The two random process $\{X(t)\}$ & $\{Y(t)\}$ are jointly WSS.

II Markov process.

Markov process

A random process or Stochastic process X(t) is said to be a Markov process if given the value of X(t), the value of X(v) for v > t does not depend on values of X(u) for u < t. In other words, the future behavior of the process depends only on the present value and not on the past value.

Example probability of raining day depends on weather conditions of last two days and not past weather condition.

Define a Markov process with an example.

If for $t_1 < t_2 < t_3 < \ldots < t_n < t$,

$$P\left\{X(t) \le x / X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = X_n\right\} = P\left\{X(t) \le x / X(t_n) = x_n\right\}$$

then the process $\{X(t)\}$ is called a markov process.

Example: The Poisson process is a Markov Process.

Define a Markov chain and give an example.

If for all n, $P\left\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots X_0 = a_0\right\} = P\left\{X_n = a_n / X_{n-1} = a_{n-1}\right\}$,

then the process $\{x_n\}$, $n = 0,1,\ldots$ is called a Markov chain.

Example: Poisson Process is a continuous time Markov chain.

What is a stochastic matrix? When is it said to be regular?

A sequence matrix, in which the sum of all the elements of each row is 1, is called a stochastic matrix. A stochastic matrix P is said to be regular if all the entries of P^m (for some positive integer m) are positive.

Markov chain

A Markov process is called Markov chain if the states $\{X_i\}$ are discrete no matter whether 't' is discrete or continuous. The Markov chain is irreducible if all states communicate with each other at some time.

A **regular Markov** chain is defined as a chain having a transition matrix P such that for some power of P, it has only non-zero positive probability values.

Let A be the set of all positive integers n such that $p_{ii}^{(n)} > 0$ and 'd' be the Greatest Common Divisor of the set A. We say state 'i' is periodic if d>1 and aperiodic if d = 1.

Homogeneous Markov chain

If the one-step transition probability is independent of n, i.e., $p_{ij}(n, n+1) = p_{ij}(m, m+1)$ then the Markov chain is said to have stationary transition probabilities and the process is called as homogeneous Markov chain.

Transition probability matrix.

The transition probability matrix (TPM) of the process $\{X_n, n \geq 0\}$ is defined by

$$P = [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Where the transition probabilities (elements of P) satisfy $p_{ij} \geq 0, \& \sum_{j=1}^{\infty} p_{ij} = 1, i = 1,2,3,\cdots$

Chapman-Kolmogrov Equation.

The Chapman-Kolmogrov equation provides a method to compute the n-step transition probabilities. The equation can be represented as $P_{ij}^{n+m} = \sum_{k=0}^{\infty} p_{ik}^{n} p_{kj}^{m} \, \forall n, m \geq 0$.

Problems under Markov process and morkov chain. (UQ)

1.Define a Markov chain. Explain how you would clarify the states and identify different classes of a Markov chain. Give example to each class.

Solution:

Markov Chain: If for all n,

$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ..., X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\}$ then the process $\{X_n\}$, n = 0,1, 2,... is called a Markov Chain.

<u>Classification of states of a Markov chain</u>

Irreducible: A Markov chain is said to be irreducible if every state can be reacted from every other state, where $P_{ij}^{(n)} > 0$ for some n and for all i & j.

Example: $\begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

Period: The Period d_i of a return state i is defined as the greatest common division of all m such that $P_{ij}^{(m)} > 0$

i.e., $d_i = GCD\{m : p_{ij}^{(m)} > 0\}$

State i is said to be periodic with period d_i if $d_i > 1$ and a periodic if $d_i = 1$.

Example:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ So states are with period 2.

$\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ The states are aperiodic as period of each state is 1.

Ergodic: A non null persistent and aperiodic state is called ergodic.

Example:

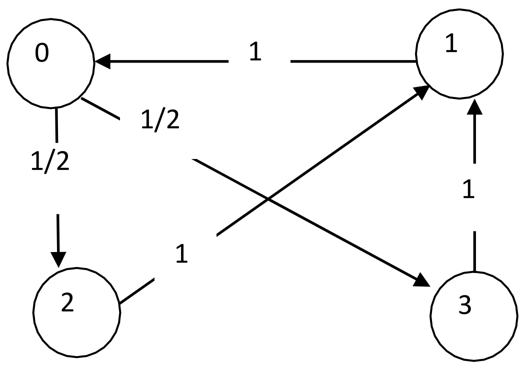
$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$ Here all the states are ergodic.

2. Let the Markov Chain consisting of the states 0, 1, 2, 3 have the transition probability matrix $P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. **Determine which states are transient and which are recurrent by defining transient and recurrent states.** **(MAY 2010)**

Solution:

Transient state: A state 'a' is transient if $F_{aa} < 1$. Recurrent state: A state 'a' is recurrent if $F_{aa} = 1$.

Here $F_{aa} = \sum_{n=0}^{\infty} f_{aa}^{(n)}$, where $f_{aa}^{(n)} =$ first time return probability of state 'a' after n steps.



