ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY
Near Anjugramam Junction, Kanyakumari Main Road, Palkulam, Variyoor P.O - 629401
Kanyakumari Dist, Tamilnadu., E-mail : admin@rcet.org.in, Website : www.rcet.org.in

DEPARTMENT OF MATHEMATICS

SUBJECT NAME : NUMERICAL METHODS

SUBJECT CODE : MA8491

REGULATION : 2017

UNIT – V : NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS:

CLASSIFICATION OF PDE OF SECOND ORDER:

Let a second order PDE in the function 'U' of two independent variable (x, y) of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{---} (1)$$

$$\Rightarrow A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

This eqn. is linear in the second order term but the form $F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ may be linear or non-linear.

Equ. (1) is classified as elliptic, parabollic, hyperbolic at the points of a given region 'R' depending on whether:

$B^2 - 4AC < 0$ (Elliptic equ.)

$B^2 - 4AC = 0$ (Parabolic eqn)

$B^2 - 4AC > 0$ (Hyperbolic eqn)

EXAMPLES:

Elliptic type : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ [Laplace eqn., in 2-D]

Parabollic type : $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t}$ [1-D Heat flow eqn]

Hyperbolic type : $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial t^2}$ [1-D wave eqn.]

CLASSIFY THE FOLLOWING EQUATION :

(1) What is the classification of $f_{xx} + 2f_{xy} +$ $=0$?

To find :

$$B^2 - 4AC$$

$A=1$; $B=2$; $C=1$

$B^2 - 4AC = 4-4 = 0.$

$\Rightarrow B^2 - 4AC = 0.$

$\therefore$ The eqn. is parabolic.

(2) $f_{xx} - f_{yy} = 0$

$A=1$; $B=0$; $C=-1$.

$B^2 - 4AC = 0+4 = 4 > 0$

$\therefore$ The eqn. is Hyperbolic.

(3) $x^2 f_{xx} + (1-y)^2 f_{yy} = 0$; $-\infty < x < \infty$ $-1 < y <$

$A = x^2$; $B=0$; $C = (1-y)^2$.

$B^2 - 4AC = 0 - 4x^2(1-y)^2$.

$= -4x^2(1-y)^2$

which is clearly negative for $-\infty < x$ ∞ and $-1 < y < 1$. Hence, the equation is ellip

(4) $y u_{xx} + u_{yy} = 0$.

$A = y$, $B=0$, $C=1$.

$B^2 - 4AC = 0 - 4y$

$= -4y$

which is hyperbolic for negative 'y' a parabo for positive 'y'

3) $f_{xx} - 2f_{xy} = 0$

$A = 1;\ B = -2,\ C = 0$

$B^2 - 4AC = (-2)^2 - 4\cdot(1)(0)$

$= 4 - 0$

$= 4 > 0$

$\therefore$ The equation is hyperbolic.

ELLIPTIC EQUATION : $(B^2 - 4AC < 0)$

Laplace eqn, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is elliptic equation.

Liebmann's Iteration Process

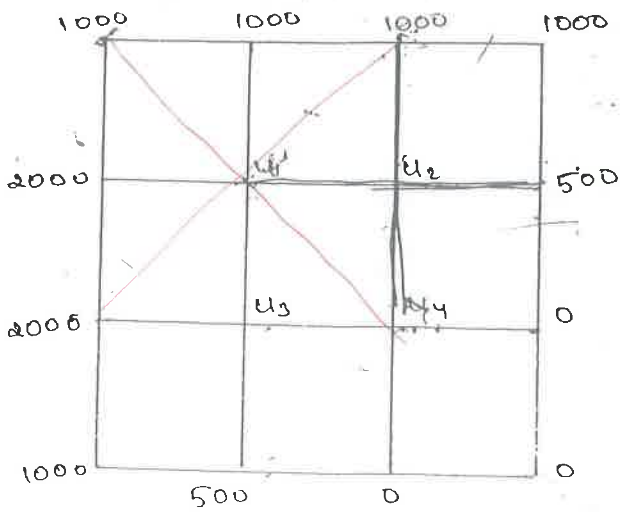
STANDARD FIVE POINT FORMULA (SFPF) :-

$$u_{ij} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$ all the v.ts except first

DIAGONAL FIVE POINT FORMULA : (DFPF)

$$u_{ij} = \frac{1}{4}\left(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right)$$

i) Determine by iteration method. Given that values of $u(x,y)$ on the boundary of the square, evaluate $u(x,y)$ satisfying Laplace eqn. $\nabla^2 u = 0$ at the pivotal points of this figure.



$u_5 \to$ SFPF

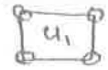
$u_1, u_3, u_7, u_9 \to$ DFPF

$u_2, u_4, u_6, u_8 \to$ SFPF

Soln :

(4)

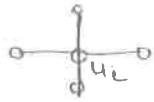
$$u_1 = \frac{1}{4}\left(1000 + 1000 + 2000 + \underset{\substack{\Downarrow \\ 0}}{u_4}\right)$$ assume



(Diagonal averaging)

$$\boxed{u_1 = 1000}$$

$$u_2 = \frac{1}{4}\left(u_1 + 1000 + 500 + u_4\right).$$



cross averaging

$$u_2 = \frac{1}{4}\left[1000 + 1000 + 500 + 0\right]$$

$$\boxed{u_2 = 625.}$$

$$u_3 = \frac{1}{4}\left(2000 + u_1 + u_4 + 500\right) \quad \text{[cross averaging]}$$

$$= \frac{1}{4}\left[2000 + 1000 + 0 + 500\right]$$

$$\boxed{u_3 = 875}$$

$$u_4 = \frac{1}{4}\left(u_3 + u_2 + 0 + 0\right)$$

$$= \frac{1}{4}\left(875 + 625 + 0 + 0\right)$$

$$\boxed{u_4 = 375}$$

Next iteration :

$$u_1' = \frac{1}{4}\left(2000 + 1000 + u_2 + u_3\right) \quad \text{[cross averaging]}$$

$$= \frac{1}{4}\left[2000 + 1000 + 625 + 875\right]$$

$$\boxed{u_1' = 1125}$$

$$u_2' = \frac{1}{4}\left(u_1' + 1000 + 500 + u_4\right) \quad \text{[cross averaging]}$$

$$= \frac{1}{4}\left(1125 + 1000 + 500 + 375\right)$$

$$\boxed{u_2' = 750}$$

$$u_3' = \frac{1}{4}\left(2000 + u_1' + u_4 + 500\right) \quad \text{[cross]}$$

$$= \frac{1}{4}\left(2000 + 1125 + 375 + 500\right)$$

$$u_3' = 1000$$

Repeat the Iteration Process until u get the required accuracy.

$$\boxed{u_3' = 1000}$$

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$$u_4' = \frac{1}{4}(u_3' + u_2' + 0 + 0) \qquad [\text{cross}]$$

$$= \frac{1}{4}(1000 + 750 + 0 + 0)$$

$$\boxed{u_4' = 437.6 \sim 438}$$

Next iteration :-

$$u_1'' = \frac{1}{4}(2000 + 1000 + u_2' + u_3') \qquad [\text{cross}]$$

$$= \frac{1}{4}(2000 + 1000 + 750 + 1000)$$

$$\boxed{u_1''. = 1187.5 \sim 1188}$$

$$u_2'' = \frac{1}{4}(u_1'' + 1000 + 500 + u_4') \qquad [\text{cross}]$$

$$= \frac{1}{4}(1188 + 1000 + 500 + 438)$$

$$\boxed{u_2'' = 781.5 \sim 782}$$

$$u_3'' = \frac{1}{4}(2000 + u_1'' + u_4' + 500) \qquad [\text{cross}]$$

$$= \frac{1}{4}(2000 + 1188 + 438 + 500)$$

$$\boxed{u_3'' = 1031.5 \sim 1032}$$

$$u_4'' = \frac{1}{4}(u_3'' + u_2'' + 0 + 0) \qquad [\text{cross}]$$

$$= \frac{1}{4}[1032 + 782]$$

$$\boxed{u_4'' = 453.5 \sim 454}$$

Next iteration :

$$u_1^{(3)} = \frac{1}{4}(2000 + 1000 + u_2'' + u_3'') \qquad [\text{cross}]$$

$$u_1^{(3)} = \frac{1}{4}(2000 + 1000 + 782 + 1032)$$

$$\boxed{u_1^{(3)} = 203.5 \sim 204}$$

$$u_2^{(3)} = \frac{1}{4}(u_1^{(3)} + 1000 + 500 + u_4'') \qquad [\text{cross}]$$

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$= \frac{1}{4}(204 + 1000 + 500 + 454)$

$\boxed{= 789.5 \sim 790}$

$u_3^{(3)} = \frac{1}{4}(2000 + u_1^{(3)} + u_4^{(2)} + 500)$ [cross]

$= \frac{1}{4}(2000 + 1204 + 454 + 500)$

$\boxed{u_3^{(3)} = 1039.5 \sim 1040}$

$u_4^{(3)} = \frac{1}{4}(u_3^{(3)} + u_2^{(3)} + 0 + 0)$ [cross]

$= \frac{1}{4}(1040 + 790)$

$\boxed{u_4^{(3)} = 457.5 \sim 458}$

Next iteration :

