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DEPARTMENT OF MATHEMATICS

SUBJECT NAME : NUMERICAL METHODS

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UNIT – I : SOLUTIONS OF EQUATIONS & EIGEN VALUE PROBLEMS

SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

UNIT I INTRODUCTION – TOPICS COVERED

* ITERATIVE METHOD
* NEWTON - RAPHSON METHOD FOR SINGLE VARIABLE
* NEWTON - RAPHSON METHOD FOR SIMULTANEOUS EQUATIONS WITH TWO VARIABLES. Regula Falsi method.
* SOLUTION OF LINEAR SYSTEM BY GAUSS JORDAN METHOD, Gauss elimination method.
* GAUSS JACOBI METHOD
* GAUSS SEIDAL METHOD
* INVERSE OF A MATRIX BY GAUSS JORDAN METHOD
* EIGEN VALUE OF A MATRIX BY POWER AND JACOBI METHOD.

ITERATIVE METHOD : [OR METHOD OF SUCCESSIVE APPROXIMATIONS]

Iterative method is a self correction method.

PROCEDURE :

Suppose we want to find the roots of the equation

$$\boxed{f(x)=0} \quad \text{-------} \rightarrow (1)$$

The equation (1) can be expressed as

$$\boxed{x = \phi(x)} \quad \text{-------} \rightarrow (2)$$

Let us consider x_0 be an initial approximation, then

$$\left.\begin{aligned} x_1 &= \phi(x_0) \\ x_2 &= \phi(x_1) \\ x_3 &= \phi(x_2) \\ &\vdots \\ x_n &= \phi(x_{n-1}) \end{aligned}\right\} \quad \text{if } \boxed{|\phi'(x)| < 1}$$

Let $f(x)=0$ be the eqn. If the eqn has a root in (a,b), and then it can be written as $x = \phi(x)$

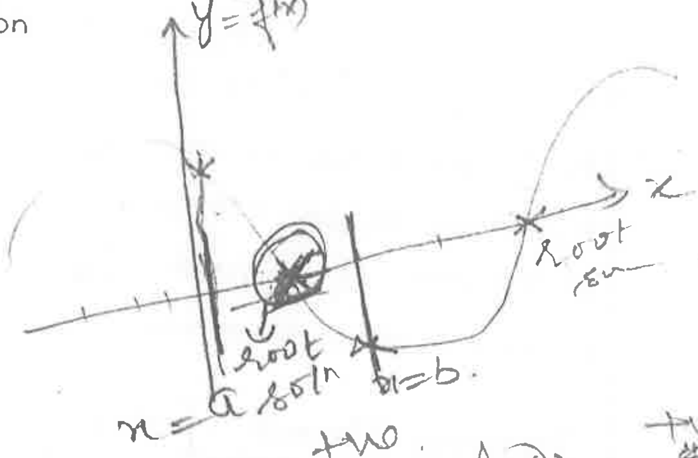
If $f(x)$ is cts in the interval (a,b) and if $f(a)$ and $f(b)$ are opp. signs then the eqn. $f(x)=0$ will have at least one real root between a & b
(It is fundamental thm. of algebra)

PROBLEMS :

① Find the root of $f(x) = x^3 + x^2 - 100$

SOLUTION :

$f(x) = 0$



Step 1 : Initial approximation

$f(0) = 0 + 0 - 100 < 0$

$f(1) = 1 + 1 - 100 < 0$

$f(2) = 8 + 2 \times 2 - 100 < 0$

$f(3) = 3^3 + 3^2 - 100 < 0$

$f(4) = 4^3 + 4^2 - 100 < 0$

$f(5) = 5^3 + 5^2 - 100 > 0$

$f(a) = +ve$ $\qquad$ $f(0) =$ +ve

$f(b) = -ve$ $\qquad$ $f(1) =$

$\therefore$ The root lies between 4 and 5

$\therefore x_0 = \dfrac{4+5}{2} = \underline{\underline{4.5}}$

Step 2 : The given equation can be written as

$x^3 + x^2 = 100 \quad [\because f(x) = 0]$

$x^2(x+1) = 100$

$x^2 = \dfrac{100}{x+1}$

$x = \dfrac{10}{\sqrt{x+1}}$

$$\boxed{x = \phi(x) = \dfrac{10}{\sqrt{x+1}}}$$

$x^3\left(1 + \frac{1}{x}\right) = 100$

$x^3 = \underline{100}$

$f(x) = 0$

$\parallel$

$x = \phi(x)$

fixed pt

Step 3 : $x' = \phi'(x) = 10(-1/2)(x+1)^{-3/2}$

$= -5(x+1)^{-3/2}$

$= \dfrac{-5}{(x+1)^{3/2}}$

fixed pt thm.