Here $P^{3}_{00} > 0,\ P^{2}_{01} > 0,\ P^{1}_{02} > 0,\ P^{1}_{03} > 0$

$P^{1}_{10} > 0,\ P^{3}_{11} > 0,\ P^{2}_{12} > 0,\ P^{2}_{13} > 0$

$P^{2}_{20} > 0,\ P^{1}_{21} > 0,\ P^{3}_{22} > 0,\ P^{3}_{23} > 0$

$P^{2}_{30} > 0,\ P^{1}_{31} > 0,\ P^{3}_{32} > 0,\ P^{3}_{33} > 0$

Therefore, the Markov chain is irreducible. And also it is finite.
So, all the states are of same nature.
Consider the state '0'

$$f^{1}_{00} = 0;\quad f^{2}_{00} = 0;\ f^{3}_{00} = \frac{1}{2} + \frac{1}{2} = 1;\ f^{4}_{00} = 0 \text{ and so on.}$$

Therefore, the state '0' is recurrent.
Since, the chain is irreducible, all the states are recurrent.

3.Three boys A,B and C are throwing a ball to each other . A always throws the ball to B and B always throws the ball to C , but C is just likely to throw the ball to B as to A .Show that the process is Markovian. Find the transition matrix and classify the states

Solution:

TPM= $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix}$ Since states of X_n depend only on states of X_{n-1} .$\{X_n\}$ is a Markov chain.

Now $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$; $P^3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$

Therefore $P^{(3)}_{11} > 0,\ P^{(2)}_{13} > 0,\ P^{(2)}_{21} > 0,\ P^{(2)}_{22} > 0,\ P^{(2)}_{33} > 0$ and all other $P^{(1)}_{ij} > 0$.
Therefore chain is irreducible.

$P^4 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$; $P^5 = \begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.125 & 0.375 & 0.5 \end{bmatrix}$ and so on.

We note that $P^{(2)}_{ii}, P^{(3)}_{ii}, P^{(5)}_{ii}$ …. are greater than zero for i=2,3,.. and GCD of 2,3,5,6,….=1
Therefore the state 1 is aperiodic. Since the chain is finite and irreducible , all its states are non–null Persistant. The period of 2 and 3 is 1. The state with period 1 is aperiodic all states are ergodic

4.Find the nature of the states of the Markov chain with the TPM $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$ **and the state space** $\{1, 2, 3\}$.

Solution:

$$P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^3 = P^2 . P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = P$$

$$P^4 = P^3 . P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = P^2$$

$\therefore P^{2n} = P^2$ & $P^{2n+1} = P$

Also $P^2_{00} > 0,\ P^1_{01} > 0,\ P^2_{02} > 0$

$P^1_{10} > 0,\ P^2_{11} > 0,\ P^1_{12} > 0$

$P^2_{20} > 0,\ P^1_{21} > 0,\ P^2_{22} > 0$

□The Markov chain is irreducible

Also $P^2_{ii} = P^4_{ii} = \ldots > 0$ for all i

□The states of the chain have period 2. Since the chain is finite irreducible, all states are non null persistent. All states are not ergodic.

5.A gambler has Rs. 2. He bets Re. 1 at a time and wins Re. 1 with probability ½. He stops playing if he loses Rs. 2 or wins Rs. 4. i) What is the tpm of the related Markov chain? ii) What is the probability that he has lost his money at the end of 5 plays?

(MAY 2013)

Solution:

Let X_n denote the amount with the player at the end of the n^{th} round of the play.

The possible values of X_n = State space = {0, 1, 2, 3, 4, 5, 6}

Initial probability distribution= $P^{(0)} = (0\ \ 0\ 1\ 0\ 0\ 0\ 0)$

(i) The transition probability matrix is =

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(ii) Probability that he has lost his money at the end of 5 plays = $P[X_5 = 0]$

To find this we need $P^{(5)}$

$$P^{(1)} = P^{(0)} P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

$$P^{(2)} = P^{(1)} P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix}$$

$$P^{(3)} = P^{(2)}P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix}$$

$$P^{(4)} = P^{(3)}P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{3}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{3}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{pmatrix}$$

$$P^{(5)} = P^{(4)}P = \begin{pmatrix} \frac{3}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{5}{32} & 0 & \frac{9}{32} & 0 & \frac{1}{8} & \frac{1}{16} \end{pmatrix}$$

$\therefore \boxed{P(X_5 = 0) = 3/8}$

6. A raining process is considered as a two state Markov chain. If it rains the state is 0 and if it does not rain the state is 1. The TPM is $P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$. If the Initial distribution is $\left[0.4,\ 0.6\right]$. Find it chance that it will rain on third day assuming that it is raining today.

