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DEPARTMENT OF MATHEMATICS

SUBJECT NAME : NUMERICAL METHODS

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UNIT – III : NUMERICAL DIFFERENTIATION & INTEGRATION

III - Numerical Differentiation and Integration

* Numerical differentiation with interpolation polynomials.

(i) Numerical differentiation by Newton's forward difference formula

(ii) Numerical differentiation by Newton's Backward difference formula.

(iii) Derivative using Sterling's Formula

(iv) Derivative using Bessel's Formula

* Numerical Integration

(i) by Trapezoidal Rule

(ii) by Simpson's one-third Rule.

(iii) by Simpson's three-eight (3/8) Rule.

* Numerical methods of Double Integration

(i) by Trapezoidal Rule

(ii) by Simpson's Rule

* Gauss Quadrature Formula:

(i) 2-point Formula.

(ii) 3-point Formula.

Newton's forward difference formula to get derivative when $x = x_0$

$n = n_0$.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

$$\frac{dy}{dx} = \frac{dy}{dn}\cdot\frac{dn}{dx} \quad \because n = \frac{x - x_0}{h}, \quad \frac{dn}{dx} = \frac{1}{h}$$

$$\frac{dy}{dn} = \Delta y_0 + \left(\frac{2n-1}{2!}\right)\Delta^2 y_0 + \left(\frac{3n^2 - 6n + 2}{3!}\right)\Delta^3 y_0 + \left(\frac{4n^3 - 18n^2 + 22n - 6}{4!}\right)\Delta^4 y_0 + \cdots$$

Newton's Backward difference formula to get the derivative: when $x=x_n$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\Delta^4 y_n + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \dots\right]$$

Derivative using Stirling's Formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \dots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \dots\right]$$

Type 2: $x \neq x_0$

Newton's Forward differentiation formula when $x \neq x_0$

$$P = \frac{x-x_0}{h}$$

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2P-1}{2}\right)\Delta^2 y_0 + \frac{3P^2-6P+2}{6}\Delta^3 y_0 + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 + (P-1)\Delta^3 y_0 + \left(\frac{6P^2-18P+11}{12}\right)\Delta^4 y_0 + \dots\right]$$

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^3 y_0 + \left(\frac{12P-18}{12}\right)\Delta^4 y_0 + \dots\right]$$

Newton's Backward differentiation formula when ($x \neq x_n$)

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{2}\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \left(\frac{4p^3+18p^2+22p+6}{24}\right)\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \left(\frac{6p^2+18p+11}{12}\right)\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{12p+18}{12}\nabla^4 y_n + \cdots\right]$$

Bessel's Formula

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{4}\right)\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) + \left(\frac{3p^2-3p+\frac{1}{2}}{6}\right)\Delta^3 y_{-1} + \right.$$

Numerical Integration

(i) Trapezoidal Rule:

$$\int_a^b y\,dx = \frac{h}{2}\left[A+2B\right]$$

where $y = f(x)$ $\quad h = \frac{b-a}{n}$

A = Sum of the first and last ordinates.

B = Sum of the remaining ordinates.

(ii) Simpson's ($\frac{1}{3}$)rd Rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+y_5+\cdots\right.$$

Simpson's three-eighth rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})$$
$$+2(y_3+y_6+y_9+\cdots)\Big]$$

Numerical Methods of Double Integration:

(i) Trapezoidal Rule:

$$I = \frac{hk}{4}\big[\text{Sum of the values of } (x,y) \text{ at the four corner points}\big]$$

$$I = \frac{hk}{4}\big[f_{i,j} + F_{i,j+1} + F_{i+1,j} + F_{i+1,j+1}\big]$$

(ii) Extension to general form of trapezoidal Rule:

$I = \frac{hk}{4}$ [Sum of the values of the f at 4 corners + 2(Sum of the values of f at the remaining nodes on the boundary) + 4(Sum of the values of f at the interior nodes)

(iii) Simpson's Rule for Double Integration

$I = \frac{hk}{9}$ [Sum of the values of f at the 4 corners + 4(Sum of the values of f at remaining nodes on the boundary + 16(Value of f at the centre)]

(iv) Extension to general form of Simpson's rule

$I = \frac{hk}{9}$ [Sum of the values of (f) at 4 corners + 2 (Sum of the values of (f) at odd position on the boundary except at corners) + 4 (Sum of the values of (f) at even position on boundary) + 4 (Sum of the values of (f) at odd positions) + 8 (Sum of the values of (f) at even position on the odd row of the matrix except boundary rows) + 8 (Sum of the values of (f) at the odd position) + 16 (Sum of the values of values of (f) at even position on the even row of the Matrix)]

Type 1:

1. Find the first two derivatives of $(x)^{1/3}$ at $x=50$ and $x=56$ given the table below

x :	50	51	52	53	54	55	56
$y=(x)^{1/3}$:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
50	3.6840						
		0.0244					
51	3.7084		-0.0003				
		0.0241		0			
52	3.7325		-0.0003		0		
		0.0238		0		0	
53	3.7563		-0.0003		0		0
		0.0235		0		0	
54	3.7798		-0.0003		0		
		0.0232		0			
55	3.8030		-0.0003				
		0.0229					
56	3.8259						

To get $f'(x)$ at $x=50$ we use Newton's Forward formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

Here $h = 1$

$$\left(\frac{dy}{dx}\right)_{x=50} = \frac{1}{1}\left(0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=50} = 0.02455$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{1^2}\left[-0.0003\right] = -0.0003$$

case (ii)

To get $f'(x)$ and $f''(x)$ at $x=56$ we use Newton's Backward Formula:

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=56} = \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=56} = 0.02275$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=56} = \frac{1}{1^2}(-0.0003) = -0.0003$$

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2. The table given below reveals the velocity v of a body during the time t specified, find the acceleration at t = 1.1.

t :	1.0	1.1	1.2	1.3	1.4
v :	43.1	47.7	52.1	56.4	60.8

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Solution:

Step 1:

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

In this problem $h = 0.1$

h = Difference between 2 values

$$\left(\frac{dy}{dx}\right)_{x=1.1} \Rightarrow \left(\frac{dv}{dt}\right)_{t=1.1} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0	43.1				
		4.6			
1.1	47.7		-0.2		
		4.4		0.1	
1.2	52.1		-0.1		0.1
		4.3		0.2	
1.3	56.4		0.1		
		4.4			
1.4	60.8				

$$\left(\frac{dv}{dt}\right)_{1.1} = \frac{1}{0.1}\left[4.4 - \tfrac{1}{2}(-0.1) + \tfrac{1}{3}(0.2)\right]$$

$$= \frac{1}{0.1}\left[4.4 + 0.05 + 0.0667\right]$$

$$\left(\frac{dv}{dt}\right)_{1.1} = 45.167 \ m/s^2$$

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3. Find the value of $f'(0.5)$ and $f''(0.5)$ using Stirling's formula from the following data:

x:	0.35	0.40	0.45	0.50	0.55	0.60	0.65
$y=f(x)$:	1.521	1.506	1.488	1.467	1.444	1.418	1.389

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.35	1.521						
		−0.015					
0.40	1.506		−0.003				
		−0.018		0			
0.45	1.488		−0.003		0.001		
		$\boxed{-0.021}$		$\boxed{0.001}$		$\boxed{-0.003}$	
0.50	$\boxed{1.467}$		−0.002		−0.002		0.006
		$\boxed{-0.023}$		$\boxed{-0.001}$		$\boxed{0.003}$	
0.55	1.444		−0.003		0.001		
		−0.026		0			
0.60	1.418		−0.003				
		−0.029					
0.65	1.389						

Solution:

Step 1: $$\left(\frac{dy}{dx}\right)_{x=0.5} = \frac{1}{h}\left[\tfrac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \tfrac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \tfrac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0.5} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \tfrac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