Putting $x = 4$,

$$|\phi'(x)| = |\phi'(4)| = \left| \frac{-5}{(4+1)^{3/2}} \right| < 1$$

Putting $x = 5$,

$$|\phi'(5)| = \left| \frac{-5}{(5+1)^{3/2}} \right| < 1$$

<u>Step 4</u>: By applying the iteration method,

$$x_1 = \phi(x_0) = \frac{10}{\sqrt{1+4.5}} = 4.26$$

$$x_2 = \phi(x_1) = \frac{10}{\sqrt{1+4.26}} = 4.3602$$

$$x_3 = \phi(x_2) = \frac{10}{\sqrt{1+4.3602}} = 4.3192$$

$$x_4 = \phi(x_3) = \frac{10}{\sqrt{1+4.3192}} = 4.3358$$

$$x_5 = \phi(x_4) = \frac{10}{\sqrt{1+4.3358}} = 4.3291$$

$$x_6 = \phi(x_5) = \frac{10}{\sqrt{1+4.3291}} = 4.3318$$

$$x_7 = \phi(x_6) = 4.3307$$

$$x_8 = \phi(x_7) = 4.3312$$

$$x_9 = \phi(x_8) = 4.3309$$

$$x_{10} = \phi(x_9) = 4.3310$$

$$x_{11} = \phi(x_{10}) = 4.3310$$

<u>SOLUTION</u> : $\boxed{x = 4.3310}$

(2) Find the root of $\cos x = 3x - 1$ Trignometric → radian mode

SOLUTION:

$f(1) = \cos 1 - (3) + 1$

$= \cos 1 - 2$

$= .5403 - 2$

$= -$

Let $f(x) = \cos x - 3x + 1$

Step 1: Initial approximation

$f(0) = \cos 0 - 3(0) + 1 = 2 > 0$

$f(1)$ + $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - 3\left(\frac{\pi}{2}\right) + 1 = -3.712 < 0$

$\therefore$ The root of $f(x)$ lies between 0 and $\pi/2$.

$\therefore x_0 = \frac{\pi/2 + 0}{2} = \frac{\pi}{4}$ $\frac{1+0}{2} = \frac{1}{2}$

Step 2: The given equation can be written as

$x = \phi(x)$

$\boxed{\phi(x) = \frac{\cos x + 1}{3}}$

$\phi'(x) = -\frac{\sin x}{3}$

$\phi'(0) = |0| < 1$; $\phi'(\pi/2) = |-0.333| < 1$

Step 3: $x_1 = \phi(x_0) = \frac{\cos(\pi/4) + 1}{3} = 0.569$

$x_2 = \phi(x_1) = \frac{\cos(0.569) + 1}{3} = 0.614$

$x_3 = \phi(x_2) = \frac{\cos(0.614) + 1}{3} = 0.6057$

$x_4 = \phi(x_3) = \frac{\cos(0.6057) + 1}{3} = 0.6073$

$x_5 = \phi(x_4) = \frac{\cos(0.6073) + 1}{3} = 0.6071$

$x_6 = \phi(x_5) = \frac{\cos(0.6071) + 1}{3} = 0.6071$

Solution: $x = 0.6071$

(3) Find the root of $f(x) = e^{x} - 3x$

Solution:

Step 1: Initial approximation.

$$f(0) = e^{0} - 3(0) = 1 > 0$$

$$f(1) = e^{1} - 3(1) = -0.282 < 0$$

$\therefore$ The root lies between 0 and 1

$$x_0 = \frac{0+1}{2} = 0.5$$

Step 2: The given equation can be written as

$$e^{x} = 3x$$

$$\boxed{x = e^{x}/3}$$

$$\phi(x) = e^{x}/3$$

$$\phi'(x) = \frac{1}{3} e^{x}$$

$$\phi'(0) = \left|\frac{e^{0}}{3}\right| < 1 ; \quad \phi'(1) = \left|\frac{e^{1}}{3}\right| < 1.$$