Solution:

$$P^2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.32 \end{pmatrix}$$

P[rains on third day / it rains today $= P\left[X_3 = 0 / X_1 = 0\right]$

$$= P_{00}^{2} = 0.44$$

7. The transition probability matrix of a Markov chain { X(t)}, n = 1,2,3,….} having three states 1, 2 and 3 is $P=\begin{bmatrix}0.1 & 0.5 & 0.4\\ 0.6 & 0.2 & 0.2\\ 0.3 & 0.4 & 0.3\end{bmatrix}$ **and the initial distribution is** $P^{(0)}=\begin{pmatrix}0.7 & 0.2 & 0.1\end{pmatrix}$**.Find**

$(i)\ P\left[X_2=3\right]$ $(ii)\ P\left[X_3=2, X_2=3, X_1=3, X_0=2\right]$**.**

(MAY 2012,2014, DEC 2013)

Solution:

We have $P^2=P.P=\begin{pmatrix}0.43 & 0.31 & 0.26\\ 0.24 & 0.42 & 0.34\\ 0.36 & 0.35 & 0.29\end{pmatrix}$

(I) $P\left(X_2=3\right)=\sum_{i=1}^{3}P\left(X_2=3/X_0=i\right)P\left(X_0=i\right)$

$=P\left(X_2=3/X_0=1\right)P\left(X_0=1\right)+P\left(X_2=3/X_0=2\right)P\left(X_0=2\right)\quad+P\left(X_2=3/X_0=3\right)P\left(X_0=3\right)$

$=P^2_{13}P\left(X_0=1\right)+P^2_{23}P\left(X_0=2\right)+P^2_{33}P\left(X_0=3\right)=0.26\times0.7+0.34\times0.2+0.29\times0.1=0.279$

(II) $P\left(X_3=2, X_2=3, X_1=3, X_0=2\right)$

$=P\left[X_0=2, X_1=3, X_2=3\right]P\left[X_3=2/X_0=2, X_1=3, X_2=3\right]$

$=P\left[X_0=2, X_1=3, X_2=3\right]P\left[X_3=2/X_2=3\right]$

$=P\left[X_0=2, X_1=3\right]P\left[X_2=3/X_0=2, X_1=3\right]P\left[X_3=2/X_2=3\right]$

$=P\left[X_0=2, X_1=3\right]P\left[X_2=3/X_1=3\right]P\left[X_3=2/X_2=3\right]$

$=P\left[X_0=2\right]P\left[X_1=3/X_0=2\right]P\left[X_2=3/X_1=3\right]P\left[X_3=2/X_2=3\right]$

$=(0.2)(0.2)(0.3)(0.4)=0.0048$

8. Let $\{X_n;\ n=1,2,3,.....\}$ **be a Markov chain with state space** $S=\{0,1,2\}$ **and 1 – step Transition probability matrix** $P=\begin{bmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{bmatrix}$ **(i) Is the chain ergodic? Explain (ii) Find the invariant probabilities.**

Solution:

$$P^2=P.P=\begin{bmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8}\\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4}\end{bmatrix}$$

$$P^3 = P^2 P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{3}{16} & \frac{5}{8} & \frac{3}{16} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{pmatrix}$$

$P_{11}^{(3)} > 0,\ P_{13}^{(2)} > 0,\ P_{21}^{(2)} > 0,\ P_{22}^{(2)} > 0,\ P_{33}^{(2)} > 0$ and all other $P_{ij}^{(1)} > 0$

Therefore the chain is irreducible as the states are periodic with period 1

i.e., aperiodic since the chain is finite and irreducible, all are non null persistent

∴ The states are ergodic.

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix}$$

$$\frac{\pi_1}{4} = \pi_0 \text{ -------------(1)}$$

$$\pi_0 + \frac{\pi_1}{2} + \pi_2 = \pi_1 \text{ -------------(2)}$$

$$\frac{\pi_1}{4} = \pi_2 \text{ -------------(3)}$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \text{ -------------(4)}$$

From (2) $\pi_0 + \pi_2 = \pi_1 - \frac{\pi_1}{2} = \frac{\pi_1}{2}$

$$\therefore \pi_0 + \pi_1 + \pi_2 = 1$$

$$\frac{\pi_1}{2} + \pi_1 = 1$$

$$\frac{3\pi_1}{2} = 1$$

$$\pi_1 = \frac{2}{3}$$

From (3) $\frac{\pi_1}{4} = \pi_2$

$$\pi_2 = \frac{1}{6}$$

Using (4) $\pi_0 + \frac{2}{3} + \frac{1}{6} = 1$

$$\pi_0 + \frac{4+1}{6} = 1$$

$$\pi_0 + \frac{5}{6} = 1 \Rightarrow \pi_0 = \frac{1}{6}$$

$\therefore \pi_0 + \frac{1}{6}, \pi_1 = \frac{2}{3}$ & $\pi_2 = \frac{1}{6}$.

9. A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that one the first day of the week, the man tossed a fair dice and drove to work iff a 6 appeared. Find the probability that he takes a train on the third day and also the probability that on the third day and also the probability that he drives to work in the long run.