Step 2:

$$\left(\frac{dy}{dx}\right)_{0.5} = \frac{1}{0.05}\Big[\tfrac{1}{2}(-0.083-0.081) - \tfrac{1}{12}(-0.001+0.001) + \tfrac{1}{60}(0.003-0.003)\Big]$$

$$= \frac{1}{0.05}\left[-0.022\right] = -0.44$$

$$\left(\frac{d^2y}{dx^2}\right)_{0.5} = \frac{1}{0.05^2}\left[-0.002 - \tfrac{1}{12}(-0.002)\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{0.5} = -0.733$$

4. Find the first and second derivative of the function tabulated below at $x=0.6$

x: 0.4 0.5 0.6 0.7 0.8

y: 1.5836 1.7974 2.0442 2.3275 2.6511

Solution: Since $x=0.6$ is in the middle of the table, we will use Stirling's formula

Step 1: $$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\tfrac{1}{2}(\Delta y_0+\Delta y_{-1}) - \tfrac{1}{12}(\Delta^3 y_{-1}+\Delta^3 y_{-2}) + \cdots\right]$$

Step 2: $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \tfrac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836				
		0.2138			
0.5	1.7974		0.033		
		$\boxed{0.2468}$		$\boxed{0.0035}$	
0.6	$\boxed{2.0442}$		0.0365		0.0003
		$\boxed{0.2833}$		$\boxed{0.0038}$	
0.7	2.3275		0.0403		
		0.3236			
0.8	2.6511				

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{0.1}\left[\frac{1}{2}(0.2833+0.2468) - \frac{1}{12}(0.0038+0.0035)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.6} = 10\left[0.26505 + (-0.0006083)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.6} = 2.64442.$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0.6} = \frac{1}{(.01)}\left[0.0365 - \frac{1}{12}(0.000\ldots\right] = 3.6475$$

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5. Find the gradient of the road at the middle point of the elevation above a datum line of seven points of road which are given below.

x:	0	300	600	900	1200	1500	1800
y:	135	149	157	183	201	205	193

Solution:

Step 1: Since $x = 900$ is in the middle of the table we use one of the central difference formula, in particular Stirling's formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3})\right]$$

Step 2:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	135						
		14					
			-6				
300	149						
		8		24			
600	157		18		-50		
		$\boxed{26}$		$\boxed{-26}$		$\boxed{70}$	
900	183						-86
			-8		20		
		$\boxed{18}$		$\boxed{-6}$		$\boxed{-16}$	
1200	201						
			-14		4		
		4		-2			
1500	205						
			-16				
		-12					
1800	193						

Step 3:

$$\left(\frac{dy}{dx}\right)_{x=900} = \frac{1}{300}\left[\frac{1}{2}(18+26)+\left(\frac{-1}{12}\right)(-6-26)+\frac{1}{60}(70-16)\right]$$

$$= \frac{1}{300}\left[22+2.6666+0.9\right]$$

$$\left(\frac{dy}{dx}\right)_{x=900} = 0.085222$$

Hence, the gradient of the road at the middle point is 0.085222

6. Find the value of $f'(0.5)$ using Stirling's formula from the following data.

x :	0.35	0.40	0.45	0.50	0.55	0.60	0.65
$y=f(x)$:	1.521	1.506	1.488	1.467	1.444	1.418	1.389

Solution:

Since $x=0.5$ is in the middle of the table, we use Stirling's formula

Step 1: $$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0+\Delta y_{-1})-\frac{1}{12}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+\frac{1}{60}(\Delta^5 y_{-2}+\Delta^5 y\right.$$

Step 2:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.35	1.521						
		−0.015					
0.40	1.506		−0.003				
		−0.018		0			
0.45	1.488		−0.003		0.001		
		−0.021		0.001		−0.003	
0.50	1.467		−0.002		−0.002		0.006
		−0.023		−0.001		+0.003	
0.55	1.444		−0.003		−0.001		
		−0.026		0			
0.60	1.418		−0.003				
		−0.029					
0.65	1.389						

Step 3:

$$\left(\frac{dy}{dx}\right)_{x=0.5} = \frac{1}{0.05}\Big[\tfrac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \tfrac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \tfrac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots$$

$$= \frac{1}{0.05}\Big[\tfrac{1}{2}(-0.023 - 0.021) - \tfrac{1}{12}(-0.001 + 0.001) + \frac{1}{60}(0.003 - 0.003)\Big]$$

$$= \frac{1}{0.05}\Big[-\tfrac{1}{2} \times 0.044\Big]$$

$$\left(\frac{dy}{dx}\right)_{0.5} = -0.44.$$

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7. Find y', y'' at $x=1.5$ given

x:	1.5	2.0	2.5	3.0	3.5	4.0
y:	3.375	7.0	13.625	24.0	38.875	59

Solution:

$h =$ difference between 2 adjacent values

$h = 2 - 1.5 = 2.5 - 2 = 0.5$

$h = 0.5$

Step 1: Formula

(i) $$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\Big[\Delta y_0 - \tfrac{1}{2}\Delta^2 y_0 + \tfrac{1}{3}\Delta^3 y_0 - \tfrac{1}{4}\Delta^4 y_0 + \tfrac{1}{5}\Delta^5 y_0 + \cdots\Big]$$

(ii) $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\Big[\Delta^2 y_0 - \Delta^3 y_0 + \tfrac{11}{12}\Delta^4 y_0 - \cdots\Big]$$

Step 2 :

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2.0	7.0		3.0		
		6.625		0.75	
2.5	13.625		3.75		0
		10.375		0.75	
3.0	24.0		4.5		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			
4.0	59.0				

Step 3: Substitution

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^3 y_0 - \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3.0) + \frac{1}{3}(0.75) - \frac{1}{4}(0)\right]$$

$$= \frac{1}{0.5}\left[3.625 - 1.5 + 0.25\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = 4.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{(0.5)^2}\left[3 - 0.75 + \frac{11}{12}(0)\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.5} = 9$$

Type 2 :

Newton's Forward differentiation when $x \neq x_0$

Newton's Backward differentiation when $x \neq x_n$ //

1. The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971

Year : x :	1931	1941	1951	1961	1971
Population in thousands (y) :	40.62	60.80	79.95	103.56	132.65

Solution:

Step 1:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
1951	79.95		4.46		-4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

Step 2: Formula:

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

Step 3: Substitution

We use the same table for backward and forward differences.

(i) To get $f'(1931)$ and $f'(1941)$ we use forward formula

$$\left(\frac{dy}{dx}\right)_{x=1931} = \frac{1}{10}\left[20.18 - \tfrac{1}{2}(-1.03) + \tfrac{1}{3}(5.49) - \tfrac{1}{4}(-4.47)\right]$$

$$= \frac{1}{10}\left[20.18 + 0.515 + 1.83 + 1.1175\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1931} = \frac{1}{10}\left[23.6425\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1931} = 2.36425.$$

$$\left(\frac{dy}{dx}\right)_{x=1941} = \frac{1}{10}\left[19.15 - \tfrac{1}{2}(4.46) + \tfrac{1}{3}(1.02)\right]$$

$$= \frac{1}{10}\left[19.15 - 2.23 + 0.34\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1941} = 1.7260$$

(ii) To get $f'(1961)$ and $f'(1971)$ we use Backward Formula

$$\left(\frac{dy}{dx}\right)_{1971} = \frac{1}{10}\left[29.09 + \tfrac{1}{2}(5.48) + \tfrac{1}{3}(1.02) + \tfrac{1}{4}(-4.47)\right]$$

$$\left(\frac{dy}{dx}\right)_{1971} = \frac{1}{10}\left[31.0525\right] = 3.10525$$

$$\left(\frac{dy}{dx}\right)_{x=1961} = \frac{1}{10}\left[23.61+\tfrac{1}{2}(4.46)+\tfrac{1}{3}(5.49)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1961} = \frac{1}{10}\left[23.61+2.23+1.83\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1961} = 2.7670$$