$u_1^{(4)} = \frac{1}{4}(2000 + 1000 + u_2^{(3)} + u_3^{(3)})$

$= \frac{1}{4}(3000 + 790 + 1040)$

$\boxed{u_1^{(4)} = 1207.5 \sim 1208}$

$u_2^{(4)} = \frac{1}{4}(u_1^{(4)} + 1000 + 500 + u_4^{(3)})$

$= \frac{1}{4}(1500 + 1208 + 458)$

$\boxed{u_2^{(4)} = 791.5 \sim 792}$

$u_3^{(4)} = \frac{1}{4}(2000 + u_1^{(4)} + u_4^{(3)} + 500)$

$= \frac{1}{4}(2500 + 1208 + 458)$

$\boxed{u_3^{(4)} = 1041.5 \sim 1042}$

$$u_4^{(4)} = \frac{1}{4}\left(u_3^{(4)} + u_2^{(4)} + 0 + 0\right)$$

$$= \frac{1}{4}(1042 + 792)$$

$$\boxed{u_4^{(4)} = 458.5 \sim 459}$$

Next iteration :

$$u_1^{(5)} = \frac{1}{4}\left(2000 + 1000 + u_2^{(4)} + u_3^{(4)}\right)$$

$$= \frac{1}{4}(3000 + 792 + 1042)$$

$$\boxed{u_1^{(5)} = 1208.5 \sim 1209}$$

$$u_2^{(5)} = \frac{1}{4}\left(u_1^{(5)} + 1000 + 500 + u_4^{(4)}\right)$$

$$= \frac{1}{4}(1209 + 1500 + 459)$$

$$\boxed{= 792}$$

$$u_3^{(5)} = \frac{1}{4}\left(2000 + u_1^{(5)} + u_4^{(4)} + 500\right)$$

$$= \frac{1}{4}(2500 + 1209 + 459)$$

$$\boxed{u_3^{(5)} = 1042}$$

$$u_4^{(5)} = \frac{1}{4}\left(u_3^{(5)} + u_2^{(5)} + 0 + 0\right)$$

$$= \frac{1}{4}(1042 + 792)$$

$$\boxed{u_4^{(5)} = 458.5 \sim 459}$$

Gauss Seidal

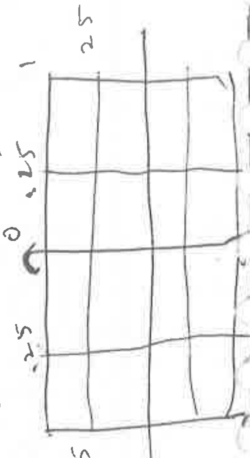
$\therefore u_1 = 1209;\ u_2 = 792;\ u_3 = 1042;\ u_4 = 459.$

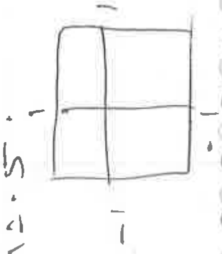
Iteration	0	1	2	3	4	5
u_1	1000	1125	1188	1204	1208	1209
u_2	625	750	782	790	792	792
u_3	875	1000	1032	1040	1042	1042
u_4	375	438	454	458	459	459

(2) Solve $\nabla^2 u = 0$, $|x| < 1$, $|y| < 1$

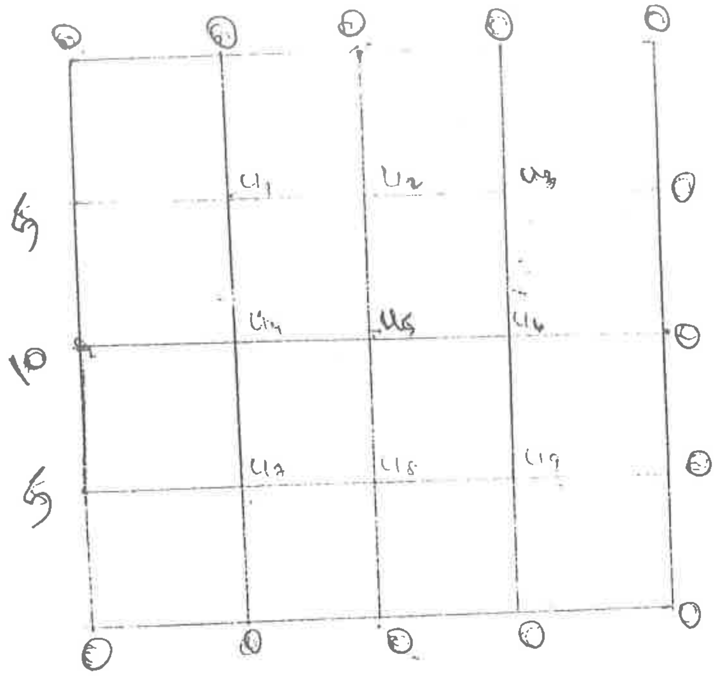
$u(x, \pm 1) = x^2$, $u(\pm 1, y) = 2y^2$, choose

$h = .5$





(2) Solve the elliptic eqn, $u_{xx} + u_{yy} = 0$ for the following square mesh, with boundary values given using Leibmann's iteration method. Iterate until the max. difference between successive values at any point is less than 0.001.



Let $u_1, u_2 \ldots u_9$ be the values of u at the interior lattice points,

we first determine the initial values;

First u_5, u_1, u_3, u_7, u_9 by Diagonal five point formula.
(odd).

Then u_2, u_4, u_6, u_8 by standard five point formula.

$u_5^{(0)} = \frac{1}{4}(10+0+0+0)$ [100 — u_5 — 00 (cross averaging) 0]

$\boxed{= 2.5 \sim 3}$

I → U_5 → use +

All odd → u_1, u_3 → use ×

All even → u_2, u_4 → use +

From 2nd Itera... use

From 2nd Itali use + only.

+ average (S.F.P)

$= \frac{1}{4}(5+5+2) = 3.$ $\boxed{u_7^{(1)} = 3}$

$u_8^{(1)} = \frac{1}{4}\left(u_7^{(1)} + u_5^{(1)} + u_9^{(0)} + 0\right)$

$= \frac{1}{4}(3+3+1) = 1.75 \sim 2.$ $\boxed{u_8^{(1)} = 2}$

$u_9^{(1)} = \frac{1}{4}\left(u_8^{(1)} + u_6^{(1)} + 0 + 0\right)$ $\boxed{u_9^{(1)} = 1}$

$= \frac{1}{4}(2+1) = 0.75 \sim 1$

$\therefore u_1 = 3,\ u_2 = 2;\ u_3 = 1,\ u_4 = 5;\ u_5 = 3;\ u_6 = 1$

$u_7 = 3;\ u_8 = 2;\ u_9 = 1.$

Poisson equation

Another important PDE is $\nabla^2 u = F(x,y)$

(i.e.,) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y)$ where $F(x,y)$ is the given functions of x and y is known as poisson's equation.

Solve the PDE $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x = y = 0$, $x = y = 3$, with $u = 0$ on the boundary and mesh length $= 1$.

(i)

$i = x$ $j = y$ $i = 1$ $j = 2$

Let the values of 'u' at the four mesh points A, B, C, D be u_1, u_2, u_3, u_4 respectively.

The differential equation is $\nabla^2 u = -10(x^2+y^2+10)$

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$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = F(ih, jh)$$

(ie.,) $u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = h^2 F(ih, jh)$

$\Rightarrow u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 F(ih, jh)$

Initial iteration:

At A, $i=1, j=2$.

$0 + 0 + u_2 + u_3 - 4u_1 = -10(1^2+2^2+10)$.

$u_2 + u_3 - 4u_1 = -150 \longrightarrow (1)$.

At B, $i=2, j=2$.

$u_1 + 0 + 0 + u_4 - 4u_2 = -10(2^2+2^2+10)$

$u_1 + u_4 - 4u_2 = -180 \longrightarrow (2)$.

At C, $i=1, j=1$.

$0 + u_1 + u_4 + 0 - 4u_3 = -10(1^2+1^2+10)$

$u_1 + u_4 - 4u_3 = -120 \longrightarrow (3)$

At D, $i=2, j=1$.

$u_3 + u_2 + 0 + 0 - 4u_4 = -10(2^2+1^2+10)$

$u^2 + u_3 - 4u_4 = -150 \longrightarrow (4)$

From (1), (2), (3) & (4)

$u_1 = \frac{1}{4}(150 + \underset{(0)}{u_2} + \underset{(0)}{u_3}) = 37.5 \longrightarrow (5)$

$u_2 = \frac{1}{4}(\underset{37.5}{u_1} + \underset{37.5}{u_4} + 180) = \longrightarrow (6)$

u_1 //
u_2 //
u_3 // //
u_4

4

$\longrightarrow$ (7)

=

(4) $u_4 = \frac{1}{4}(u_2 + u_3 + 180)$

$\longrightarrow$ (8)

=

From (5) & (8)

$u_1 = u_4$

$\therefore$ To find u_1, u_2, u_3 only.