Solution:

State Space = (train, car)

The TPM of the chain is

$$P = \begin{matrix} & T & C \\ T & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} \end{matrix}$$

P (traveling by car) = P (getting 6 in the toss of the die)= $\frac{1}{6}$

& P (traveling by train) = $\frac{5}{6}$

$$P^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$$

$$P^{(2)} = P^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$P^{(3)} = P^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

P (the man travels by train on the third day) = $\frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

By the property of π, $\pi P = \pi$

$$(\pi_1 \ \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1 \ \pi_2)$$

$$\frac{1}{2}\pi_2 = \pi_1$$

$$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$$

& $\pi_1 + \pi_2 = 1$

Solving $\pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$

P{The man travels by car in the long run }= $\frac{2}{3}$.

10. Three are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are inter changed. Let the state a_i of the system be the number of red marbles in A after i changes. What is the probability that there are 2 red marbles in A after 3 steps? In the long run, what is the probability that there are 2 red marbles in urn A?

Solution:

State Space $\{X_n\} = \{0, 1, 2\}$ Since the number of ball in the urn A is always 2.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

$X_n = 0$, $A = 2W$ (Marbles) $B = 3R$ (Marbles)

$X_{n+1} = 0 \quad P_{00} = 0$

$X_{n+1} = 1 \quad P_{01} = 1$

$X_{n+1} = 2 \quad P_{02} = 0$

$X_n = 0$, $A = 1W \& 1R$ (Marbles) $B = 2R \& 1W$ (Marbles)

$X_{n+1} = 0 \quad P_{10} = \frac{1}{6}$

$X_{n+1} = 1 \quad P_{11} = \frac{1}{2}$

$X_{n+1} = 2 \quad P_{12} = \frac{1}{3}$

$X_n = 2$, $A = 2R$ (Marbles) $B = 1R \& 2W$ (Marbles)

$X_{n+1} = 0 \quad P_{20} = 0$

$X_{n+1} = 1 \quad P_{21} = \frac{2}{3}$

$X_{n+1} = 2 \quad P_{22} = \frac{1}{3}$

$P^{(0)} = (1, 0, 0)$ as there is not red marble in A in the beginning.

$P^{(1)} = P^{(0)} P = (0, 1, 0)$

$P^{(2)} = P^{(1)} P = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$

$P^{(3)} = P^{(2)} P = \left(\frac{1}{12}, \frac{23}{36}, \frac{5}{18} \right)$

$\therefore$ P (There are 2 red marbles in A after 3 steps) $= P\{X_3 = 2\} = P_2^{(3)} = \frac{5}{18}$

Let the stationary probability distribution of the chain be $\pi = (\pi_0, \pi_1, \pi_2)$.

By the property of π, $\pi P = \pi$ & $\pi_0 + \pi_1 + \pi_2 = 1$

$$\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix}$$

$\frac{1}{6}\pi_1 = \pi_0$

$\pi_0 + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2 = \pi_1$

$\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 = \pi_2$

& $\pi_0 + \pi_1 + \pi_2 = 1$

Solving $\pi_0 = \frac{1}{10}, \pi_1 = \frac{6}{10}, \pi_2 = \frac{3}{10}$

P {here are 2 red marbles in A in the long run} = 0.3.

11.A salesman territory consists of three cities A, B and C. He never sells in the same city on successive days. If he sells in city A, then the next day he sells in city B. However if he sells in either city B or city C, the next day he is twice as likely to sell in city A as in the other city.In the long run how often does he sell in each of the cities?

(MAY 2012,DEC 2013)

Solution:

States: A, B and C

The transition probability matrix is given by $P = \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$

The long run probability is given by $\pi = \begin{bmatrix} a & b & c \end{bmatrix}$, where π P = π .

Now $\pi \mathrm{P} = \pi \Rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$

$\therefore\ 0.a + \frac{2}{3}b + \frac{2}{3}c = a \Rightarrow -a + \frac{2}{3}b + \frac{2}{3}c = 0 \cdots (1)$

$1.a + 0.b + \frac{1}{3}c = b \Rightarrow a - b + \frac{1}{3}c = 0 \cdots (2)$

$0.a + \frac{1}{3}b + 0.c = c \Rightarrow \frac{1}{3}b - c = 0 \cdots (3)$

Also, we know that a + b + c = 1 $\cdots(4)$

From (3), c = b/3

From (2), a - b + b/9 = 0 $\Rightarrow$ a = 8b/9

From (4), 8b/9 + b + b/3 = 1 $\Rightarrow$ 20b/9 = 1 $\Rightarrow$ $\boxed{b = 9/20}$

$\therefore$ $\boxed{c = 3/20 \text{ and } a = 8/20}$

$\therefore$ Long run probability = $[8/20 \quad 9/20 \quad 3/20]$

12. The transition probability matrix of a Markov chain { X(t)}, n = 1,2,3,….} having three states 1, 2 and 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ **and the initial distribution is** $P^{(0)} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \end{pmatrix}$ **.Find**

$(i)\ P[X_2 = 3]$ $(ii)\ P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$.