Ans:

$$\left(\frac{dy}{dx}\right)_{x=1931} = 2.36425$$

$$\left(\frac{dy}{dx}\right)_{x=1941} = 1.7260$$

$$\left(\frac{dy}{dx}\right)_{x=1971} = 3.10525$$

$$\left(\frac{dy}{dx}\right)_{x=1961} = 2.7670$$

2. The table below gives the results of an observation: Θ is the observed temperature in degrees centigrade of a vessel of cooling water; t is the time in minutes from the beginning of observation.

t :	1	3	5	7	9
Θ :	85.3	74.5	67.0	60.5	54.3

Find the approximate rate of cooling at $t=3$ and 3.5

Solution:

Step 1: Difference table

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
1	85.3				
		-10.8			
3	74.5		3.3		
		-7.5		-2.3	
5	67.0		1.0		1.6
		-6.5		-0.7	
7	60.5		0.3		
		-6.2			
9	54.3				

Here $h = 2$

At $t=3$ $\quad P = \dfrac{x - x_0}{h} = \dfrac{3-1}{2} = 1$

At $t=3.5$ $\quad P = \dfrac{3.5-1}{2} = 1.25$

Step 2: Formula

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0\right.$$

Step 3: Substitution:

$$\left(\frac{dy}{dx}\right)_{P=1} = \left(\frac{d\theta}{dt}\right)_{t=3} = \frac{1}{2}\left[-10.8 + \frac{1}{2}(3.3) - \frac{1}{6}(-2.3) + \frac{1}{12}(1.6)\right]$$

$$\left(\frac{d\theta}{dt}\right)_{t=3} = \frac{1}{2}\left[-10.8 + 1.65 + 0.38333 + 0.13333\right]$$

$$\left(\frac{d\theta}{dt}\right)_{t=3} = \frac{1}{2}\left[-8.63334\right] = -4.31667$$

$$\left(\frac{d\theta}{dt}\right)_{t=3} = \frac{1}{2}\left[-7.5 \mp \frac{1}{2}(1.0) + \frac{1}{3}(-0.\right.$$

$= -4.116$

(ii) placing $p = 1.25$ in formula

$$\left(\frac{d\theta}{dt}\right)_{p=1.25} = \frac{1}{2}\left[-10.8 + 0.75(3.3) - 0.1354(-2.3) + (0.04948)(1.6)\right]$$

$$\left(\frac{d\theta}{dt}\right)_{p=1.25} = \frac{1}{2}\left[-7.934412\right]$$

$$\left(\frac{d\theta}{dt}\right)_{p=1.25} = -3.9672$$

A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t (seconds). Calculate the angular velocity and angular acceleration of the rod at $t = 0.6$ seconds.

t :	0	0.2	0.4	0.6	0.8	1.0
θ :	0	0.12	0.49	1.12	2.02	3.20

Solution:

Step 1: Difference table.

t	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.90		0.01	
0.8	2.02		0.28		
		1.18			
1.0	3.20				

Step 2: Formula

By Newton's backward difference formula,

$$\left(\frac{dy}{dx}\right)_{x=x} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \left(\frac{4p^3+18p^2+22p+6}{24}\right)\nabla^4 y_n + \cdots\right]$$

Here $p = \frac{x-x_n}{h} = \frac{0.6-1.0}{0.2} = -2$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \cdots\right]$$

Step 3: Substitution

$$\left(\frac{d\theta}{dt}\right)_{t=0.6} = \frac{1}{0.2}\left[1.18 - \tfrac{3}{2}(0.28) + \tfrac{1}{3}(0.01)\right]$$

$$= 5\left[1.18 - 0.42 + 0.00333\right]$$

$$\left(\frac{d\theta}{dt}\right)_{t=0.6} = 3.81665 \text{ radian/sec.}$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.6} = \frac{1}{0.04}\left[0.28 - 0.01\right]$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.6} = 6.75 \text{ radian/sec}^2$$

---

① Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 51$ from the following data

x : 50 60 70 80 90

y : 19.96 36.65 58.81 77.21 94.61

$h = 10$

x	y	$\cdots$	$\Delta^4 y$
			11.99

$n = \frac{x - x_0}{h} = \frac{51-50}{10} = .1$

$$\left(\frac{dy}{dx}\right)_{x=51} = 1.0316$$

$$\left(\frac{d^2y}{dx^2}\right) = 0.2303$$

Type 3:

Obtain the value of $f'(0.04)$ using Bessel's formula given the table below:

x :	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Solution:

Step 1:

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-3	0.01	0.1023					
			0.0024				
-2	0.02	0.1047		0			
			0.0024		0.0001		
-1	0.03	0.1071		0.0001		-0.0001	
			0.0025		0		0
0	0.04	$\boxed{0.1096}$		$\boxed{0.0001}$		$\boxed{-0.0001}$	
			$\boxed{0.0026}$		$\boxed{-0.0001}$		
1	0.05	$\boxed{0.1122}$		$\boxed{0}$			
			0.0026				
2	0.06	0.1148					

Step 2: Formula

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}(\Delta^3 y_{-1}) + \frac{1}{24}(\Delta^4 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right) = \frac{1}{0.01}\left[0.0026 - \frac{1}{4}[0.0001+0] + \frac{1}{12}(-0.0001) + \frac{1}{24}(-0.0001)\right)$$

$$\left(\frac{dy}{dx}\right) = \frac{1}{0.01}\left[0.0026 - 2.5\times10^{-5} - 8.333\times10^{-6} - 4.1667\times10^{-6}\right]$$

$$\left(\frac{dy}{dx}\right) = \frac{1}{0.01}\left[0.002625\right]$$

$$\left(\frac{dy}{dx}\right) = 0.25625.$$

Given the following data, find $y'(6)$ and the maximum value of y

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922

Solution:

Since the arguments are not equally spaced, we will use Newton's divided difference formula (or even Lagrange's formula).

Step 1:

<u>Divided difference table</u>

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

Step 2:

By Newton's divided difference formula.

$$y = f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \cdots$$

Step 3:

$$y = 4 + (x-0)11 + (x-0)(x-2)7 + (x-0)(x-2)(x-3)1$$

$$y = x^3 + 2x^2 + 3x + 4$$

Therefore $y'(x) = 3x^2 + 4x + 3$

$$y'(6) = 3(6)^2 + 4(6) + 3 = 135$$

$y(x)$ is maximum if $y'(x) = 0$ $\therefore$ $3x^2 + 4x + 3 = 0$. But the roots are imaginary. Therefore there is no extreme value in the range. In fact, it is an increasing curve. //

From the following table, find the value of x for which $f(x)$ is a maximum. Also find the maximum value of $f(x)$ from the table of values given below.

x:	60	75	90	105	120
$f(x)$:	28.2	38.2	43.2	40.9	37.7

Solution: The maximum value appears to be in the neighbourhood of $x=90$. Hence, we will use stirling's formula.