FIRST ITERATION:

$u_1^{(1)} = \frac{1}{4}(150 + 0 + 0) \boxed{= 37.5}$

$u_2^{(1)} = \frac{1}{4}(u_1^{(1)} + u_4 + 180)$

$= \frac{1}{4}(37.5 + 180 + 37.5)$, $(\because u_1 = u_4)$

$\boxed{u_2^{(1)} = 63.75}$

$u_3^{(1)} = \frac{1}{4}(u_1^{(1)} + u_4^{(1)} + 120)$

$= \frac{1}{4}(75 + 120) \boxed{= 48.75}$

$\boxed{u_4^{(1)} = u_1^{(1)} = 37.5}$

SECOND ITERATION:

$u_1^{(2)} = \frac{1}{4}(u_2^{(1)} + u_3^{(1)} + 150)$

$= \frac{1}{4}(63.75 + 48.75 + 150)$

$\boxed{u_1^{(2)} = 65.625 \simeq 66}$

$u_2^{(2)} = \frac{1}{4}(u_1^{(2)} + u_4^{(2)} + 180)$

$= \frac{1}{4}(66 + 66 + 180) = 78$

$\boxed{u_2^{(2)} = 78}$

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$$u_3^{(2)} = \frac{1}{4}\left(u_1^{(2)} + u_4^{(2)} + 120\right)$$

$$= \frac{1}{4}(66 + 66 + 120)$$

$$\boxed{= 63.}$$

THIRD ITERATION:

$$u_1^{(3)} = \frac{1}{4}\left(160 + u_2^{(2)} + u_3^{(2)}\right)$$

$$= \frac{1}{4}(160 + 78 + 63) = 72.5 \sim 73.$$

$$u_2^{(3)} = \frac{1}{4}\left(u_1^{(3)} + u_4^{(3)} + 180\right)$$

$$= \frac{1}{4}(73 + 73 + 180) = 81.5 \sim 82$$

$$u_3^{(3)} = \frac{1}{4}\left(u_1^{(3)} + u_4^{(3)} + 120\right)$$

$$= \frac{1}{4}(73 + 73 + 120)$$

$$\boxed{= 66.5 \sim 67}$$

$$\boxed{u_4^{(3)} = 73}$$

FOURTH ITERATION:

$$u_1^{(4)} = \frac{1}{4}\left(u_2^{(3)} + u_3^{(3)} + 160\right)$$

$$= \frac{1}{4}[82 + 67 + 160) = 75$$

$$u_2^{(4)} = \frac{1}{4}\left(u_1^{(4)} + u_4^{(4)} + 180\right)$$

$$= \frac{1}{4}(75 + 75 + 180) = 82.5 \sim 83$$

$$u_3^{(4)} = \frac{1}{4}\left(u_1^{(4)} + u_4^{(4)} + 120\right)$$

$$= \frac{1}{4}(75 + 75 + 120) = 67.5 \sim 68$$

FIFTH ITERATION:

$$u_1^{(5)} = \frac{1}{4}\left(u_2^{(4)} + u_3^{(4)} + 160\right)$$

$$= \frac{1}{4}(83 + 68 + 160) = 75.25 \sim 75$$

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$$u_2^{(5)} = \frac{1}{4}\left(u_1^{(4)} + u_4^{(4)} + 180\right)$$

$$= \frac{1}{4}(75 + 75 + 180) = 82.5 \sim 83$$

$$u_3^{(5)} = \frac{1}{4}\left(u_1^{(4)} + u_4^{(4)} + 120\right)$$

$$= \frac{1}{4}(75 + 75 + 120) = 67.5 \sim 68$$

$$\boxed{u_4^{(5)} = 75}$$

$\therefore u_1 = 75, \ u_2 = 83; \ u_3 = 68; \ u_4 = 75$ //

PARABOLIC EQUATIONS:

$B^2 - 4AC = 0$ at all the points of the region.

Eg: 1-D heat equation.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

where: $\alpha^2 = \frac{K}{c\rho}$

1) Find the solution of the parabolic equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t} = 0.$$

when $u(0,t) = 0$; $u(4,t) = 0$; $u(x,0) = x(4-x)$

Assume $h=1$. Find the values upto $t=5$.

The general heat equation is

$$u_{xx} = a \cdot u_t.$$

Given:

$u(0,t) = 0.$

$u(4,t) = 0.$

$u(x,0) = x(4-x)$

Here $a = 2$, $h = 1$

$$\lambda = \frac{k}{h^2 a} = \frac{k}{2}$$

If $k = 1$; $\lambda = \frac{1}{2}$

Bender-Schmidt's Recurrence relation:

(19)

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

$u(x,0) = x(4-x)$

$u(1,0) = 1(4-1) = 3$

$u(2,0) = 2(4-2) = 4$

$u(3,0) = 3(4-3) = 3$

$u(x,0) = x(4-x)$

$u(1,0) = 1(4-1) = 3$

$u(2,0) = 2(4-2) = 4$

$u(3,0) = 3(4-3) = 3$

$i=1,\ j=2$

$u_{1,2+1} = \frac{1}{2}[u_{1+1,0} + u$

$= \frac{1}{2}[u_{2,0} + u$

$= \frac{1}{2}[4+0$

Formula:

S.T, G.S, RMT, HV, AHS — Bad boys

Put $j=0$ in ①

$$u_{i,j+1} = \frac{1}{2}\left[u_{i+1,0} + u_{i-1,0}\right].$$

$$u_{i,j+1} = \frac{1}{2}\left[u_{i+1,j} + u_{i-1,j}\right] \rightarrow ①$$

$3 = \frac{1}{2}[u_{1+1,0} + u_{1-1,}$

$= \frac{1}{2}[\overset{4}{2}, u_{0}]$

$u_{2,1+1} = \frac{1}{2}[u_{2+1,0} + u_{2-1}$

$= \frac{1}{2}[u_{3,0} + u_{1,0}$

$= \frac{1}{2}[3+3]$

$u_{1,2+1} = \frac{1}{2}[u_{1+1,0} + u_{1-1,}$

$= \frac{1}{2}[u_{2,0} + u_{0,0}$

$= \frac{1}{2}[4].$

Hyperbolic equations:

$$B^2 - 4AC > 0$$

Eg: 1-D wave eqn,

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0.$$

Boundary conditions:

$u(0,t) = 0$

$u(l,t) = 0$

$u(x,0) = f(x)$

$u_t(x,0) = 0.$

① $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$. Given $u(0,t) = 0 = u(4,t)$; $u_t(x,0) = 0$ and $u(x,0) = x(4-x)$ taking $h=1$ and $k=\frac{1}{2}$.

$j \backslash i$	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	$0+3-3=0$	$2+2-4=0$	$3+0-3=0$	0
3	0	$0+0-2=-2$	-3	-2	0
4	0	-3	-4	-3	0
5	0	-2	-3	-2	0
6	0	0	0	0	0
7	0	2	3	2	0
8	0	3	4.	3	0

$u(x,0) = x(4-x)$

$$\boxed{u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}}$$

— x —

Boundary value problems for ordinary and partial differential equation:

Finite diff — ord, partial

~~Type 2~~

Finite difference method to solve ordinary differential equation

Finite difference method to solve partial differential equation

Type: ~~2~~

Partial differential equation

Parabolic - one dimensional heat eqn

Hyperbolic - one dimensional wave eqn

elliptic - poisson and Laplace eqns (2-Dimension)

Type: 1

Finite difference method to solve ordinary differential eqn:-

Consider a second order differential eqn of the form

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = f(x) \quad - (1)$$

$$y'' + a_1 y' + a_2 y = f(x)$$

By finite difference method replace $y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$

and $y' = \frac{y_{i+1} - y_{i-1}}{2h}$, where 'h' is the step size.

$y(x_0) = y_0$

Solve the differential eqn $\frac{d^2y}{dx^2} - y = x$ with $y(0)=0$, $y(1)=0$ with $h = 1/4$

Solution:- Given $y'' - y = x \to (1)$ with $y(0) = 0$, $y(1) = 0$.

$x_0 = 0$, $y_0 = 0$.

$x_1 = 0.25$, $y_1 = ?$

$x_2 = 0.5$, $y_2 = ?$

$x_3 = 0.75$, $y_3 = ?$

$x_4 = 1$, $y_4 = 0$

x_0	x_1	x_2	x_3	x_4
0	.25	.5	.75	1
0	y_1	y_2	y_3	0

sub $y'' = \dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$ in ①

Replace y by y_i and x by x_i

$$\therefore y'' - y = x \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = x_i$$

$$y_{i-1} - 2y_i + y_{i+1} - h^2 y_i = h^2 x_i$$

Since $h = 1/4$, the above eqn becomes,

$$y_{i-1} - 2y_i + y_{i+1} - \frac{1}{16} y_i = \frac{1}{16} x_i$$

$$16 y_{i-1} - 32 y_i + 16 y_{i+1} - y_i = x_i$$

$$16 y_{i+1} - 33 y_i + 16 y_{i-1} = x_i \quad - ②$$

Sub $i = 1, 2, 3$ (Since at $i = 0$ & 4, we know y_i values)

when $i = 1$,

$$16 y_2 - 33 y_1 + 16 y_0 = x_1 \quad - ③$$

when $i = 2$,

$$16 y_3 - 33 y_2 + 16 y_1 = x_2 \quad - ④$$

when $i = 3$,

$$16 y_4 - 33 y_3 + 16 y_2 = x_3 \quad - ⑤$$

③ $\Rightarrow 16 y_2 - 33 y_1 + 16(0) = 0.25 \Rightarrow 16 y_2 - 33 y_1 = 0.25$ — ⑥

④ $\Rightarrow 16 y_3 - 33 y_2 + 16 y_1 = 0.5$ — ⑦

⑤ $\Rightarrow 16(0) - 33 y_3 + 16 y_2 = 0.75 \Rightarrow 16 y_2 - 33 y_3 = 0.75$ — ⑧

Eqns ⑥, ⑦, ⑧ can be solved by any one of the followin methods. ① ordinary simultaneous method ② Gauss eliminatio method ③ Gauss Jordan ④ Gauss Seidal method.

above system.

from ⑥ & ⑧ $\quad 16y_2 - 33y_1 = 0.25$

$$\frac{\begin{array}{r} 16y_2 - 33y_3 = 0.75 \\ {\scriptstyle(-)} \quad\quad {\scriptstyle(+)} \quad\quad {\scriptstyle(-)} \end{array}}{-33y_1 + 33y_3 = -0.5} \quad - ⑨$$

On solving for y_1, y_2 and y_3 we get,

$\boxed{y_1 = -0.03488}$ $\quad \boxed{y_2 = -0.05632}$ $\quad \boxed{y_3 = -0.05003}$

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2) Solve $\frac{d^2y}{dx^2} = y$, $y(1) = 1.1752$ $\quad y(3) = 10.0179$, $h = 1/2$.

Solution: Let $h = 1/2$ then $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$.