(MAY 2012,2014, DEC 2013)

Solution:

We have $P^2 = P.P = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$

(I) $P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3/X_0 = i)P(X_0 = i)$

$= P(X_2 = 3/X_0 = 1)P(X_0 = 1) + P(X_2 = 3/X_0 = 2)P(X_0 = 2) + P(X_2 = 3/X_0 = 3)P(X_0 = 3)$

$= P^2_{13}P(X_0 = 1) + P^2_{23}P(X_0 = 2) + P^2_{33}P(X_0 = 3) = 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 = 0.279$

(II) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

$= P[X_0 = 2, X_1 = 3, X_2 = 3]P[X_3 = 2/X_0 = 2, X_1 = 3, X_2 = 3]$

$= P[X_0 = 2, X_1 = 3, X_2 = 3]P[X_3 = 2/X_2 = 3]$

$= P[X_0 = 2, X_1 = 3]P[X_2 = 3/X_0 = 2, X_1 = 3]P[X_3 = 2/X_2 = 3]$

$= P[X_0 = 2, X_1 = 3]P[X_2 = 3/X_1 = 3]P[X_3 = 2/X_2 = 3]$

$= P[X_0 = 2]P[X_1 = 3/X_0 = 2]P[X_2 = 3/X_1 = 3]P[X_3 = 2/X_2 = 3]$

$= (0.2)(0.2)(0.3)(0.4) = 0.0048$

13. If the transition probability matrix of a markov chain is $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ find the steady-state distribution of the chain.

$$\left(2 \quad 2 \right)$$

Solution:

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution on stationary state distribution of the markov chain.

By the property of π, $\pi P = \pi$

i.e., $(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$

$\frac{1}{2}\pi_2 = \pi_1$ ----------- (1)

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$ ------ (2)

Equation (1) & (2) are one and the same.

Consider (1) or (2) with $\pi_1 + \pi_2 = 1$, since π is a probability distribution.

$\pi_1 + \pi_2 = 1$

Using (1), $\frac{1}{2}\pi_2 + \pi_2 = 1$

$\frac{3\pi_2}{2} = 1$

$\pi_2 = \frac{2}{3}$

$\pi_1 = 1 - \pi_2 = 1 - \frac{2}{3} = \frac{1}{3}$

$\pi_2 = 1 - \pi_1 = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$.

III Poisson process

postulates of a poisson process

If $\{X(t)\}$ represents the number of occurrences of a certain event in $(0, t)$ then the discrete random process $\{X(t)\}$ is called the poisson process, provided the following postualates are satisfied

(i) $P[1 \; occumence \; in \; (t, t+\Delta t)] = \lambda \Delta t + o(\Delta t)$

(ii) $P[no \; occurrence \; in \; (t, t+\Delta t)] = 1 - \lambda \Delta t + o(\Delta t)$

(iii) $P[2 \; or \; more \; occurrences \; in \; (t, t+\Delta t)] = \lambda \Delta t + o(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.

(v) The probability that the event occurs in a specified number of times $(t_0, t_0 + t)$ depends only on t, but not on t_0

State and establish the properties of Poisson process.

(i). Sum of two independent poisson process is a poisson process.

Proof:

The moment generating function of the Poisson process is

$$M_{X(t)}(u) = E\left[e^{uX(t)}\right]$$
$$= \sum_{x=0}^{\infty} e^{ux} P[X(t)=x]$$
$$= \sum_{x=0}^{\infty} e^{ux} \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$
$$= e^{-\lambda t}\left[1 + \frac{e^u(\lambda t)^1}{1!} + \frac{e^{2u}(\lambda t)^2}{2!} + \dots\right]$$
$$= e^{-\lambda t} e^{\lambda t e^u}$$
$$M_{X(t)}(u) = e^{\lambda t(e^u - 1)}$$

Let $X_1(t)$ and $X_2(t)$ be two independent Poisson processes

$\therefore$ Their moment generating functions are,

$$M_{X_1(t)}(u) = e^{\lambda_1 t(e^u-1)} \text{ and } M_{X_2(t)}(u) = e^{\lambda_2 t(e^u-1)}$$
$$\therefore M_{X_1(t)+X_2(t)}(u) = M_{X_1(t)}(u)\, M_{X_2(t)}(u)$$
$$= e^{\lambda_1 t(e^u-1)} e^{\lambda_2 t(e^u-1)}$$
$$= e^{(\lambda_1+\lambda_2)t(e^u-1)}$$

$\therefore$ By uniqueness of moment generating function, the process $\{X_1(t)+X_2(t)\}$ is a Poisson process with occurrence rate $(\lambda_1+\lambda_2)$ per unit time.

Prove that (i) difference of two independent Poisson processes is not a Poisson process and (ii) Poisson process is a Markov process. **(MAY 2013)**.