Step 1

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
60	28.2				
		10			
75	38.2		-5		
		5		-2.3	
90	43.2		-7.3		8.7
		-2.3		6.4	
105	40.9		-0.9		
		-3.2			
120	37.7				

Step 2 :

By Stirling's Formula

$$y(x) = y(x_0+ph) = y_0 + \frac{p}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2})$$

$$+ \frac{p^2(p^2-1)}{24}\Delta^5 y_{-2} + \cdots$$

$$y(x) = 43.2 + \frac{p}{2}(-2.3+5) + \frac{p^2}{2}(-7.3) + \frac{(p^3-p)}{12}(-2.3+6.4)$$

$$= 43.2 + 1.35p - 3.65p^2 + 0.3417(p^2-1)p$$

$$y(x) = 0.3417p^3 - 3.65p^2 + 1.0083p + 43.2$$

If y is maximum, $\frac{dy}{dp} = 0$ Hence

$$3 \times 0.3417p^2 - 2 \times 3.65p + 1.0083 = 0$$

$$1.0251p^2 - 7.30p + 1.0083 = 0$$

$$p = \frac{7.30 \pm \sqrt{7.30^2 - 4(1.0251)(1.0083)}}{2 \times 1.0251}$$

$p = 6.9803$ (or) 0.1409

$p = 6.9803$ goes beyond the range

Therefore take $p = 0.1409$

$$x = x_0 + ph = 90 + 15(0.1409) = 92.1135$$

Maximum $y = 0.3417(0.1409)^3 - 3.65(0.1409)^2$

$$+1.0083(0.1409) + 43.2$$

$$y = 43.27$$

$f(x)$ is maximum at $x = 92.1135$ and the maximum value is 43.2

Numerical Integration

Trapezoidal Rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\Big[(\text{Sum of the first and the last ordinates}) + 2(\text{Sum of the remaining ordinates})\Big]$$

$$= \frac{h}{2}\Big[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})$$

Truncation error in Trapezoidal rule

The error in the trapezoidal rule is the order h^2

Simpson's one-third rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\Big[(\text{Sum of the first and last ordinates}) + \overset{4}{\cancel{2}}(\text{Sum of remaining odd ordinates}) + \overset{2}{\cancel{4}}(\text{Sum of even ordinates})\Big]$$

$$= \frac{h}{3}\Big[(y_0 + y_n) + \cancel{4}($$

Simpson's three-eighth rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_n)\Big]$$

Note: Simpson's three-eighth rule is applicable only when n is a multiple of 3

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\Big[(y_0 + y_n) + 4(y_1 + y_3 + \cdots +) + 2(y_2 + y_4 + \cdots)\Big]$$

Error = difference between the exact value and approximate value

Notes:

1. In trapezoidal rule, $y(x)$ is a linear function of x. The rule is the simplest one but it is least accurate.

2. In Simpson's one third rule, $y(x)$ is a polynomial of degree two. To apply this rule, n, the number of intervals must be even. That is the number of ordinates must be odd.

3. In Simpson's three-eighth rule, $y(x)$ is a polynomial of degree three. This rule is applicable if n, the number of intervals is a multiple of 3.

Evaluate $\int_{-3}^{3} x^4 dy$ by using (1) Trapezoidal rule (2) Simpson's rule. Verify your results by actual integration.

Solution:

Here $f(x) = x^4$.

Interval length $= (b-a) = 3-(-3) = 3+3=6$ (multiple of 3)

So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$.

$h = \frac{b-a}{n}$
$= \frac{6}{6} = 1$

Step 1: Table Formation

x:	-3	-2	-1	0	1	2	3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6
$y = f(x)$:	81	16	1	0	1	16	81

(i) By Trapezoidal Rule

$$\int_{-3}^{3} f(x)dx = \frac{h}{2}\left[\text{Sum of the first and last ordinates} + 2(\text{Sum of the remaining ordinates})\right]$$

$$= \frac{1}{2}\left[(81+81) + 2(16+1+0+1+16)\right]$$

$$\int_{-3}^{3} f(x)dx = 115$$

(ii) By Simpson's one-third Rule (Since no. of ordinates is odd)

$$\int_{-3}^{3} f(x)dx = \frac{h}{3}\left[\text{Sum of the first and last ordinates} + 2(\text{Sum of remaining odd ordinates}) + 4(\text{Sum of even ordinates})\right]$$

$+2(y_1+y_3+\ldots)$ $+4(y_2+y_4+\ldots)$

$$= \frac{h}{3}\left[(81+81) + 2(1+1) + 4(16+0+16)\right]$$

$$\int_{-3}^{3} f(x)dx = 98$$

(iii) Since $n=6$, (Multiple of 3) we can also use Simpson's three-eighth rule.

$$\int_{-3}^{3} x^4 dx = \frac{3}{8}\left[(81+81) + 3(16+1+1+16) + 2(0)\right] = 99$$

(iv) By actual Integration

$$\int_{-3}^{3} x^4 dx = 2\left[\frac{x^5}{5}\right]_0^3 = \frac{2 \times 243}{5} = 97.2$$

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with $h = 0.2$.

Hence obtain an approximate value of π

Solution: Let $f(x) = \frac{1}{1+x^2}$

Interval length : $1 - 0 = 1$

$\because$ The value of y are calculated as points taking $h = 0.2$

x	:	0	0.2	0.4	0.6	0.8	1.0
		y_0	y_1	y_2	y_3	y_4	y_5
$y = \frac{1}{1+x^2}$	:	1	0.96154	0.86207	0.73529	0.60976	0.5

(i) By trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.2}{2}\left[(1+0.5)+2(0.96154+0.86207+0.73529+0.60976)\right]$$

$$= 0.1\left[1.5 + 6.33732\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.783732$$

By actual Integration:

$$\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1}x)_0^1 = \pi/4$$

$$\therefore \pi/4 = 0.783732$$

$$\therefore \pi = 3.13493 \text{ (approximately)}$$

In this case, we cannot use Simpson's rule (both).
(Since no. of intervals is 5).

3. From the following table, find the area bounded by the curve and the x axis from $x = 7.47$ to $x = 7.52$

x	:	7.47	7.48	7.49	7.50	7.51	7.52
$y = f(x)$	:	1.93	1.95	1.98	2.01	2.03	2.06

Solution:

Since only 6 ordinates (no. of intervals = 5) are given, we cannot use Simpson's rule. $h = 7.48 - 7.47 = 0.01$

$$\text{Area} = \int_{7.47}^{7.52} f(x)dx = \frac{0.01}{2}\left[(1.93+2.06) + 2(1.95+1.98+2.01+2.03)\right]$$

$$\int_{7.47}^{7.52} f(x)dx = 0.09965$$

---

4. Evaluate the integral $I = \int_{4}^{5.2} \log_e x \, dx$ using Trapezoidal, Simpson's rule.

Solution: Here $b - a = 5.2 - 4 = 1.2$. We shall divide the interval into 6 equal parts.

Hence, $h = \frac{1.2}{6} = 0.2$

		0	2	3	4	5
x	:	4	4.2	4.4	4.6	4.8
$f(x) = \log_e x$	:	1.3862944	1.4350845	1.4350845	1.5260563	1.5686159

		6	7
x	:	5.0	5.2
$f(x) = \log_e x$	:	1.6094379	1.6486586

(i) By Trapezoidal rule,

$$\int_4^{5.2} \log_e x\, dx = \frac{0.2}{2}\Big[(1.3862944+1.6486586)+2(1.4350845+1.4816045$$
$$+1.5260563+1.5686159+1.6094379)\Big]$$

$$= 1.82765512$$

(ii) Since n=6, we can use simpson's rule

By simpson's one-third rule,

$$I = \frac{0.2}{3}\Big[(1.3862944+1.6486586)+2(1.4816045+1.5686159)$$
$$+4(1.4350845+1.5260563)\Big]$$

$$I = 1.82784724$$

(iii) By simpson's three-eighths rule,

$$I = \frac{3(0.2)}{8}\Big[(1.3862944+1.6486586)+3(1.4350845+1.4816045+$$
$$1.5686159+1.6094379)+2(1.5260563+\cancel{1.6486586})\Big]$$

$$= \frac{0.6}{8}\big[3.034953+3(6.0947428)+3.0521126\big]$$

$$I = 1.82784705$$

5. Evaluate $I = \int_0^6 \frac{1}{1+x}\, dx$ using (i) Trapezoidal rule (ii) Simpson's rule. Also check up by direct integration.