$e^{1} = 2.718$

Step 3: $x_1 = \phi(x_0) = e^{0.5}/3 = 0.549$

$$x_2 = \phi(x_1) = e^{0.549}/3 = 0.5775$$

$$x_3 = \phi(x_2) = \frac{e^{0.5775}}{3} = 0.594$$

$$x_4 = \phi(x_3) = 0.6036$$

$$x_5 = \phi(x_4) = 0.6096$$

$$x_6 = \phi(x_5) = 0.6132$$

$$x_7 = \phi(x_6) = 0.615$$

$$x_8 = \phi(x_7) = 0.617$$

$$x_9 = \phi(x_8) = 0.6177$$

$$x_{10} = \phi(x_9) = 0.6177$$

Solution: $x = 0.6177$

Find the root of $x^2 - 400x + 1 = 0$.

SOLUTION :

Let $f(x) = x^2 - 400x + 1$

Step 1 : Initial approximation.

$f(0) = 0 - 400(0) + 1 = 1 > 0$

$f(1) = 1 - 400 + 1 = -398 < 0$.

$\therefore$ The root of $f(x)$ lies between 0 and 1

$x_0 = \frac{1+0}{2} = 0.5$

Step 2 : The given equation can be written as

$x = \phi(x)$

$\phi(x) = x^2 - 400x + 1$.

$\Rightarrow x^2 - 400x = -1$

$x(x-400) = -1$

$$\boxed{\phi(x) = -\frac{1}{x-400}}$$

$\phi'(x) = +\frac{1}{(x-400)^2}$

$\phi'(0) = \left|\frac{1}{400^2}\right| < 1$; $\phi(1) = \left|\frac{1}{(-399)^2}\right| < 1$

Step 3 : $x_1 = \phi(x_0) = \frac{-1}{0.5-400} = 2.5031 \times 10^{-3}$

$x_2 = \phi(x_1) = \frac{-1}{(2.5031 \times 10^{-3} - 400)} = 2.50001 \times 10^{-3}$

$x_3 = \phi(x_2) = \frac{-1}{(2.5001 \times 10^{-3} - 400)} = 2.50003 \times 10^{-3}$

$x_4 = \phi(x_3) = \frac{-1}{(2.50003 \times 10^{-3} - 400)} = 2.50005 \times 10^{-3}$

Solution : $x = 2.50005 \times 10^{-3}$

(5) Solve the equation $x^3+x^2-1=0$ for the positive root

SOLUTION :

$x^3 + x^2 = 1$

$x^2(x+1) = 1$

$x^2 = \frac{1}{x+1}$

$x = \frac{1}{\sqrt{x+1}}$

STEP 1 : $f(x) = x^3+x^2-1$

$f(0) = -1 = -ve$

$f(1) = 1 = +ve$

$\therefore$ The root lies between 0 and 1

$$x_0 = \frac{0+1}{2} = 0.5$$

STEP 2 : The given equation can be written as

$$x^2(x+1) = 1$$

$$\boxed{x = \frac{1}{\sqrt{x+1}} = \phi(x)}$$

$$\phi'(x) = -\frac{1}{2}\frac{1}{(x+1)^{3/2}}$$

$$|\phi'(0)| = \frac{1}{2} < 1 \text{ and } |\phi'(1)| < 1$$

$x^3 + x^2 = 1$

$x^2(1+x) = 1$

$x^2 = \frac{1}{1+x}$

$x = \frac{1}{\sqrt{1+x}}$

STEP 3 : $$x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = 0.81649$$

$$x_2 = \frac{1}{\sqrt{1+0.81649}} = 0.74196$$

$$x_3 = \frac{1}{\sqrt{1+0.74196}} = 0.75769$$

$$x_4 = \frac{1}{\sqrt{1+0.75769}} = 0.754277$$

$$x_5 = \frac{1}{\sqrt{1+0.754277}} = 0.755006$$

SOLUTION: $x = 0.75488$

(6) Solve for x from $\cos x - xe^{x} = 0$

SOLUTION:

STEP 1: $f(x) = \cos x - x e^{x}$

Initial approximation (x_0):

$f(0) = 1 = +ve$

$f(1) = \cos 1 - e^{1} = -2.1780 = -ve$

$\therefore$ The root lies between 0 and 1

$$x_0 = \frac{0+1}{2} = \underline{0.5}$$

STEP 2: $\cos x - xe^{x} = 0$

$$\boxed{x = \frac{\cos x}{e^{x}} = \phi(x)}$$

$$\phi'(x) = \frac{e^{x}(-\sin x) - \cos x\, e^{x}}{e^{2x}}$$

$$\phi'(x) = -\frac{(\sin x + \cos x)}{e^{x}}$$

$$\phi'(x) = \frac{-\sqrt{2}\sin\left(\pi/4 + x\right)}{e^{x}}$$

$$|\phi'(x)| < \frac{\sqrt{2}}{e^{x}}$$

$$|\phi'(0)| = \frac{\sqrt{2}}{1} = 1.414 > 1 \rightarrow \frac{1}{2} < 1$$

$|\phi'(0.5)| = 0.85776 < 1.$

$|\phi'(1)| = 0.52026 < 1$

$\therefore$ The root lies between 0·5 and 1.

STEP 3: $x = \frac{\cos x}{e^{x}}$

$x_1 = \frac{\cos(0.5)}{e^{0.5}} = 0.5322$

$x_2 = \frac{\cos(0.532)}{e^{0.532}} = 0.50624$

$x_3 = \frac{\cos(0.50624)}{e^{0.50624}} = 0.527$

$x_4 = \frac{\cos(0.527)}{e^{0.527}} = 0.51027$

$x_5 = \frac{\cos(0.51027)}{e^{0.51027}} = 0.5288$

$x_6 = \frac{\cos(0.5238)}{e^{0.5238}} = 0.5128$

$x_7 = \frac{\cos(0.5128)}{e^{0.5128}} = 0.52179$

$x_8 = \frac{\cos(0.522)}{e^{0.522}} = 0.5145$

$x_9 = \frac{\cos(0.514)}{e^{0.514}} = 0.5208$

$x_{10} = \frac{\cos(0.5208)}{e^{0.5208}} = 0.5159$

$x_{11} = \frac{\cos(0.5159)}{e^{0.5159}} = 0.51924$

$x_{12} = \frac{\cos(0.51924)}{e^{0.51924}} = 0.51655$

$$x_{13} = \frac{\cos(0.51655)}{e^{0.51655}} = 0.51874$$

$$x_{14} = \frac{\cos(0.51874)}{e^{0.51874}} = 0.51696$$

$x_{16} = 0.51723$

$x_{18} = 0.51741$

$x_{20} = 0.51753$.

$x_{22} = 0.51761$

$x_{24} = 0.51766$.

$x_{26} = 0.51769$

$x = 0.5176$

SOLUTION : $\boxed{x = 0.5176}$

(7) Solve $x^3 = 2x + 5$ for the positive root

SOLUTION :

STEP 1 : Initial approximation (x_0)

$f(x) = x^3 - 2x - 5 = 0$

$f(2) = 2^3 - 4 - 5 = -1 = (-ve)$

$f(3) = 3^3 - 6 - 5 = 16 = +ve$.

$\therefore$ The root lies between 2 and 3.

$$x_0 = \frac{2+3}{2} = 2.5$$

STEP 2 : $f(x) = x^3 - 2x - 5$

$x^3 = 2x + 5$

$x = (2x+5)^{1/3} = \phi(x)$

$$\phi'(x) = \frac{1}{3}(2x+5)^{-2/3}(2)$$

$$\phi'(2) = \frac{1}{3}(4+5)^{-2/3}(2) < 1$$

$$\phi'(3) = \frac{2}{3}(6+5)^{-2/3} < 1$$

STEP 3: $x_1 = (2x_0+5)^{1/3} = (2(2.5)+5)^{1/3} = 2.1544$

$$x_2 = (2\times 2.1544+5)^{1/3} = 2.105606$$

$$x_3 = (2\times 2.105606+5)^{1/3} = 2.09593$$

$$x_4 = (2\times 2.09593+5)^{1/3} = 2.09476$$

$$x_5 = (2\times 2.09476+5)^{1/3} = 2.0945$$

$\therefore x = 2.0945$

SOLUTION: $\boxed{x = 2.0945}$

NEWTON- RAPHSON METHOD FOR A SINGLE VARIABLE :