$y_0 = 1.1752$, $y_1 = ?$, $y_2 = ?$, $y_3 = ?$, $y_4 = 10.0179$

By using finite difference method, the given differential eqn will be,

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = y_i$$

(ie) $y_{i-1} - 2y_i + y_{i+1} = h^2 y_i \quad - ①$

Let $i = 1$, $\quad y_0 - 2y_1 + y_2 = \frac{1}{4}y_1 \Rightarrow 4y_0 - 8y_1 + 4y_2 = y_1$

$4(1.1752) - 8y_1 + 4y_2 = y_1$

$-9y_1 + 4y_2 = -4.7008$

$9y_1 - 4y_2 = 4.7008 \quad - ②$

Let $i = 2$, $\quad y_1 - 2y_2 + y_3 = \frac{1}{4}y_2$

$4y_1 - 8y_2 + 4y_3 = y_2$

$4y_1 - 9y_2 + 4y_3 = 0 \quad - ③$

$$y_2 - 2y_3 + y_4 = \frac{1}{4} y_3$$

$$y_2 - 2y_3 + 10.0179 = \frac{1}{4} y_3$$

$$4y_2 - 8y_3 + 40.0716 = y_3$$

$$4y_2 - 9y_3 = -40.0716 \quad - (4)$$

(3) × 9 $\quad 36y_1 + 36y_3 - 81y_2 = 0$

(4) × 4 $\quad 16y_2 - 36y_3 = -160.2864$

$$36y_1 - 65y_2 = -160.2864 \quad - (5)$$

Using (2) and (5)

(5) $\Rightarrow 36y_1 - 65y_2 = -160.2864$

(2) × 4 $\Rightarrow 36y_1 - 16y_2 = 18.8032$

$$49y_2 = 179.0896$$

$$y_2 = \frac{179.0896}{49} = 3.65489$$

(2) $\Rightarrow 9y_1 - 4y_2 = 4.7008$

$$9y_1 = 4.7008 + (4 \times 3.65489)$$

$$9y_1 = 19.32036$$

$$y_1 = 2.1467$$

from (4) $\Rightarrow$

$$4y_2 - 9y_3 = -40.0716$$

$$4(3.65489) - 9y_3 = -40.0716$$

$$-9y_3 = -54.69116$$

$$y_3 = 6.0768$$

$y_1 = 2.1467 \qquad y_2 = 3.65489 \qquad y_3 = 6.0768$

3). Solve the b.v.p at $x = 0.5$, $\frac{d^2y}{dx^2} + y = 0$, $y(0) = y(1) = 0$ with $h = 1/4$

$y_0 = 0$, $y_4 = 0$

0	.25	.5	1	
0		y_2	0	

$y_2 = 0.14031$

$y(0.5) = y$

taking $h = 1/2$ Ans: $y_1 = 0.14285$

Solve $y'' = y$, $y(0) = 0$ $y(2) = 3.627$ taking $h = 1/2$

Ans: $y(0.5) = 0.5262$ $y(1) = 1.1843$ $y(1.5) = 2.1332$ //

Type : 2

Boundary value problems of partial differential equation:

A general two dimensional second order partial differential eqn is of the form $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$

(Or) $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

where A, B, C, D, E, F, G are functions of x and y only.

The equation is said to be

1. Elliptic if $B^2 - 4AC < 0$
2. parabolic if $B^2 - 4AC = 0$
3. Hyperbolic if $B^2 - 4AC > 0$

classify the following partial differential equation:

(i) $xf_{xx} + yf_{yy} = 0$ $x > 0$, $y > 0$

$A = x$, $B = 0$, $C = y$ $\therefore B^2 - 4AC = 0 - 4xy < 0$ for $x, y > 0$

$\therefore$ The equation is elliptic

(ii) $f_{xx} + 2f_{xy} + 4f_{yy} = 0$

$A = 1$, $B = 2$, $C = 4$ $\therefore B^2 - 4AC = 4 - 4(1)(4)$

$= 4 - 16 < 0$

$\therefore$ The equation is elliptic.

③ $\partial xx - 2\partial xy + \partial yy = 0$

$A = 1, \ B = -2, \ C = 1$ $\quad f_{xx} - 2f_{xy} + f_{yy} = 0.$

$B^2 - 4AC = 4 - 4(1)(1) = 0$

$\therefore$ The equation is parabolic.

④ $U_{xx} + U_{yy} = 0$

$A = 1, \ B = 0, \ C = 1$

$B^2 - 4AC = 0 - 4 < 0$ The eqn is elliptic. This eqn is also known as Laplace eqn in two dimension.

⑤ $U_{xx} + U_{yy} = f(x, y)$ $\quad A = 1, \ B = 0 \quad C = 1$

$B^2 - 4AC = -4 < 0$ The eqn is elliptic and is called poisson equation.

⑥ $U_t = \alpha^2 \dfrac{\partial^2 U}{\partial x^2}$

$\alpha^2 \dfrac{\partial^2 U}{\partial x^2} - U_t = 0$ $\quad A = \alpha^2, \ B = 0, \ C = 0, \ E = -1$

$B^2 - 4AC = 0$ The eqn is parabolic. This eqn is called one-dimensional heat eqn.

⑦ $\alpha^2 \dfrac{\partial^2 U}{\partial x^2} - \dfrac{\partial^2 U}{\partial t^2} = 0$

$A^2 = \alpha^2, \ B = 0, \ C = -1$

$B^2 - 4AC = 0 - 4(\alpha^2)(-1) > 0$

$\therefore$ The eqn is hyperbolic. This eqn is called one-dimensional wave equation.

finite difference method :-

I Parabolic equations :

One dimensional heat equation :

The one-dimensional heat eqn is given by, $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

where $\alpha^2 = \frac{K}{C\rho}$, where C - sp. heat of the material

ρ - density K - thermal conductivity Assume $h = k =$

put $U_{xx} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$

$U_t = \frac{U_{i,j+1} - U_{i,j}}{k}$

Explicit The given eqn becomes,

$$\boxed{U_{i,j+1} = \lambda U_{i+1,j} + (1-2\lambda) U_{i,j} + \lambda U_{i-1,j}} \quad (1)$$

is called the explicity formula. This is valid if $0 < \lambda \leq 1/2$

where $\lambda = \frac{k}{h^2 \rho}$ (or) $\lambda = \frac{k}{a h^2}$

If $\lambda = 1/2$ we have $U_{i,j+1} = \frac{1}{2}\left[U_{i+1,j} + U_{i-1,j}\right]$ — (2) is called Bender Schmidt recurrence eqn or relation. The solution obtained by using (2) is stable only for $\lambda = 1/2$ otherwise unstable.

Type I

Find the soln of the parabolic eqn $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$ with $u(0,t) = 0$ $u(4,t) = 0$ $u(x,0) = x(4-x)$ by taking $h = 1$ find the value of 'u' upto $t = 5$

(ii) $\left[\dfrac{\partial^2 u}{\partial x^2} = a\dfrac{\partial u}{\partial t}\right]$.

$\therefore \alpha^2 = \dfrac{1}{2}$ (ie) $a = 2$

Wkt, $\lambda = \dfrac{k}{h^2 a} = \dfrac{k}{1\times 2}$ //

Choose $k=1$, so that $\lambda = 1/2$ // now the Bender Schmidt recurrence eqn can be applied.

Given: $u(4,t) = 0$; $u(0,t) = 0$; $u(x,0) = x(4-x)$ Take $h=1$; $k=1$

$u(4,t) = 0 \Rightarrow$ when $x=4$ and for any values of 't' $u(4,t)$

$u(0,t) = 0 \Rightarrow$ when $x=0$ and for any values of 't', $u(0,t)$

we have $u(x,0) = x(4-x)$

when $x=0$, $u(0,0) = 0$

$x=1$, $u(1,0) = 1(4-1) = 3$

$x=2$, $u(2,0) = 2(4-2) = 4$

$x=3$, $u(3,0) = 3(4-3) = 3$

$x=4$, $u(4,0) = 4(4-4) = 0$

$\therefore$ To find all values of u upto 5, we proceed as follows:

$x=i$ / $t=j$	0 ($u(0,t)$)	1	2	3	4 ($u(4,t)$)
0	0 ($u_{i-1,j}$)	3	4 ($u_{i+1,j}$)	3	0
1	0	2 // ($u_{i,j+1}$)	3	2	0
2	0	1.5	2	1.5 //	0
3	0	1	1.5	1	0
4	0	0.75	1 //	0.75	0
5	0	0.5	0.75	0.5	0

when $\lambda = 1/2$

eg: $\dfrac{0+4}{2} = 2$

$u(2,4) = \dfrac{1+1}{2} = 1$

$u(3,2) = \dfrac{3+0}{2} = 1.5$

differential eqn: $U_t = 4U_{xx}$ and the boundary conditions $U(0,t) = 0 = U(8,t)$ and $U(x,0) = 4x - x^2/2$ at the pts $x = i$, $i = 0 \ldots 8$ and $\boxed{t = \frac{1}{8} j}$, $j = 0, 1, 2, 3, 4, 5$.

Solution:

$h = 1$, $k = 1/8$, $\alpha^2 = 4 = 1/a$; $a = 1/4$

$u_{xx} = a u_t$. $u_t = \frac{1}{a} u_{xx}$. $u_t = \alpha^2 u_{xx}$. $\alpha^2 = 1/a$. $\alpha^2 = 4$. $a = 1/4$

$$\lambda = \frac{k}{h^2 * a} = \frac{1/8}{1 * 1/4} = \frac{1}{2}$$

∴ The Bender Schmidt relation can be applied.

(ie) $U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right]$

$$\boxed{U(x,0) = 4x - x^2/2}$$

$U(0,0) = 0$; $U(1,0) = 3.5$; $U(2,0) = 6$; $U(3,0) = 7.5$; $U(4,0) = 8$

$U(5,0) = 7.5$; $U(6,0) = 6$; $U(7,0) = 3.5$; $U(8,0) = 0$

$x = ih$ / $t = jk = \frac{1}{8}j$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

$\partial t \quad 2 \quad \partial x^2$

condition and initial condition $U(x,0) = \frac{1}{4}x(15-x)$, $0 \le x \le 12$

$U(0,t) = 0$, $U(12,t) = 9$, $0 \le t \le 12$. using Schmidt relation

$h = K = 3$.