Solution: Take the number of intervals as 6

$$\therefore \quad h = \frac{6-0}{6} = 1$$

x	:	0	1	2	3	4	5	6
$y=\frac{1}{1+x}$	:	1	0.5	1/3	1/4	1/5	1/6	1/7

(i) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x} = \frac{1}{2}\left\{\left[1+\frac{1}{7}\right]+2\left[0.5+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right]\right\}$$

$$= 2.02142857$$

(ii) By Simpson's one-third rule,

$$I = \frac{1}{3}\left[\left(1+\frac{1}{7}\right)+2\left(\frac{1}{3}+\frac{1}{5}\right)+4\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)\right]$$

$$I = \frac{1}{3}\left(1+\frac{1}{7}+\frac{16}{15}+\frac{22}{6}\right)$$

$$I = 1.95873016$$

(iii) By Simpson's three-eighth's rule

$$I = \frac{3\times 1}{8}\left[\left(1+\frac{1}{7}\right)+3\left(0.5+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}\right)+2\left(\frac{1}{4}\right)\right]$$

$$I = 1.96607143$$

(iv) By actual integration

$$\int_0^6 \frac{dx}{1+x} = \left[\log(1+x)\right]_0^6 = \log_e 7 = 1.94591015$$

---

15 (a) (i) 6. By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x\,dx$ by Trapezoidal and Simpson's $\frac{1}{3}$ rule. Verify your answer with integration.

Simpson $3/8^{\text{th}}$ rule. ~~Gaussian two pt. form~~

Solution : Range $= \pi - 0 = \pi$

Hence $= h = \pi/10$

x :	0	$\pi/10$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$
$y = \sin x$:	0.0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511	0.809

x :	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
$y = \sin x$:	0.5878	0.3090	0

(i) By Trapezoidal Rule

$$I = \frac{\pi}{20}\left[(0+0) + 2(0.309 + 0.5878 + 0.809 + 0.9511 + 1.0 + 0.9511 + 0.8090 + 0.5878 + 0.3090)\right]$$

$$I = 1.9843$$

(ii) By Simpson's one-third rule

$$I = \frac{1}{3}\left(\frac{\pi}{10}\right)\left[(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090)\right]$$

$$I = 2.00091$$

(15) (b) (ii) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by (i) Trapezoidal rule (ii) Simpson's rule (iii) Actual integration

Solution: Here, $b-a = 6-0 = 6$. Divide into 6 equal parts

$h = \frac{6}{6} = 1$. Hence, the table is,

x :	0	1	2	3	4	5	6
$\frac{1}{1+x^2} = f(x)$:	1.00	0.5	0.2	0.1	0.058824	0.038462	0.027027

(i) By Trapezoidal Rule

$$I=\int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}\left[(1+0.027027)+2(0.5+0.2+0.1+0.058824+0.03846\right.$$

$$I = 1.41079950$$

(ii) By Simpson's one-third Rule

$$I=\frac{1}{3}\left[(1+0.027027)+2(0.2+0.058824)+4(0.5+0.1+0.038462)\right]$$

$$=\frac{1}{3}\left[1.027027+0.517648+2.553848\right]$$

$$I = 1.36617433$$

(iii) By Simpson's three-eighth Rule

$$I=\frac{3\times1}{8}\left[(1+0.027027)+3(0.5+0.2+0.058824+0.038462)+2(0.1)\right]$$

$$I=1.35708188$$

(iv) By actual integration

$$I=\int_0^6 \frac{dx}{1+x^2} = (\tan^{-1}x)_0^6 = \tan^{-1}6 = 1.40564765$$

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Romberg's method. Hence, obtain an approximate value for π.

Solution: By taking $h=0.5$, tabulate the values of $y=\frac{1}{1+x^2}$

x:	0	0.5	1
y:	1.0	0.8	0.5

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} 1 = 0.7844875$$

The difference comes only in the third digit

$I = \tan^{-1} 1 = \pi/4$

$\therefore \pi/4 \simeq 0.7854$

$\therefore \pi \simeq 3.1416$

Using Romberg's method, evaluate $I = \int_0^1 \frac{dx}{1+x}$ correct to three decimal places. Hence evaluate $\log_e 2$.

Solution:

Take $h = 0.5, 0.25$ and 0.125 Here $f(x) = \frac{1}{1+x}$

(i)

x :	0	0.5	1
$f(x) = y$:	1	0.6666	0.5

$\Rightarrow$ substitute x value in $f(x)$

I_1 (using Trapezoidal Rule) $= \frac{0.5}{2}[1.5 + 2(0.6666)] = 0.7083.$

(ii) For this $h = 0.25$

x :	0	0.25	0.5	0.75	1
y :	1	0.8	0.6666	0.5714	0.5

I_2 (using Trapezoidal Rule) $= \frac{0.25}{2}[(1+0.5) + 2(0.8 + 0.6666 + 0.5714)]$

$I_2 = 0.6970$

$\therefore I = \frac{0.5}{2}\left[1.5 + 2(0.8)\right] = 0.775$

By taking $h = 0.25$, we have the table,

$h = 0.5$
$n = 2$
$h = 0.25$
$n = 4$
$h = 0.125$
$n = 8$

x :	0	0.25	0.5	0.75	1
y :	1	0.9412	0.8	0.64	0.5

$\therefore I = \frac{0.25}{2}\left[1.5 + 2(0.9412 + 0.8 + 0.64)\right] = 0.78280$

By taking $h = 0.125$, the tabular values are

x :	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y :	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

$$I = \frac{0.125}{2}\left[(1+0.5) + 2(0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664)\right]$$

$I = 0.784750$ ✓

The different values got by Trapezoidal rule for various h's are

0.77500 0.78280 0.78475

Applying the formula $\boxed{I = I_2 + \frac{1}{3}(I_2 - I_1)}$ (I_2, I_1)

$I = 0.7828 + \frac{1}{3}(0.7828 - 0.7750)$

$I = 0.7854$

I_1 I_2 I_3

$\boxed{I = I_3 + \frac{1}{3}(I_3 - I_2)}$ (I_2, I_3)

$I = 0.78475 + \frac{1}{3}(0.78475 - 0.7828)$

$I = 0.7854$ //

h	.5	.25	.125
I	0.775	.78280	.784750
	I_1	I_2	I_3

I I

(iii) For this $h = 0.125$

x :	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x) = y$:	1	0.8889	0.8	0.7272	0.6667	0.653	0.5714	0.5333	0.5

$$I_3 = \frac{0.125}{2}\Big[1.5 + 2(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5333 + 0.5714)\Big] = \underline{0.6941}$$

(using trapezoidal Rule)

Three values are 0.7083, 0.6970, 0.6941

using these values in equation $I = I_2 + \frac{1}{3}(I_2 - I_1)$ we get — ①

$$I = 0.6970 + \frac{1}{3}(0.6970 - 0.7083)$$

$$\underline{I = 0.6932}$$

$$I = 0.6941 + \frac{1}{3}(0.6941 - 0.6970)$$

$$\underline{I = 0.6951}$$

Now taking these 2 values and using formula (1)

$$I = 0.6951 + \frac{1}{3}(0.6931 - 0.6932)$$

$$I = 0.6931$$

Hence, most approximate value is 0.6931

By actual Integration

$$I = \int_0^1 \frac{dx}{1+x} = \Big[\log(1+x)\Big]_0^1 = \log_e 2$$

Hence $\log_e 2 = 0.6931$ which is also the exact value.

A curve passes through the points (1, 2), (1.5, 2.4), (2.0, 2.7), (2.5, 2.8), (3, 3), (3.5, 2.6) and (4.0, 2.1). Obtain the area bounded by the curve, the x-axis and x = 1 and x = 4. Also find the volume of solid of revolution got by revolving this area about the x axis.

Solution: Area : $\int_a^b y\,dx = \int_1^4 y\,dx$; here $h = 0.5$

x :	1	1.5	2.0	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

By Simpson's $1/3^{rd}$ Rule :

$$\text{Area} = \int_a^b y\,dx = \frac{0.5}{3}\left[(2+2.1) + 2(2.7+3) + 4(2.4+2.8+2.6)\right]$$

$$\text{Area} = 7.7833 \text{ sq. units.}$$

$$\text{Volume} = \pi\int_a^b y^2\,dx \quad [\text{Formula}]$$

$$\int_1^4 y^2\,dx = \frac{0.5}{3}\left[(2^2+2.1^2) + 2(2.7^2+3^2) + 4(2.4^2+2.8^2+2.6^2)\right]$$

$$= \frac{1}{6}\left[(8.41) + 32.58 + 81.44\right] = 20.405$$

$$\text{Volume} = \pi \times 20.405$$

$$\text{Volume} = 64.13 \text{ cubic units.}$$

The river is 80 meters wide. The depth 'd' in meters at a distance 'x' meters from one bank is given by the following table. Calculate the area of cross-section of the river using simpson's rule

x :	0	10	20	30	40	50	60	70	80
$d=y$:	0	4	7	9	12	15	14	8	3

Solution :

Here no. of ordinates are odd. So we can use simpson's $1/3^{rd}$ Rule.