DERIVATION OF NEWTON - RAPHSON FORMULA FOR RECIPROCAL OF A NUMBER :

$\boxed{x = \frac{1}{N}}$ (or) $N = \frac{1}{x}$

$$f(x) = N - \frac{1}{x} = 0.$$

Let x be the approximate root of $f(x) = 0$.

$\therefore f(x)$ can be written as

$$f(x) = \frac{1}{x} - N = 0.$$

$$f'(x) = -\frac{1}{x^2}$$

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\left(-1/x_n^2\right)}$

$$x_{n+1} = \frac{x_n\left(-1/x_n^2\right) - \left(\frac{1}{x_n} - N\right)}{\left(-1/x_n^2\right)}$$

$$\Rightarrow x_{n+1} = \frac{-\frac{1}{x_n} - \left(\frac{1 - N x_n}{x_n}\right)}{\left(-1/x_n^2\right)}$$

$\cancel{x_n - \left(\frac{1 - x\, x_n}{x_n}\right) \quad x - x_n^2}$

$\cancel{\frac{x_n^2 - 1 + N x_n}{x_n} \quad (-x_n^2)}$

$$\Rightarrow x_{n+1} = \frac{\frac{-2 + N x_n}{x_n}}{-1/x_n^2}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^2 - N x_n^3}{x_n}.$$

$$\boxed{x_{n+1} = x_n(2 - N x_n)}$$

Derivation Of Newton-Raphson Formula To Get A Square Root Of A Positive Number:

$$\boxed{x = \sqrt{N}}$$

$$x^2 = N$$

$$f(x) = x^2 - N$$

$$f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^2 - N)}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}$$

$$\Rightarrow \boxed{x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]}$$

Problems:

① Find the root of $\cos x = x e^x$

Solution: Take initial approximation $x_0 = 0.5$

Step 1: $f(x) = \cos x - x e^x$

$f(0) = \cos 0 - 0 > 0$

$f(1) = \cos 1 - e^1 < 0$

$\therefore$ The root lies between 0 and 1

$$x_0 = \frac{0+1}{2} = 0.5$$

W.K.T. $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

Step 2: $f'(x) = -\sin x - x[e^x] - e^x = -\sin x - e^x[x+1]$

$$f'(x) = -[\sin x + x e^x + e^x]$$

Step 3 : $x_1 = x_0 - \dfrac{f(x_0)}{f_1(x_0)}$

$$x_1 = 0.5 - \frac{[\cos 0.5 - 0.5 e^{0.5}]}{-[\sin 0.5 + 0.5e^{0.5} + e^{0.5}]}$$

$$x_1 = 0.518$$

$$x_2 = 0.518 - \frac{[\cos 0.518 - 0.518 e^{0.518}]}{-[\sin 0.518 + 0.518 e^{0.518} + e^{0.518}]} = 0.5177$$

$$x_3 = 0.5177 - \frac{[\cos 0.5177 - 0.5177 e^{0.5177}]}{-[\sin 0.5177 + 0.5177 e^{0.5177} + e^{0.5177}]} = 0.5177$$

Solution : $x = 0.5177$

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(2) Find the root of $e^x = 2x + 21$

Solution : $f(x) = e^x - 2x - 21$

Step 1 : $f(0) = 1 - 0 - 21 < 0$

$f(3) = e^3 - 6 - 21 < 0$

$f(4) = e^4 - 8 - 21 > 0$

$\therefore$ The root lies between 3 and 4

$$x_0 = \frac{3+4}{2} = 3.5$$

Step 2 : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$f'(x) = e^x - 2$$

Step 3 : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3.5 - \dfrac{f(e^{3.5})}{f'(3.5)} = 3.335$

$$x_2 = 3.335 - \left[\frac{e^{3.335} - 2(3.335) - 21}{e^{3.335} - 2}\right] = 3.319$$

$$x_3 = 3.319 - \left[\frac{e^{3.319} - 2\times 3.319 - 21}{e^{3.319} - 2}\right] = 3.319$$