Type II

Solution:

Solve

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \Rightarrow \alpha^2 = \frac{1}{2} \quad \text{and} \quad a = 2$$

$$x = 0, 3, 6, 9, 12 \qquad t = 0, 3, 6, 9, 12$$

$$\lambda = \frac{K}{ah^2} = \frac{3}{2 \times 3^2} = \underline{\frac{1}{6}}$$

$\therefore$ Bender-Schmidt recurrence relation cannot be applied.

$\therefore$ By using explicit formula (Schmidt),

$$U_{i,j+1} = \lambda U_{i+1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i-1,j}$$

$$U_{i,j+1} = \frac{1}{6}U_{i+1,j} + \frac{2}{3}U_{i,j} + \frac{1}{6}U_{i-1,j}$$

$$= \frac{1}{6}\left[U_{i+1,j} + U_{i-1,j}\right] + \frac{2}{3}U_{i,j}$$

when $x = 3$, $U(3,0) = \frac{1}{4} \times 3(15-3) = 9$

$$U(6,0) = \frac{1}{4} \times 6(15-6) = 13.5$$

$$U(9,0) = \frac{1}{4} \times 9(15-9) = 13.5$$

$$U(12,0) = \frac{1}{4} \times 12(15-12) = 9$$

$t \backslash x$	0	3	6	9	12
0	0 a	9 b	13.5 c	13.5	9.
3	0	8.25	12.75	12.75	9
6	0	a 7.625	b 12	c 12.125	9
9	0	7.083	11.292	11.583	9
12	0	6.604	10.639	11.104	9

$0 \le x \le 12$

$0 \le t \le 12.$

$U(6,9) = \frac{1}{6}[7.625 + 12.125] + \frac{2}{3}(12)$
$= 11.292$

$$U_{i,j+1} = \frac{1}{6}\left[U_{i+1,j} + U_{i-1,j}\right] + \frac{2}{3}U_{i,j}$$

$$U_d = \frac{1}{6}\left[a + c\right] + \frac{2}{3}b$$

$$U(3,3) = \frac{1}{6}\left[0 + \overset{13.5}{\cancel{9}}\right] + \frac{2}{3}(9) = 8.25$$

5) Using Schmidt's method find the values of $U(x,t)$ satisfying the eqn: $4U_{xx} = U_t$ and the boundary conditions $U(0,t) = 0$, $U(8,t) = 0$, $U(x,0) = \frac{x}{2}(8-x)$ at all pts $x = i$ where $i = 0, 1, 2 \ldots$ and $t = j/8$ where $j = 0,1,2,3,4,5$.

Solution :-

given $4U_{xx} = U_t \Rightarrow \alpha^2 = 4$ and $a = 1/4$

$\lambda = \frac{k}{ah^2} = \frac{1/8}{1/4 \times 1} = \frac{1}{2}$ [If we choose $k = 1/8$, we get $\lambda = 1/2$]

∴ Bender - Schmidt recurrence relation can be applied

(ii) $U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right]$

when $x=0$, $U(0,0)=0$

$U(1,0)=3.5$; $U(2,0)=6$; $U(3,0)=7.5$; $U(4,0)=8$;
$U(5,0)=7.5$; $U(6,0)=6$; $U(7,0)=3.5$; $U(8,0)=0$

$t \backslash x$	0	1	2	3	4	5	6	7	8	
0	0	3.5	6	7.5	8	7.5	6	3.5	0	
1	0	3	5.5	7	7.5	7	5.5	3	0	
2	0	2.75	5	6.5	7	6.5	5	2.75	0	
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0	
4	0	2.3125	4.25	5.5635	6	5.5625	4.25	2.3125	0	
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0	

$$U_{i,j+1}=\frac{1}{2}\left[U_{i-1,j}+U_{i+1,j}\right]=\frac{1}{2}\left[c+a\right]$$

$$U(2,1)=\frac{1}{2}\left[7.5+3.5\right]=5.5$$

(5) Given $U_{xx}=U_t$, $u(0,t)=U(5,t)=0$, $U(x,0)=x^2(25-x^2)$ find the values of U in the range $h=1$ and upto 3 secon.

Solution:

given $U_{xx}=U_t \Rightarrow \alpha^2=1$, $a=1/1=1$, $\lambda=\frac{k}{ah^2}$

In order to apply Bender Schmidt relation, $k=1/2$

thus $\boxed{\lambda=1/2}$

The meaning is 3 seconds is divided as $t=0, 1/2, 1, 1.5, 2, 2.5, 3$. x takes the values from 0 to 5 with $h=1$.

given $U(x,0) = x^2(25-x^2)$

$U(0,0) = 0$; $U(1,0) = 24$; $U(2,0) = 84$; $U(3,0) = 144$

$U(4,0) = 144$; $U(5,0) = 0$.

	$t \backslash x$	0	1	2	3	4	5
0	0	0	24	84	144	144	0
1	0·5	0	42	84	144	72	0
2	1·0	0	42	78	78	57	0
3	1·5	0	39	60	67·5	39	0
4	2·0	0	30	53·25	49·5	33·75	0
5	2·5	0	26·625	39·75	43·5	24·75	0
	3·0	0	19·875	35·0625	32·25	21·75	0

6) Use Schmidt process to solve $25U_{xx} = U_t$ where $0 < x < 10$ and $t > 0$ with the boundary conditions $U(0,t) = 0$, $U(10,t) = 0$, $U(x,0) = \frac{x(10-x)}{25}$ and choosing ~~$h = 12k$~~ suitably find U_{ij} for $i = 1,2,3 \ldots 9$ and $j = 1,2,3,4$.

Solution:

Given $25U_{xx} = U_t$ (ie) $\alpha^2 = 25$, $a = 1/25$, $h = 1$

choose k s.t $\lambda = 1/2$ $\quad$ $K = 1/50$ so that $\lambda = \frac{K}{ah^2} = \frac{1/50}{\frac{1}{25} \times 1}$

$\boxed{\lambda = 1/2}$

Hence Bender Schmidt relation can be applied.

't' takes the value from 0, 1, 2, 3, 4

given $U(x,0) = \frac{x(10-x)}{25}$ $\quad U(0,t) = U(10,t) = 0$

$U(0,0) = 0$; $U(1,0) = 9/25$; $U(2,0) = 16/25$; $U(3,0) = 21/25$

$U(4,0) = 24/25$; $U(5,0) = 25/25$; $U(6,0) = 24/25$;

$U(7,0) = 21/25$; $U(8,0) = 16/25$; $U(9,0) = 9/25$;

$U(10,0) = 0$.

$t \backslash x$	0	1	2	3	4	5	6	7	8	9	10
0	0	0.36	0.64	0.84	0.96	1	0.96	0.84	0.64	0.36	0
1	0	0.32	0.6	0.8	0.92	0.96	0.92	0.8	0.6	0.32	0
2	0	0.3	0.56	0.76	0.88	0.92	0.88	0.76	0.64	0.3	0
3	0	0.28	0.53	0.72	0.84	0.88	0.84	0.76	0.53	0.32	0
4	0	0.265	0.5	0.685	0.8	0.84	0.82	0.685	0.54	0.265	0

(7) Solve by Schmidt's method $U_t = 5U_{xx}$ with the condition $U(0,t) = 0$, $U(5,t) = 60$ and $U(x,0) = 20x$ for $0 \leq x \leq 3$ and $U(x,0) = 60$ for $3 \leq x \leq 5$. for 5 time steps. having $h = 1$.

Solution: given $U_t = 5U_{xx} \Rightarrow \alpha^2 = 5$ and $a = 1/5$

$\lambda = \frac{k}{ah^2}$ choose K, s.t $\lambda = 1/2$

Let $K = 1/10$ st $\lambda = \frac{1/10}{1/5 \times 1} = 1/2$

also $U(0,t)=0$, $U(5,t)=60$, $U(x,0) = 20x$ for $0 \le x \le 3$
$= 60$ for $3 \le x \le 5$.

$t \backslash x$	0	1	2	3	4	5
0	0	20	40	60	60	60
1	0	20	40	50	60	60
2	0	20	35	50	55	60
3	0	17.5	35	45	55	60
4	0	17.5	31.25	45	52.5	60
5	0	15.625	31.25	41.875	52.5	60

Crank - Nicholson method for solving parabolic equation:

This method is to solve a parabolic eqn of the form $\alpha^2 U_{xx} = U_t$ (or) $U_{xx} = a U_t$ [where $\alpha^2 = 1/a$]

$$\lambda = \frac{k}{ah^2}$$

The Crank - Nicholson formula is

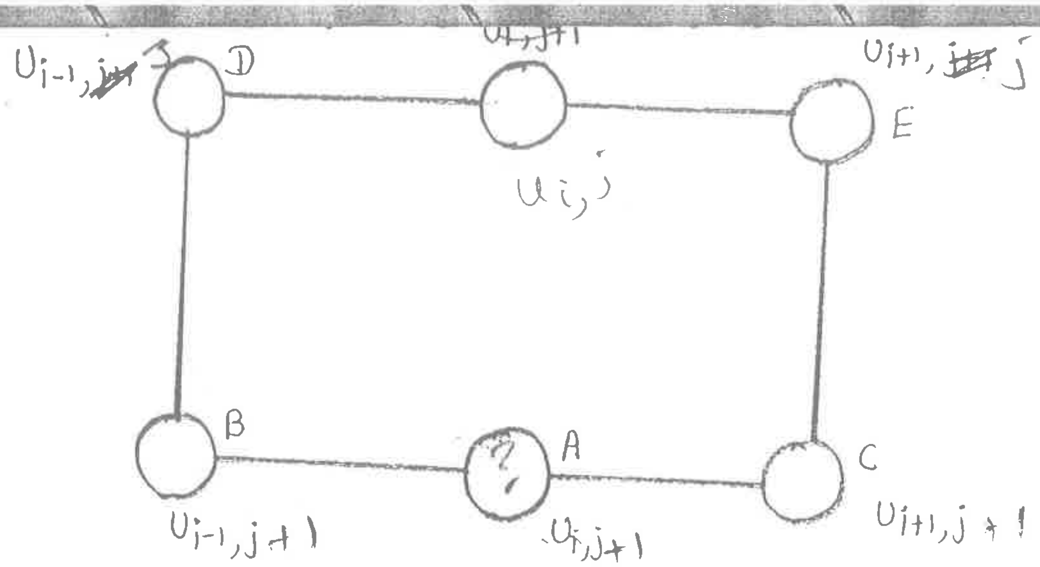
$$\lambda(U_{i+1,j+1} + U_{i-1,j+1}) - 2(\lambda+1)U_{i,j+1} = 2(\lambda-1)U_{i,j} - \lambda(U_{i+1,j} + U_{i-1,j}) \quad -(1)$$