Here $h = 10-0 = 10$

$$\text{Area} = \int_a^b y\,dx = \frac{10}{3}\left[(0+3)+2[7+12+14]+4[4+9+15+8]\right]$$

$$\text{Area} = 710 \text{ sq. meters.}$$

The table below gives the velocity v of a moving particle at time t seconds. Find the distance covered by the particle in 12 seconds and also the acceleration at t=2 sec.

t :	0	2	4	6	8	10	12
v :	4	6	16	34	60	94	136

Solution:

We know $\frac{ds}{dt} = v$ and $a = \frac{dv}{dt}$

$\therefore S = \int v\,dt$

To get distance from velocity, we have to integrate v.

Here no. of ordinates are odd (7). Therefore we can Simpson's $\frac{1}{3}$rd Rule.

$$\therefore S = \int_0^{12} v\,dt = \frac{2}{3}\left[(4+136)+2(16+60)+4(6+34+94)\right]$$

$$S = \frac{2}{3}\left[140+152+536\right]$$

$$S = 552 \text{ meters.}$$

(ii) Acceleration $= a = \left(\frac{dv}{dt}\right)_{t=2}$

$\therefore$ We require differentiation.

t	V	Δv	$\Delta^2 v$	$\Delta^3 v$
0	4			
2	6	2		
		10	8	
4	16		8	0
6	34	18	8	0
8	60	26	8	0
10	94	34	8	0
12	136	42		

$$\left(\frac{dv}{dt}\right)_{t=2} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right]$$

$$\left(\frac{dv}{dt}\right)_{t=2} = \frac{1}{2}\left[10 - \frac{1}{2}(8) + \frac{1}{3}(0)\right] = 3\,m/s^2$$

$$= \frac{0.01}{4}\left[1.5714 + 9.2864 + 13.7188\right]$$

$$= 0.0614$$

Case(ii) By Simpson's rule

$$I = \frac{(0.1)(0.1)}{9}\Big[(0.5 + 0.4167 + 0.3571 + 0.2976) + 2(0.4167 + 0.4545$$

$$+ 0.3472 + 0.3247) + 4(0.3846 + 0.4545 + 0.4762 + 0.4348$$

$$+ 0.3788 + 0.3205 + 0.3106 + 0.3401) + 4(0.3788)$$

$$+ 8(0.3968 + 0.3623) + 8(0.3497 + 0.4132)$$

$$+ 16(0.3663 + 0.3344 + 0.4529 + 0.3953)\Big]$$

$$I = \frac{0.01}{9}[55.2116]$$

$$I = 0.0613$$

Case(iii) By actual Integration.

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = \left(\int_1^{1.4} \frac{dy}{y}\right)\left(\int_2^{2.4} \frac{dx}{x}\right)$$

$$= (\log y)_1^{1.4} (\log x)_2^{2.4}$$

$$= (\log 1.4)(\log 2.4 - \log 2)$$

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = 0.0613$$

3. Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$ by using trapezoidal rule Simpson's rule and also by actual integration.

Solution: Divide the range on x and y direction in 2 equal parts and obtain the values of $f = \sin(x+y)$ at each node.

Here $h = k = \pi/4$

y\x	0	$\pi/4$	$\pi/2$
0	0	0.7071	1
$\pi/4$	0.7071	1	0.7071
$\pi/2$	~~0~~	0.7071	0

Case(i) By trapezoidal rule,

$$I = \frac{\pi/4 \times \pi/4}{4}\left[(0+1+1+0)+2(0.7071+0.7071+0.7071+0.7071)+4(1)\right]$$

$$I = 0.1542(11.6568)$$

$$I = 1.7975 \checkmark$$

Case (ii) By Simpson's Rule

$$I = \frac{\frac{\pi}{4} \times \frac{\pi}{4}}{9}\left[(0+1+1+0)+2(0)+4(0.7071+0.7071+0.7071+0.7071) + 16(1)\right]$$

$$I = 0.0685 \times 29.3136$$

$$I = 2.0080$$

evaluate $\int_0^1 \int_0^1 e^{x+y}\,dx\,dy$ using trapezoidal rule and Simpson's rule. Also evaluate directly.

Solution: Divide each side of the rectangle of integration into 2 parts. Get the values of $f(x,y) = e^{x+y}$ at each node, taking $h=k=0.5$

$y \backslash x$	0	0.5	1
0	1	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1	2.7183	4.4817	7.3851

$\frac{hk}{4}\left\{\text{sum of 4 corner values} + 2\,(\text{next box values}) + 4\,(\text{remaining values})\right\}$ = (I)

case(i) Using trapezoidal Rule,

$$\int_0^1\int_0^1 e^{x+y}\,dx\,dy = \frac{(0.5)(0.5)}{4}\Big[(1+2.7183+2.7183+7.3851)+2(1.6487+1.6487+4.4817+4.4817)+4(2.7183)\Big]$$

$$\int_0^1\int_0^1 e^{x+y}\,dx\,dy = 3.0763$$

$\frac{hk}{4} = \Big[\text{sum of the 4 corner values} + 4(\text{sum of the next box values}) + 16(\text{remaining values})\Big]$

case(ii) Using Simpson's rule,

$$\int_0^1\int_0^1 e^{x+y}\,dx\,dy = \frac{(0.5)(0.5)}{9}\Big[(1+2.7183+2.7183+7.3851)+2(0)+4(1.6487+1.6487+4.4817)+4(0)+8(0)+8(0)+16(2.7183)\Big]$$

$$\int_0^1\int_0^1 e^{x+y}\,dx\,dy = 2.9544$$

Case 3. By integrating directly.,

$$I = \left(\int_0^1 e^x dx\right)\left(\int_0^1 e^y dy\right)$$

$$I = (e^x)_0^1 (e^y)_0^1$$

$$I = (e-1)^2 = 2.9525.$$

2. Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dxdy$ using trapezoidal rule and simpson's rule. verify your result by actual integration.

Solution. divide the range of x and y into 4 equal parts

$$h = \frac{2.4-2}{4} = 0.1 \text{ and } K = \frac{1.4-1}{4} = 0.1$$

$$f(x,y) = \frac{1}{xy}$$

y\x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

case(i). By trapezoidal rule

$$I = \frac{(0.1)(0.1)}{4}\Big[(0.5+0.4167+0.3571+0.2976)$$
$$+2(0.3846+0.4167+0.4535+0.4762+0.4545+0.4348$$
$$+0.3788+0.3472+0.3205+0.3106+0.3247+0.3401)$$
$$+4(0.4329+0.4132+0.3953+0.3968+0.3788$$
$$+0.36223+0.3663+0.3497+0.3344)\Big]$$

$\int_0^{6} \frac{1}{1+x^2}dx$ evaluate this integral by using Simpson $3/8^{th}$ Rule.

Solution:

Take $h=1$ $\qquad f(x)=\frac{1}{1+x^2}$

From the table

x	:	0	1	2	3	4	5	6
$\frac{1}{1+x^2}=y$	:	1	0.5	0.2	0.10	0.0588	0.0384	0.027
		y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson $3/8^{th}$ Rule

$$\int_a^b f(x)dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+\cdots y_{n-1})+ 2(y_3+y_6+y_9+\cdots+y_n)\Big]$$

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3(1)}{8}\Big[(1+0.027)+3(0.5+0.2+0.0588+0.0384) +2(0.1)\Big]$$

$$= 3/8\,[\,1.027+2.3916+0.2\,]$$

$$\int_0^6 \frac{1}{1+x^2}dx = 1.356975$$

Numerical methods For Double Integrals.