(i) Let $X(t) = X_1(t) - X_2(t)$ where $X_1(t)$ and $X_2(t)$ are poisson processes with λ_1 and λ_2 as the parameters

$$E[X(t)] = E[X_1(t)] - E[X_2(t)] = (\lambda_1 - \lambda_2)t$$
$$E[X^2(t)] = E\left\{[X_1(t) - X_2(t)]^2\right\} = E[X_1^2(t)] + E[X_2^2(t)] - 2E[X_1(t)X_2(t)]$$
$$= E[X_1^2(t)] + E[X_2^2(t)] - 2E[X_1(t)]E[X_2(t)]$$
$$= (\lambda_1^2 t^2 + \lambda_1 t) + (\lambda_2^2 t^2 + \lambda_2 t) - 2(\lambda_1 t)(\lambda_2 t) = (\lambda_1 + \lambda_2)t + (\lambda_1^2 + \lambda_2^2)t^2 - 2\lambda_1\lambda_2 t^2$$
$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2 \neq (\lambda_1 - \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

$\therefore X_1(t) - X_2(t)$ is not a Poisson process.

(ii) Consider $P[X(t_3) = n_3 / X(t_2) = n_2, X(t_1) = n_1] = \dfrac{P[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3]}{P[X(t_1) = n_1, X(t_2) = n_2]}$

$$= \frac{\dfrac{e^{-\lambda t_3}\lambda^{n_3} t_1^{n_1}(t_2 - t_1)^{n_2-n_1}(t_3 - t_2)^{n_3-n_2}}{n_1!(n_2-n_1)!(n_3-n_2)!}}{\dfrac{e^{-\lambda t_2}\lambda^{n_2} t_1^{n_1}(t_2 - t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}} = \frac{e^{-\lambda(t_3-t_2)}\lambda^{n_3-n_2}(t_3-t_2)^{n_3-n_2}}{(n_3-n_2)!} = P\left[X(t_3)=n_3 / X(t_2)=n_2\right]$$

$$\therefore P\left[X(t_3 = n_3 / X(t_2)=n_2, X(t_1)=n_1)\right] = P\left[X(t_3)=n_3 / X(t_2)=n_2\right]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1, X(t_2) = n_2$ depends only on the most recent values $X(t_2) = n_2$.

i.e., The Poisson process possesses Markov property.Consider

$$P\left[X(t_3 = n_3 / X(t_2)=n_2, X(t_1)=n_1)\right] = \frac{P\left[X(t_1)=n_1, X(t_2)=n_2, X(t_3)=n_3\right]}{P\left[X(t_1)=n_1, X(t_2)=n_2\right]}$$

$$= \frac{\dfrac{e^{-\lambda t_3}\lambda^{n_3} t_1^{n_1}(t_2 - t_1)^{n_2-n_1}(t_3 - t_2)^{n_3-n_2}}{n_1!(n_2-n_1)!(n_3-n_2)!}}{\dfrac{e^{-\lambda t_2}\lambda^{n_2} t_1^{n_1}(t_2 - t_1)^{n_2-n_1}}{n_1!(n_2-n_1)!}}$$

$$= \frac{e^{-\lambda(t_3-t_2)}\lambda^{n_3-n_2}(t_3-t_2)^{n_3-n_2}}{(n_3-n_2)!}$$

$$P\left[X(t_3 = n_3 / X(t_2)=n_2, X(t_1)=n_1)\right] = P\left[X(t_3)=n_3 / X(t_2)=n_2\right]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1, X(t_2) = n_2$ depends only on the most recent values $X(t_2) = n_2$ .i.e.,

The Poisson process possesses Markov property.

(iii). The inter arrival time of a poisson process i.e., with the interval between two successive occurrences of a poisson process with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$.

Proof:

Let two consecutive occurrences of the event be E_i & E_{i+1}.

Let E_i take place at time instant t_i and T be the interval between the occurrences of E_i E_{i+1}.

Thus T is a continuous random variable.

$$P(T > t) = P\{Interval\ between occurrence of\ E_i\ and\ E_{i+1}\ exceeds t\}$$

$$= P\{E_{i+1}\ does\ not\ occur\ upto\ the\ instant\ (t_i + t)\}$$

$$= P\{No\ event\ occurs\ in\ the\ interval\ (t_i, t_i + t)\}$$

$$= P\{X(t) = 0\} = P_0(t)$$

$$= e^{-\lambda t}$$

$\therefore$The cumulative distribution function of T is given by

$$F(t) = P\{T \le t\} = 1 - e^{-\lambda t}$$

∴ The probability density function is given by

$$f(t) = \lambda e^{-\lambda t},\ (t \ge 0)$$

Which is an exponential distribution with mean $\frac{1}{\lambda}$.

When is a poisson process said to be homogenous?

The rate of occurrence of the event λ is a constant, then the process is called a homogenous poisson process

Problems under **poisson process**

1. If the process $\{N(t): t \ge 0\}$ is a Poisson process with parameter λ, obtain $P[N(t) = n]$ and $E[N(t)]$.

Solution:

Let λ be the number of occurrences of the event in unit time.