T. If $\lambda = 1$, then (1) becomes,

$$4U_{i,j+1} - U_{i+1,j+1} - U_{i-1,j+1} = U_{i+1,j} + U_{i-1,j}$$

(or).

$$U_{i,j+1} = \frac{1}{4}\left[U_{i-1,j+1} + U_{i+1,j+1} + U_{i-1,j} + U_{i+1,j}\right] \quad -(2)$$



Also the value of U at A is the average of the values at B, C, E, D [(ie) ② when $\lambda = 1$]

II The Crank-Nicholson formula is convergent for all values of λ

① Use Crank-Nicholson method to solve $U_t = U_{xx}$ subject $U(x,0) = 0$ $U(0,t) = 0$ and $U(1,t) = t$ taking $h = 1/4$, $K = 1/8$ for one step in t-direction.

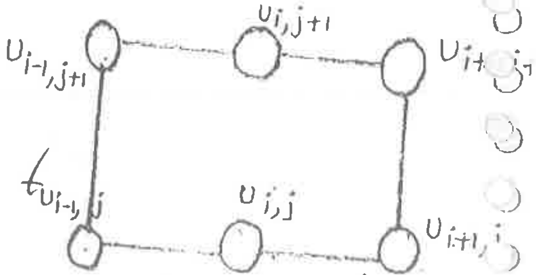
Solution: given $U_t = U_{xx}$, $\alpha^2 = 1$, $a = \cancel{1}$, $h = 1/4$, $K = 1/8$

$\lambda = \frac{k\alpha^2}{h^2}$

$U(x,0) = 0$

$U(0,t) = 0$ and $U(1,t) = t$

The crank-Nicholson formula is



$$\lambda(U_{i-1,j+1} + U_{i+1,j+1}) - 2(\lambda+1)U_{i,j+1} = 2(\lambda-1)U_{i,j} - \lambda(U_{i-1,j} + U_{i+1,j})$$

$= \lambda(u$ — ①

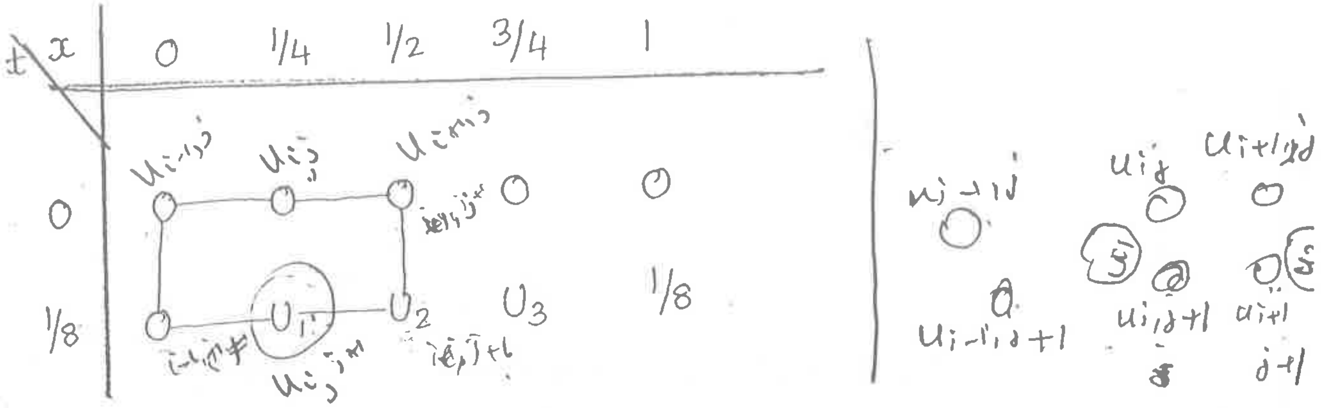
Now $\lambda = \frac{K}{ah^2} = \frac{1/8}{1 \times 1/16} = 2.$ $\quad \lambda = \frac{1/8}{1/16} = \frac{16}{8} = 2$

$\therefore \lambda = 2$ ① becomes,

$$2(U_{i-1,j+1} + U_{i+1,j+1}) - 2(3)U_{i,j+1} = 2U_{i,j} - 2(U_{i-1,j} + U_{i+1,j})$$

$$u_{i-1,j+1} + u_{i+1,j+1} - 3u_{i,j+1} = u_{i,j} - (u_{i-1,j} + u_{i+1,j})$$

$$\therefore\ U_{i-1,j+1} + U_{i+1,j+1} - 3U_{i,j+1} = U_{i,j} - (U_{i-1,j} + U_{i+1,j}) \quad - (2)$$



at U_1 ($\cancel{i=1, j=1}$) we get by (2)

$$0 + U_2 - 3U_1 = 0 - (0+0) \Rightarrow U_2 - 3U_1 = 0 \quad - (3)$$

at U_2,

$$U_1 + U_3 - 3U_2 = 0 - (0+0) \Rightarrow U_1 + U_3 - 3U_2 = 0 \quad - (4)$$

at U_3,

$$U_2 + 1/8 - 3U_3 = 0 - (0+0) \Rightarrow U_2 - 3U_3 = -1/8 \quad - (5)$$

On solving,

$$U_2 = 3U_1$$
$$U_2 = 3U_3 - 1/8$$

$$(4) \Rightarrow U_1 + U_3 - 3U_2 = 0$$
$$3 \times (5) \Rightarrow 3U_2 - 9U_3 + 3/8 = 0$$
$$U_1 - 8U_3 = -3/8 \quad - (6)$$

$$(4) \Rightarrow -3/8 + 8U_3 + U_3 - 3U_2 = 0$$
$$9U_3 - 3U_2 = 3/8$$
$$3U_3 - U_2 = 1/8$$

using (3) & (5)

$$-3U_1 + 3U_3 = 1/8$$
$$3U_1 - 24U_3 = -9/8 \quad (6)\times 3$$
$$21U_3 = 1$$
$$\boxed{U_3 = 0.04762}$$
$$-3U_1 = 1/8 - 3(0.04762)$$
$$\boxed{U_1 = 0.00595}$$
$$U_2 = 3U_1$$
$$\boxed{U_2 = 0.01786}$$

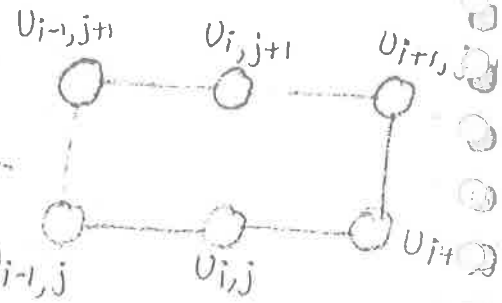
On solving we get $U_1 = 0.00595$; $U_2 = 0.01786$; $U_3 = 0.04762$

$U(0,t)=0$, $U(1,t)=0$, $U(x,0)=\sin\pi x$ for $0<x<1$ and $h=1/3$, $K=1/36$ compute 't' for 2 steps.

Solution:

given: $U_{xx}=U_t \Rightarrow a=1$

$$\lambda=\frac{K}{ah^2}=\frac{1/36}{1\times 1/9}=\frac{1}{4}$$



By crank-nicholson formula,

$$\lambda(U_{i-1,j+1}+U_{i+1,j+1})-2(\lambda+1)U_{i,j+1}=2(\lambda-1)U_{i,j}-\lambda(U_{i-1,j}+U_{i+1,j}) \quad -(1)$$

put $\lambda=1/4$ in (1)

$$\frac{1}{4}(U_{i-1,j+1}+U_{i+1,j+1})-2(5/4)U_{i,j+1}=2(-3/4)U_{i,j}-\frac{1}{4}(U_{i-1,j}+U_{i+1,j})$$

$$\frac{1}{4}(U_{i-1,j+1}+U_{i+1,j+1})-\frac{5}{2}U_{i,j+1}=-\frac{3}{2}U_{i,j}-\frac{1}{4}(U_{i-1,j}+U_{i+1,j})$$

t \ x	0	1/3	2/3	1
0	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0
1/36	0	U_1	U_2	0
2/36	0	U_3	U_4	0

when $x=1/3$, $U(1/3,0)=\sin\pi/3$
$=\sqrt{3}/2$

$x=2/3$, $U(2/3,0)=\sin(2\pi/3)$
$=\sin(\frac{\pi}{2}+\frac{\pi}{6})$
$=\cos 30$
$=\sqrt{3}/2$

at U_1,

$$\frac{1}{4}(0+U_2)-\frac{5}{2}U_1=-\frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{4}(0+\sqrt{3}/2)$$