Formula:

(i) Trapezoidal Rule:

$$I = \frac{hk}{4}$$ [Sum of the values of (x, y) at the four corner points]

(ii) Extension to general form of trapezoidal Rule.

$$I = \frac{hk}{4}$$ [Sum of values of f at the 4 corners + 2(Sum of the values of f at the remaining nodes on the boundary) + 4(Sum of the values of f at the interior nodes)]

(iii) Simpson's rule for double integration

$$I = \frac{hk}{9}$$ [Sum of the values of f at the 4 corners + 4(Sum of the values of f at the remaining nodes in the boundary) + 16(Value of f at the central point)]

(iv) Extension to general form of Simpson's Rule.

$$I = \frac{hk}{9}$$ [Sum of the values of f at 4 corners (1) + 2(Sum of the values of f at odd position on the boundary except at corners) (2) + 4(Sum of the values of f at even position on boundary) (3) + 4(Sum of the values of f at odd position on int + 8(Sum of the values of f at even position on the odd row of the matrix except boundary rows + 8(Sum of the values of f at the odd position + 16(Sum of the values of f at even position on the even row of the matrix)]

$$\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dx\,dy=\int_0^{\pi/2}\int_0^{\pi/2}(\sin x\cos y+\cos x\sin y)\,dx\,dy$$

$$=\left(\int_0^{\pi/2}\sin x\,dx\right)\left(\int_0^{\pi/2}\cos y\,dy\right)+\left(\int_0^{\pi/2}\cos x\,dx\right)\left(\int_0^{\pi/2}\sin y\,dy\right)$$

$$=(-\cos x)_0^{\pi/2}(\sin y)_0^{\pi/2}+(\sin x)_0^{\pi/2}(-\cos y)_0^{\pi/2}$$

$$=(1\times 1)+(1\times 1)$$

$$=2$$

The value got by Simpson's rule differs from the exact value only by 0.008 while the error in the trapezoidal rule is 0.2185

4. Evaluate $\int_1^{1.2}\int_1^{1.4}\frac{1}{x+y}\,dx\,dy$ by trapezoidal rule and Simpson's rule.

Solution: Range for x is $(1, 1.4)$ and range for y is $(1, 1.2)$

$\therefore$ Divide the x-range into 4 equal parts ($h=0.1$) and y range into 2 equal parts ($k=0.1$)

Values of $f(x,y)=\frac{1}{x+y}$

$y \backslash x$	1	1.1	1.2	1.3	1.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4762	0.4545	0.4348	0.4167	0.4
1.2	0.4545	0.4348	0.4167	0.4	0.3846

$$I=\int_{1}^{1.2}\int_{1}^{1.4}\frac{1}{x+y}\,dx\,dy=\frac{(0.1)(0.1)}{4}\Big[(0.5+0.4167+0.4545+0.3846)$$

$$+2(0.4545+0.4762+0.4348+0.4+0.4+0.4167)$$

$$+2(0.4348+0.4762)+4(0.4545+0.4348+0.4167)\Big]$$

$$=\frac{0.01}{4}\left[1.7558+6.9864+5.2224\right]$$

$$\int_{1}^{1.2}\int_{1}^{1.4}\frac{1}{x+y}\,dx\,dy=0.0349$$

Case (ii) By Simpson's rule,

$$I=\frac{0.01}{9}\Big[(0.5+0.4167+0.4545+0.3846)+2(0.4545+0.4167)$$

$$+4(0.4762+0.4762+0.4348+0.4+0.4+0.4348)$$

$$+8(0.4348)+16(0.4545+0.4167)\Big]$$

$$I=\frac{0.01}{9}\left[1.7558+1.7424+10.488+3.4784+13.9392\right]$$

$$I=0.0349$$

5. Evaluate $\int_{1}^{2}\int_{3}^{4}\frac{1}{(x+y)^2}\,dx\,dy$ taking $h=k=0.5$ by both trapezoidal rule and Simpson's rule.

Solution: $f(x,y)=\frac{1}{(x+y)^2}$; range for x is $(3,4)$

$y \backslash x$	3	3.5	4
1	0.0625	0.0494	0.04
1.5	0.0494	0.04	0.0331
2	0.04	0.0331	0.0278

Case(i) By Trapezoidal Rule

$$\int_1^2 \int_3^4 \frac{1}{(x+y)^2}\,dx\,dy = \frac{hk}{4}\Big[(0.0625+0.04+0.04+0.0278)$$

$$+2(0.0494+0.0494+0.0331+0.0331)+4(0.04)\Big]$$

$$=\frac{0.25}{4}\left[0.1703+0.33+0.16\right]$$

$$=0.0413$$

case(ii) By Simpson's rule

$$I=\frac{0.25}{9}\Big[(0.0625+0.04+0.04+0.0278)+2(0)$$

$$+4(0.0494+0.0494+0.0331+0.0331)+16(0.04)\Big]$$

$$=\frac{0.25}{9}\left[0.1703+0.66+0.64\right]$$

$$I = 0.0408$$

Gauss Quadrature formula: Gaussian quadrature formulae

Two points Gauss Quadrature formula:

$$\int_{-1}^{1} f(t)\,dt = f\left(-\tfrac{1}{\sqrt{3}}\right)+f\left(\tfrac{1}{\sqrt{3}}\right)$$

Gauss 3-point formula. $\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$

$$\int_{a}^{b} f(x)\,dx = \int_{-1}^{1} f(x)\,dx$$

For Gen range

$$x = \left(\frac{a+b}{2}\right)+\left(\frac{b-a}{2}\right)t$$

$\frac{5}{9} = .5555$

$$\int_{-1}^{1} f(t)\,dt = A_1 f(t_1)+A_2 f(t_2)+A_3 f(t_3)$$

$A_1 = A_3 = \overset{5/9}{0.5555}$ $\quad A_2 = \overset{8/9}{0.8888}$

$t_1 = -0.7745 \quad t_2 = 0 \quad t_3 = 0.7745$ $\quad \sqrt{3/5} = \sqrt{.6}$

$\int_{-1}^{1} \frac{3\,dt}{1+(3+3t)^2}$

Apply Gauss 2-point formula to evaluate

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx$$

$x = \frac{6}{2} \times \frac{6}{2}t$, $= 3+3t$, $dx = 3dt$

Solution: Here $f(x) = \frac{1}{1+x^2}$; $f(1/\sqrt{3}) = f(-1/\sqrt{3}) = 3/4$

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = f\left(-1/\sqrt{3}\right)+f\left(1/\sqrt{3}\right) = 3/2 = 1.5$$

$\int_0^6$

But $\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 2(\tan^{-1}x)_0^1 = 1.5708$ //

Here, the error due to 2-point formula is 0.0708

$\int_{-1}^{1} \frac{1}{1+9}$ ... $\frac{1}{3}$

Using Gaussian three-point formula evaluate

(i) $\int_{-1}^{1}(3x^2+5x^4)dx$ (ii) $\int_{0}^{1}(3x^2+5x^4)dx$

Also compare with exact values.