Let $P_n(t)$ represent the probability of n occurrences of the event in the interval $(0, t)$. i.e., $P_n(t) = P\{X(t) = n\}$

$$\therefore P_n(t+\Delta t) = P\{X(t+\Delta t) = n\}$$

$$= P\{n \textit{ occurences in the time } (0, t+\Delta t)\}$$

$$= P\begin{cases} n \textit{ occurences in the interval } (0,t) \textit{ and no occurences in } (t, t+\Delta t) \textit{ or} \\ n-1 \textit{ occurences in the interval } (0,t) \textit{ and } 1 \textit{ occurences in } (t, t+\Delta t) \textit{ or} \\ n-2 \textit{ occurences in the interval } (0,t) \textit{ and } 2 \textit{ occurences in } (t, t+\Delta t) \textit{ or...} \end{cases}$$

$$= P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t + 0 + \ldots$$

$$\therefore \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda\{P_{n-1}(t) - P_n(t)\}$$

Taking the limits as $\Delta t \to 0$

$$\frac{d}{dt}P_n(t) = \lambda\{P_{n-1}(t) - P_n(t)\} \quad (1)$$

This is a linear differential equation.

$$\therefore\ P_n(t)e^{\lambda t} = \int_0^t \lambda P_{n-1}(t)e^{\lambda t} \text{ ------------------------ } (2)$$

Now taking $n = 1$ we get

$$e^{\lambda t}P_1(t) = \lambda\int_0^t P_0(t)e^{\lambda t}dt \text{ -------------- } (3)$$

Now, we have,

$$P_0(t+\Delta t) = P\{0 \textit{ occurences in } (0, t+\Delta t)\}$$

$$= P\{0 \textit{ occurences in } (0,t) \textit{ and } 0 \textit{ occurences in } (t, t+\Delta t)\}$$

$$= P_0(t)(1-\lambda\Delta t)$$

$$P_0(t+\Delta t) - P_0(t) = -\lambda P_0(t)\Delta t$$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

∴ Taking limit $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$$

$$\frac{dP_0(t)}{P_0(t)} = -\lambda\, dt$$

$$\log P_0(t) = -\lambda t + c$$

$$P_0(t) = e^{-\lambda t + c}$$

$$P_0(t) = e^{-\lambda t} e^{c}$$

$$P_0(t) = e^{-\lambda t} A \text{ ---------------(4)}$$

Putting $t = 0$ we get

$$P_0(0) = e^{0} A = A$$

i.e., $A = 1$

∴(4) we have

$$P_0(t) = e^{-\lambda t}$$

∴substituting in (3) we get

$$e^{\lambda t} P_1(t) = \int_0^t \lambda e^{-\lambda t} e^{\lambda t} dt$$

$$= \lambda \int_0^t dt = \lambda t$$

$$P_1(t) = e^{-\lambda t} \lambda t$$

Similarly $n = 2$ in (2) we have,

$$P_2(t) e^{\lambda t} = \lambda \int_0^t P_1(t) e^{\lambda t} dt$$

$$= \lambda \int_0^t e^{-\lambda t} \lambda t e^{\lambda t} dt$$

$$= \lambda^2 \left(\frac{t^2}{2} \right)$$

$$P_2(t) e^{\lambda t} = \frac{e^{-\lambda t} (\lambda t)^2}{2!}$$

Proceeding similarly we have in general

$$P_n(t) = P\{X(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!},\ n = 0,1,\ldots$$

Thus the probability distribution of $X(t)$ is the Poisson distribution with parameter λt.

$$E[X(t)] = \lambda t.$$

2. Find the mean and autocorrelation and auto covariance of the Poisson process.

Solution:

The probability law of the poisson process $\{X(t)\}$ is the same as that of a poisson distribution with parameter λt

$$E\left[X(t)\right] = \sum_{n=0}^{\infty} nP\left(X(t) = n\right)$$

$$= \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda t(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t} e^{\lambda t}$$

$$E\left[X(t)\right] = \lambda t$$

$$Var\left[X(t)\right] = E\left[X^2(t)\right] - E\left[X(t)\right]^2$$

$$E\left\{X^2(t)\right\} = \sum_{n=0}^{\infty} n^2 P\left(X(t) = n\right)$$

$$= \sum_{n=0}^{\infty} \left(n^2 - n + n\right) \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda t}(\lambda t)^n}{n!} + \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda t}(\lambda t)^n}{n!} + \lambda t$$

$$= e^{-\lambda t}\left[\frac{(\lambda t)^2}{1} + \frac{(\lambda t)^3}{1!} + \frac{(\lambda t)^4}{2!} + \dots\right] + \lambda t$$

$$= (\lambda t)^2 e^{-\lambda t} e^{\lambda t} + \lambda t$$

$$= (\lambda t)^2 + \lambda t$$

$$E\left\{X^2(t)\right\} = \lambda t + \lambda^2 t^2$$

$$\therefore Var\left[X(t)\right] = \lambda t$$

$$R_{XX}(t_1, t_2) = E\left\{X(t_1) X(t_2)\right\}$$

$$= E\left\{X(t_1)\left[X(t_2) - X(t_1) + X(t_1)\right]\right\}$$

$$= E\left\{X(t_1)\left[X(t_2) - X(t_1)\right] + E\left[X^2(t_1)\right]\right\}$$

$$= E\left[X(t_1)\right] E\left[X(t_2) - X(t_1)\right] + E\left[X^2(t_1)\right]$$