$$\frac{U_2}{4}-\frac{5}{2}U_1=-\frac{3\sqrt{3}}{4}-\frac{1}{4}(\sqrt{3}/2)$$

multiplying by 4,

$$U_2-10U_1=-3\sqrt{3}-\frac{\sqrt{3}}{2} \quad -(2)$$

at U_2,

$$\frac{1}{4}\left[U_1+0\right] - 5U_2 = -\frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(0+\frac{\sqrt{3}}{2}\right)$$

$$\frac{U_1}{4} - 5U_2 = -\frac{3\sqrt{3}}{4} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right).$$

multiplying by 4,

$$U_1 - 20U_2 = -3\sqrt{3} - \frac{\sqrt{3}}{2} \quad - (3)$$

On solving U_1 and U_2 we get,

$$(2) \Rightarrow U_2 - 10U_1 = -3\sqrt{3} - \sqrt{3}/2.$$

$$10\times(3) \Rightarrow 10U_1 - 200U_2 = -30\sqrt{3} - 5\sqrt{3}.$$

$$199U_2 = 66.684$$

$U_2 = 0.335$ $\quad$ $U_2 = 0.6736$

$U_1 = 0.6397$ $\quad$ $U_1 = 0.6736$

at U_3,

$$\frac{1}{4}\left[0+U_4\right] - \frac{5}{2}U_3 = -\frac{3}{2}(U_1) - \frac{1}{4}(0+U_2)$$

$$\frac{U_4}{4} - \frac{5}{2}U_3 = -\frac{3}{2}(0.6397) - \frac{1}{4}(0.335).$$

multiplying by 4 we get,

$U_3 = 0.5239$

$U_4 = 0.5239$

$$U_4 - 10U_3 = -4.1732 \quad - (4).$$

at U_4, $\frac{1}{4}\left[U_3+0\right] - \frac{5}{2}U_4 = -\frac{3}{2}U_2 - \frac{1}{4}(0+U_1).$

$$\frac{U_3}{4} - \frac{5}{2}U_4 = -\frac{3}{2}(0.335) - \frac{0.6397}{4}$$

$$U_3 - 10U_4 = -0.662425$$

$$10U_4 - U_3 = 0.662425 \quad - (5)$$

$10\times$ (4) $\Rightarrow 10U_4 - 100U_3 = -41.732$

(5) $\Rightarrow 10U_4 - U_3 = 0.662425$

$$99U_3 = 42.394425$$

$$\boxed{U_3 = 0.4282}$$

$$10U_4 = 0.662425 + 0.4282$$

$$\boxed{U_4 = 0.1090625}$$

$\therefore U_1 = 0.6397 \quad U_2 = 0.335 \quad U_3 = 0.4282 \quad U_4 = 0.1090625$

---

(3) Solve by crank Nicholson method $\frac{\partial u}{\partial t} = \frac{1}{16}\frac{\partial^2 u}{\partial x^2}$ $0 \le x \le 1$, $t > 0$ $u(x,0) = 0 = u(0,t)$, $u(1,t) = 100t$ compute u for one step with $h = 1/4$.

$\lambda \geq 1$

Type 1

$\alpha^2 = 1/16$

$\lambda = \frac{k \times 16}{1/16}$

Latin

given $\frac{\partial u}{\partial t} = \frac{1}{16}\frac{\partial^2 u}{\partial x^2}$ $\alpha^2 = \frac{1}{16}$, $a = 16$

$\lambda = \frac{k}{ah^2}$ choose k, (such that) s.t $\lambda = 1$ [$\because$ k is not given, the choice is ours]

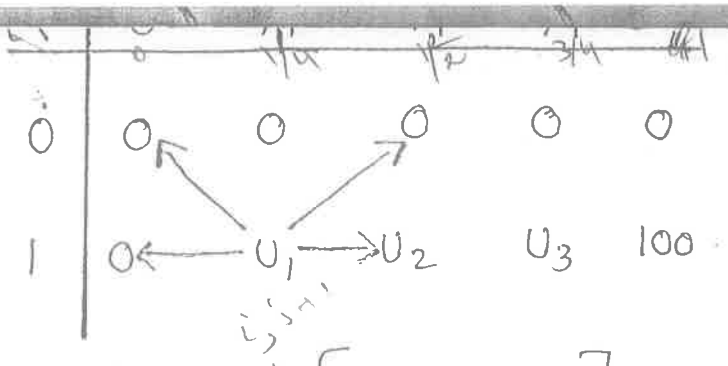
Let $k = 1$

$$\lambda = \frac{1}{16 \times 1/16} = 1$$

The Crank-nicholson formula will take a simple form when $\lambda = 1$

$$\therefore U_{i,j+1} = \frac{1}{4}\left[U_{i-1,j+1} + U_{i+1,j+1} + U_{i-1,j} + U_{i+1,j}\right]$$

given $u(x,0) = 0 = u(0,t)$ and $u(1,t) = 100t$



Since $K = 1$ in one step. from '0' we move to '1'.

At $U_1 \Rightarrow U_1 = \frac{1}{4}\left[0 + U_2 + 0 + 0\right] \Rightarrow 4U_1 = U_2$ — (1)

At $U_2 \Rightarrow U_2 = \frac{1}{4}\left[U_1 + U_3 + 0 + 0\right] \Rightarrow 4U_2 = U_1 + U_3$

$4(4U_1) = U_1 + U_3$

$15U_1 = U_3$ — (2)

at $U_3 \Rightarrow U_3 = \frac{1}{4}\left[U_2 + 100 + 0 + 0\right] \Rightarrow 4U_3 = U_2 + 100$

$4U_3 - 4U_1 = 100$ — (3)

Solving (2) & (3),

(2) × 4 $\Rightarrow 60U_1 - 4U_3 = 0$

(3) $\Rightarrow -4U_1 + 4U_3 = 100$

$56U_1 = 100$

$\boxed{U_1 = 1.7857}$

$U_2 = 4(1.7857)$

$\boxed{U_2 = 7.1428}$

$U_3 = 15(1.7857)$

$\boxed{U_3 = 26.7855}$

---

(2) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = t$, compute u for $\underbrace{t = 1/8 \text{ in } 2}_{0 \quad \frac{1}{16} \quad \frac{2}{16}}$ steps using crank-nicholson formula

$\lambda = \frac{\frac{1}{16}}{\left(\frac{1}{4}\right)^2}$

Solution:

$U_{xx} = U_t \Rightarrow a = 1$, $t = 1/8$ in 2 steps $\therefore K = 1/16$

choose $h = 1/4$ $\quad \lambda = K \quad = \frac{1/16}{} = 1$

t are $0, 1/16, 1/8$

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1/16	0	U_1	U_2	U_3	1/16
1/8	0	U_4	U_5	U_6	1/8

Since $\lambda = 1$, the crank - nicholson formula will be,

$$U_{i,j+1} = \frac{1}{4}\left[U_{i-1,j+1} + U_{i+1,j+1} + U_{i-1,j} + U_{i+1,j}\right]$$

At U_1, $U_1 = \frac{1}{4}\left[0 + U_2 + 0 + 0\right] \Rightarrow 4U_1 = U_2$ - (1)

At U_2, $U_2 = \frac{1}{4}\left[U_1 + U_3 + 0 + 0\right] \Rightarrow U_2 = \frac{1}{4}\left[\frac{U_2}{4} + U_3\right]$

$$\Rightarrow U_2 - \frac{U_2}{16} = \frac{U_3}{4}$$

$$\frac{15U_2}{16} = \frac{U_3}{4}$$

$15U_2 = 4U_3$ - (2)

at U_3,

$$U_3 = \frac{1}{4}\left[U_2 + \frac{1}{16} + 0 + 0\right]$$

$$\Rightarrow 4U_3 = U_2 + 1/16$$

$$15U_2 - U_2 = 1/16$$

$$14U_2 = 1/16$$

$$\boxed{U_2 = 0.004464}$$

$$① \Rightarrow U_1 = \frac{U_2}{4} = 0.001116$$

$$② \Rightarrow U_3 = \frac{15}{4}(0.004464) = \underline{\underline{0.01674}}$$

at U_4, $$U_4 = \frac{1}{4}\left[0 + U_5 + U_2 + 0\right]$$

$$= \frac{1}{4}\left[U_5 + 0.004464\right]$$

$$4U_4 = U_5 + 0.004464 \quad - ③$$

at U_5, $$U_5 = \frac{1}{4}\left[U_4 + U_6 + U_1 + U_3\right]$$

$$= \frac{1}{4}\left[U_4 + U_6 + 0.001116 + 0.01674\right]$$

$$4U_5 = U_4 + U_6 + 0.017856 \quad - ④$$

at U_6, $$U_6 = \frac{1}{4}\left[U_5 + \frac{1}{8} + \frac{1}{16} + U_2\right]$$

$$= \frac{1}{4}\left[U_5 + \frac{1}{16} + \frac{1}{8} + 0.004464\right]$$

$$4U_6 = U_5 + 0.191964 \quad - ⑤$$

$$③ \Rightarrow 4U_4 = 4U_6 - 0.191964 + 0.004464$$

$$4U_4 - 4U_6 = -0.1875 \quad - ⑥$$

$$④ \Rightarrow 4(4U_6 - 0.191964) = U_4 + U_6 + 0.017856$$

$$15U_6 - U_4 = 0.785712 \quad - ⑦$$

$$⑥ \Rightarrow \quad -4U_6 + 4U_4 = -0.1875$$

$$56U_6 = 2.955348$$

$$\boxed{U_6 = 0.052774}$$

$$⑦ \Rightarrow U_4 = 15(0.052774) - 0.785712$$

$$\boxed{U_4 = 0.005898}$$

$$⑤ \Rightarrow 4U_6 = U_5 + 0.191964$$

$$U_5 = 4(0.052774) - 0.191964$$

$$\boxed{U_5 = 0.019132}$$

$\therefore$ $U_1 = 0.00116$; $U_2 = 0.004464$; $U_3 = 0.01674$

$U_4 = 0.005898$; $U_5 = 0.019132$; $U_6 = 0.052774$

---

⑤ Solve by crank - Nicholson method $U_{xx} = U_t$ for $0 < x < 1$, t given that $u(0,t) = 0$ $u(1,t) = 0$ and $u(x,0) = 100(x - x^2)$ compute 'u' for one time step with $h = 1/4$.