(i) Let $f(x) = 3x^2 + 5x^4$

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t$$

Here $f(t) = 3t^2 + 5t^4$

$$\left.\begin{aligned} f(t_1) &= 3(-0.7745)^2 + 5(-0.7745)^4 = 3.5986 \\ f(t_2) &= 3(0)^2 + 5(0)^4 = 0 \end{aligned}\right\} \text{using values of } t_1, t_2, t_3.$$

$$f(t_3) = 3(0.7745)^2 + 5(0.7745)^4 = 3.5986$$

$$I = \int_{-1}^{1}(3t^2+5t^4)dt = (3.5986)(0.5555) + 0.5555(3.5986)$$

$$\int_{-1}^{1}(3t^2+5t^4)dt = 3.998$$

(ii) $\int_{0}^{1}(3x^2+5x^4)dx$

This can be written as $= \frac{1}{2}\int_{-1}^{1}(3x^2+5x^4)dx$

$$= \frac{1}{2}(3.998) = 1.999$$

Use Gaussian three-point formula and evaluate

$$I = \int_1^5 \frac{dz}{z}$$

$a = 1 \quad b = 5$

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t \implies x = 3+2t$$

when $z = 1$, $t = -1$ | when $z = 5$, $t = 1$ | Here $z = 3+2t$, $dz = 2dt$

$$\implies \int_{-1}^{1} \frac{2dt}{3+2t} \qquad \text{Here } f(t) = \frac{2}{3+2t}$$

$$f_1 = \frac{2}{3+2(-0.7745)} = 1.3783$$

$$f_2 = \frac{2}{3+2(0)} = 2/3 = 0.6667$$

$$f_3 = \frac{2}{3+2(0.7745)} = 0.43965$$

$$I = 0.5555(1.3783) + (0.6667 \times 0.8888) + (0.43965 \times 0.5555)$$

$$I = 1.6024$$

V) a)(ii) Evaluate $\int_{-2}^{2} e^{-x/2} dx$ by Gauss 2 point formula. (3)

Solution: $a = -2 \quad b = 2$

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)t \implies x = \left(\frac{2-2}{2}\right) + \left(\frac{2+2}{2}\right)t$$

$x = 2t$

$dx = 2dt$

when $x=2$ when $x=\infty$

$t=-1$ $t=1$

$$\int_{-1}^{1} e^{-2t/2}\, 2\,dt = 2\int_{-1}^{1} e^{-t}\,dt$$

$$= 2e^{-1/\sqrt{3}} + 2e^{1/\sqrt{3}}$$

$$= 1.1287 + 3.5626$$

$$\int_{-2}^{2} e^{-x/2}\,dx = 4.6853$$

Evaluate $\int_0^{\pi/2} \sin t\,dt$ by Gaussian 2 point formula

Solution: Let $t = \pi/4\, x + \pi/4 = \pi/4(x+1)$; $\therefore dt = \pi/4\, dx$

15)b)ii $I = \int_0^{\pi/2} \sin t\,dt$

$$= \pi/4 \int_{-1}^{1} \sin\left(\frac{\pi x + \pi}{4}\right) dx$$

$$= \pi/4 \left[\sin \pi/4(1.57735) + \sin \pi/4 (0.42265)\right]$$

$$= \pi/4 \left[\sin(0.39434\pi) + \sin(0.10566\pi)\right]$$

$$I = 0.99847$$

By Gaussian formula evaluate $\int_2^3 \frac{dt}{1+t}$

Solution:

$$t = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)x \implies t = 5/2 + x/2$$

$$dt = \frac{dx}{2} \quad dx = 2dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{2\,dx}{7+x}$$

when $t=2$ $\quad$ when $t=3$,

$x=-1$ $\quad$ $x=1$

$$1+t = \frac{7+x}{2}$$

$$= \int_{-1}^{1} \frac{dx}{7+x}$$

By 2 point formula,

$$I = f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

$$= \frac{1}{(7-1/\sqrt{3})} + \frac{1}{(7+1/\sqrt{3})}$$

$$I = 0.28188$$

By Three point formula

$$f(t) = \frac{1}{7+x}$$

$$f(t_1) = \frac{1}{7-0.7745} = 0.1606$$

$$f(t_2) = \frac{1}{7-0} = 0.1428$$

$$f(t_3) = \frac{1}{7+0.7745} = 0.1286$$

$$\int_{-1}^{1} \frac{dx}{7+x} = (0.1609 \times 0.5555) + (0.1428 \times 0.8888) + (0.1286 \times 0.5555)$$

$$\int_{-1}^{1} \frac{dx}{7+x} = 0.28773$$

Evaluate $\int_{1} \frac{dx}{x}$ using Gaussian quadrature formula.

Solution

$$x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$$

$$x = 3/2 + \left(1/2\right)t$$

$$dx = \frac{dt}{2}$$

when $x=1$ when $x=2$

$t=-1$ $t=1$

$$I = \int_{-1}^{1} \frac{dt}{2\left(3/2 + t/2\right)}$$

$$f_1 = \frac{1}{2\left(\frac{3-0.7745}{2}\right)} = 0.4493$$

$$f_2 = \frac{1}{2\left(\frac{3-0}{2}\right)} = 0.3333$$

$$f_3 = \frac{1}{2\left(\frac{3+0.7745}{2}\right)} = 0.2649$$

$$I = (0.4493 \times 0.5555) + (0.3333 \times 0.8888) + (0.2649 \times 0.5555)$$

$$I = 0.6929.$$

Find the value of sec 31° using the following table.

θ :	31	32	33	34
$\tan\theta$:	0.6008	0.6249	0.6494	0.6745

<u>Solution:</u>

θ	$\tan\theta = y$	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008			
		0.0241		
32	0.6249		0.0004	
		0.0245		0.0002
33	0.6494		0.0006	
		0.0251		
34	0.6745			

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \tfrac{1}{2}\Delta^2 y_0 + \tfrac{1}{3}\Delta^3 y_0 - \tfrac{1}{4}\Delta^4 y_0\right]$$

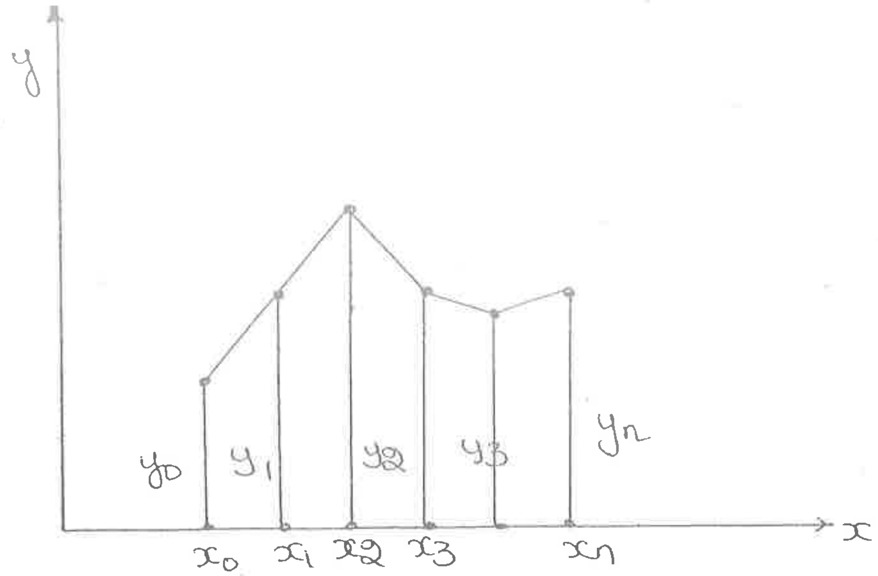
$$h = \frac{\pi}{180} \text{ rad}$$

$$\frac{dy}{dx} = \frac{1}{\pi/180}\left[0.0241 - \frac{0.0004}{2} + \frac{0.0002}{3}\right]$$

$$\left(\frac{dy}{dx}\right)_{31} = 1.373$$

$$\sec^2\theta = 1.373$$

$$\sec\theta = 1.17. \quad \left(\sqrt{1.373}\right)$$



Geometrically, if the ordered pairs (x_i, y_i), $i = 0, 1, 2 \ldots, n$ are plotted, and if any 2 consecutive points are joined by straight lines, we get the fig as shown.

The area between $f(x)$, x-axis and ordinates $x = x_0$ and $x = x_n$ is approximated to the sum of the trapeziums as shown in the fig.

__Note:__ Though this method is very simple for calculation purposes of numerical integration, the error in this case is significant. The accuracy of the result can be improved by increasing the number of intervals and decreasing the value of h.