Since $\{X(t)\}$ is a process of independent increments.

$$= \lambda t_1\left[\lambda(t_2 - t_1)\right] + \lambda t_1 + \lambda t_1^2 \; if \; t_2 \geq t_1 \qquad (by - (1))$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 \; if \; t_2 \geq t_1$$

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

Auto Covariance

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E\{X(t_1)\} E\{X(t_2)\}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2$$

$$= \lambda t_1, \; if \; t_2 \geq t_1$$

$$= \lambda \min(t_1, t_2)$$

3. If the customers arrive at a bank according to a poisson process with a mean rate of 2 per minute, find the probability that, during an 1- minute interval no customer arrives.

Solution:

Here $\lambda = 2$, t □1

$$\therefore P\{X(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \; n = 0, 1, 2, ...$$

Probability during 1-min interval, no customer arrives = $P\{X(t) = 0\} = e^{-2}$.

4. Examine whether the Poisson process {X(t)} is stationary or not. (DEC2010, 2012, APR 2015)

A random process to be stationary in any sense, its mean must be a constant. We know that the mean of a Poisson process with rate λ is given by **E{X(t)} = λt** which depends on the time t. Thus the Poisson process is not a stationary process.

5. **Is a Poisson process a continuous time Markov chain? Justify your answer (MAY2010)**

We know that Poisson process has the Markovian property. Therefore, it is a Markov chain as the states of Poisson process are discrete. Also, the time‘t’ in a Poisson process is continuous.Therefore, the Poisson process a continuous time Markov chain.

6. A radioactive source emits particles at a rate of 5 per min in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in 4 min period. (DEC 2014)

The number of particles N(t) emitted is Poisson with parameter λ= np = 5(0.6) = 3

$$P(N(t) = m) = \frac{e^{-3t} (3t)^m}{m!} \Rightarrow$$

$$P(N(4) = 10) = \frac{e^{-3(4)} (3(4))^{10}}{10!} = 0.1048.$$

7.**On the average a submarine on patrol sights 6 enemy ships per hour. Assuming that the number of ships sighted in a given length of time is a Poisson variate. Find the probability Of sighting 6 ships in the next half-an-hour , 4 ships in the next 2 hours and at lest 1 ship in the next 15 minutes**. (APRIL 2017)

Solution:

Mean arrival rate = mean number of arrivals per minute (unit time) = λ=6/hr

Given λ = 6. $P\{X(t) = k\} = \frac{e^{-\lambda t} (\lambda t)^k}{\lfloor k}$

(i) $P\left\{X\left(\frac{1}{2}\right)=6\right\} = \frac{e^{-3}3^6}{\lfloor 6} = 0.0504$

(ii) $P\{X(2)=4\} = \frac{e^{-12}(12)^4}{\lfloor 4} = 0.053$

(iii) $P\left\{X\left(\frac{1}{4}\right)\geq 1\right\} = 1-P\left\{X\left(\frac{1}{4}\right)=0\right\} = 1-\left\{\frac{e^{-\frac{3}{2}}\left(\frac{3}{2}\right)^0}{\lfloor 0}\right\} = 1-e^{-\frac{3}{2}} = 0.776.$

8.Suppose that customers arrive at a bank according to Poisson process with mean rate of 3 per minute. Find the probability that during a time interval of two minutesi) exactly 4 customers arrive ii) greater than 4 customers arrive iii) fewer than 4 customers arrive. (MAY 2012, DEC 2013)

Solution:

Mean of the Poisson process = λt.

Mean arrival rate = mean number of arrivals per minute (unit time) = λ

Given λ = 3. $P\{X(t)=k\} = \frac{e^{-\lambda t}(\lambda t)^k}{\lfloor k}$

(i) Probability that exactly 4 customers arrive $P\{X(2)=4\} = \frac{e^{-6}6^4}{\lfloor 4} = 0.133$

(ii) Probability that greater than 4 customers arrive

$$P\{X(2)>4\} = 1-\{P[X(2)=0]+P[X(2)=1]+P[X(2)=2]+P[X(2)=3]+P[X(2)=4]\}$$

$$= 1-\left\{\frac{e^{-6}6^0}{\lfloor 0}+\frac{e^{-6}6^1}{\lfloor 1}+\frac{e^{-6}6^2}{\lfloor 2}+\frac{e^{-6}6^3}{\lfloor 3}+\frac{e^{-6}6^4}{\lfloor 4}\right\} = 0.715$$

(iii) Probability that fewer than 4 customers arrive

$$P\{X(2)<4\} = P[X(2)=0]+P[X(2)=1]+P[X(2)=2]+P[X(2)=3]$$

$$= \frac{e^{-6}6^0}{\lfloor 0}+\frac{e^{-6}6^1}{\lfloor 1}+\frac{e^{-6}6^2}{\lfloor 2}+\frac{e^{-6}6^3}{\lfloor 3} = 0.151$$