③

Solution:

given $U_{xx} = U_t$ $h = 1/4$, K is not given so choose K s.t $\lambda = 1$.

If $K = 1/16$ then $\lambda = \frac{K}{ah^2} = \frac{1/16}{1 \times 1/16} = 1$

$\therefore$ The crank - nicholson simplified formula is

$$U_{i,j+1} = \frac{1}{4}\left[U_{i-1,j+1} + U_{i+1,j+1} + U_{i-1,j} + U_{i+1,j}\right]$$

Since $h = 1/4$, x takes ...

For one time step, t takes values 0 and $1/16$

$t \backslash x$	0	0.25	0.5	0.75	1.0
0	0	18.75	25	18.75	0
1/16	0	U_1	U_2	U_3	0

$$U(x,0) = 100(x - x^2).$$

$$U(1/4, 0) = 100\left(\frac{1}{4} - \frac{1}{16}\right) = 18.75$$

$$U(1/2, 0) = 100\left(\frac{1}{2} - \frac{1}{4}\right) = 25$$

$$U(3/4, 0) = 100\left(\frac{3}{4} - \frac{9}{16}\right) = 18.75$$

$$U(1,0) = 100(1-1) = 0.$$

at U_1,

$$U_1 = \frac{1}{4}\left[0 + 25 + U_2 + 0\right] \Rightarrow 4U_1 = 25 + U_2 \quad - (1)$$

at U_2,

$$U_2 = \frac{1}{4}\left[18.75 + 18.75 + U_1 + U_3\right] \Rightarrow 4U_2 = 37.5 + U_1 + U_3 \quad - (2)$$

at U_3,

$$U_3 = \frac{1}{4}\left[25 + 0 + U_2 + 0\right] \Rightarrow 4U_3 = 25 + U_2 \quad - (3)$$

(3) $\Rightarrow 4U_3 = 25 + 4U_1 - 25$

$4U_3 = 4U_1$

$\boxed{U_3 = U_1}$

(2) $\Rightarrow 4U_2 = 37.5 + 2U_1$

(3) $\Rightarrow 4U_1 = 25 + U_2$

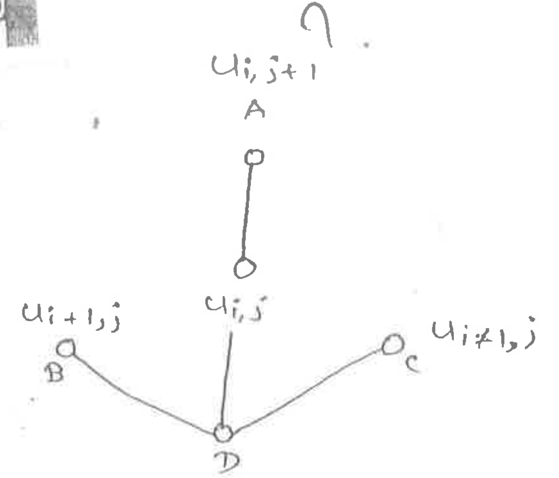
(3) $\times 4 \Rightarrow 16U_1 - 4U_2 = 100$

(2) $\Rightarrow 4U_2 - 2U_1 = 37.5$

$14U_1 = 137.5$

$\boxed{U_1 = U_3 = 9.8214}$

The value of u at A = (The value at B + The value at C) − value at D



NOTE : 2

If $1-\lambda^2 a^2 < 0$ (ie) $\frac{ak}{h} > 1$, the soln. is unstable.

if $\frac{ak}{h} = 1$, the soln. is stable.

if $\frac{ak}{h} < 1$, it is stable but the accuracy of the soln. decreases as $\frac{ak}{h}$ decreases.

$\therefore$ If $\lambda \leq 1/a$ solution is stable

Solve $u_{tt} = u_{xx}$ given $u(0,t) = 0$, $u(4,t) = 0$,

$u(x,0) = \frac{x(4-x)}{2}$, $u_t(x,0) = 0$. Take $h = 1$, find

$u(1,0)$
$u(2,0)$

the solution upto 5 steps in 't' direction.

soln: Given $u_{tt} = u_{xx} \Rightarrow a^2 = 1$; $h = 1$, Now, if $1 - \lambda^2 a^2 = 0$, then the problem will be simple, so choose k

s.t. $\lambda = k/h = \frac{1}{1} = 1$.

$\therefore k = 1$ $\because 1 - \lambda^2 a^2 = 0$ the explicit eqn. is of the form.

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}.$$

Now x takes the values from 0 to 4 with $h = 1$.
t takes the " 0 to 5 with $k = 1$

Also $u(0,t) = 0 = u(4,t)$ & $u(x,0) = \frac{x(4-x)}{2}$

when $x = 1$, $u(1,0) = 3/2$
$x = 2$, $u(2,0) = \frac{2 \times 2}{2} = 2$
$x = 3$, $u(3,0) = \frac{3 \times 1}{2} = 1.5$

$t \backslash x$	0	1	2	3	4	
0	0	1.5	2	1.5	0	Bender schmidt
1	0	$\left(\frac{0+2}{2}\right)$ 1	1.5	1	0	$\frac{0+2}{2} = 1$, $\frac{1.5+1.5}{2} = 1.5$; $\frac{2+0}{2} = 1$.
2	0	0	0	0	0	$(0+1.5) - 1.5 = 0$, $1+1-2 = 0$; $1.5+0-1.5 = 0$.
3	0	-1	-1.5	-1	0	$(0+0)-1$, $0+0-1.5$, $0+0-1$
4	0	-1.5	-2	-1.5	0	$(0-1.5)-0$; , ...
5	0	-1	-1.5	-1	0	

Solve $25u_{xx} = u_t$ for u at the pivotal pts given $u(0,t) = u(5,t) = 0$, $u_t(x,0) = 0$ & $u(x,0) = 2x$ for $0 \le x \le 2.5$.
$= 10-2x$ for $2.5 \le x \le 5$ for one half period of vibration.

Soln :-

Given $25u_{xx} = u_t \Rightarrow a^2 = 25$.

The period of vibration, $= \boxed{\frac{2l}{a}} = \frac{2 \times 5}{5}$
$= 2$ secs.

$\therefore$ Half period of vibration $= \frac{1}{2} \times 2 = t$.

$\boxed{t = 1}$

Now $\lambda = \frac{k}{h}$ also if $1 - \lambda^2 a^2 = 0$, then the procedure is simple.

Period of Vib $\rightarrow \frac{2l}{a}$

$\therefore \lambda^2 = \frac{1}{25}$, so that $1 - \lambda^2 a^2 = 1 - \frac{1}{25} \times 25$
$= 0$

$\lambda^2 = \frac{1}{25} \Rightarrow \lambda = \frac{1}{5}$.

$\therefore \lambda = \frac{1/5}{1} \Rightarrow (\lambda = k/h)$

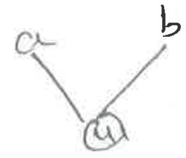
(a) $h = 1$, $k = \frac{1}{5}$

$\therefore x$ takes the values from 0 to 5 and

t takes the values from 0 to 1 with $k = \frac{1}{5}$

$\therefore$ x takes the values from 0 to 5 and 't' takes values from 0 to 1 with $k = \frac{1}{5}$. @ $n = 0, 1, 2, 3, 4$ & 5

$t = 0, 1/5, 2/5, 3/5, 4/5, 5/5$.



$\frac{a+b}{2} = u_1$

$t \backslash x$	0	1	2	3	4	5
0	0	2	4	4	2	0
1/5	0	2	3	3	2	0
2/5	0	1	1	1	1	0
3/5	0	-1	-1	-1	-1	0
4/5	0	-2	-3	-3	-2	0
5/5	0	-2	-4	-4	-2	0

$\rightarrow \frac{0+4}{2}, \frac{2+4}{2}, \frac{4+2}{2}, \frac{4+0}{2}$

$\rightarrow (0+3)-2, (2+3)-4, (3+2)-4, (3+0)-2.$

ELLIPTIC EQUATIONS
- Laplace $\nabla^2 u = 0$
- Poission. $\nabla^2 u = f(x, y)$

Laplace Equations :-

Let 'u' be a function of x and y s.t.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is called Laplace eqn. in 2-Dimension. By using finite difference method we replace $u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$,

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

If $h=k$ (1) becomes,

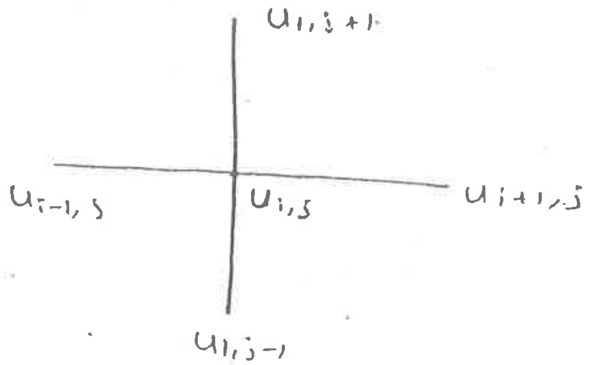
$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1}\right]$$ is called

the standard five pt formula. (a) the value of u

at any interior grid pt is the arithmetic

mean of the values of u at the four grid points

near to it.



DIAGONAL FIVE POINT FORMULA :-

Since the laplace equation is invariant we use another formula called diagonal five pt formula.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$