Solution : $x = 3.319$

③ Find the value of $1/31$

Solution :

$N = 31$

Step 1: $x_{n+1} = x_n[2 - Nx_n]$

$\boxed{f(x) = x - 1/31}$

Step 2: Let $x_0 = 0.03$

$x_1 = 0.03(2 - 31(0.03)) = 0.0321$

$x_2 = 0.0321\,(2 - 31(0.0321)) = 0.03226$

$x_3 = 0.03226\,(2 - 31(0.03226)) = 0.03226$

Solution : $\boxed{x = 0.03226}$

④ Find the value of $\sqrt{5}$

Solution: Let $x_0 = 2$ approximation

$N = 5$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

$x_1 = \frac{1}{2}\left[x_0 + N/x_0\right] = \frac{1}{2}\left[2 + 5/2\right] = 2.25$

$x_2 = \frac{1}{2}\left[2.25 + 5/2.25\right] = 2.2361$

$x_3 = \frac{1}{2}\left[2.2361 + 5/2.2361\right] = 2.2361$

Solution : $x = \sqrt{5} = 2.2361$

(5) Find the root of $x^3 = x+1$

Step 1 : $f(x) = x^3 - x - 1$

$f(0) = 0 - 0 - 1 < 0$

$f(1) = 1 - 1 - 1 < 0$

$f(2) = 8 - 2 - 1 > 0$

$\therefore$ The root lies between 1 and 2

$$x_0 = \frac{1+2}{2} = 1.5$$

Step 2 : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f(x) = x^3 - x - 1$

$f'(x) = 3x^2 - 1$

Step 3 : $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1.5 - \dfrac{(1.5^3 - 1.5 - 1)}{[3\times(1.5)^2 - 1]} = 1.3478$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.3478 - \dfrac{(1.3478^3 - 1.3478 - 1)}{(3\times(1.3478)^2 - 1)} = 1.3252$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 1.3252 - \dfrac{(1.3252^3 - 1.3252 - 1)}{(3\times 1.3252^2 - 1)} = 1.3247$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} = 1.3247 - \dfrac{(1.3247^3 - 1.3247 - 1)}{(3\times 1.3247^2 - 1)} = 1.3247$

Solution : $x = 1.3247$

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(6) Find the value of $f(x) = x^3 - 6x + 4 = 0$.

SOLUTION :

STEP 1 : $f(0) = 0 - 0 + 4 > 0$

$f(1) = 1 - 6 + 4 < 0$

$\therefore$ The root lies between 0 and 1

$$x_0 = \frac{0+1}{2} = 0.5$$

STEP 2 : $f'(x) = 3x^2 - 6$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

STEP 3 : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{[0.5^3 - 6(0.5) + 4]}{[3 \times 0.5^2 - 6]} = 0.7143$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.7143 - \frac{[0.7143^3 - 6(0.7143) + 4]}{[3 \times 0.7143^2 - 6]}$$

$$x_2 = 0.732$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.732 - \frac{[0.732^3 - 6(0.732) + 4]}{[3 \times 0.732^2 - 6]}$$

$$x_3 = 0.732$$

$\therefore \boxed{x = 0.732}$

* NEWTON - RAPHSON METHOD FOR SIMULTANEOUS EQUATIONS WITH TWO VARIABLES

PROCEDURE :

* Let us consider two non-linear equations $f(x,y) = 0$ — ① , $g(x,y) = 0$ — ② in terms of x and y.
* Assume the initial approximation (x_0, y_0)
* Then $\frac{\partial f}{\partial x} = f_x$; $\frac{\partial f}{\partial y} = f_y$

  $\frac{\partial g}{\partial x} = g_x$; $\frac{\partial g}{\partial y} = g_y$

* $J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}_{x_0, y_0}$

* $h_0 = \left(\dfrac{g f_y - f g_y}{J} \right)_{x_0, y_0}$; $k_0 = \left(\dfrac{f g_x - g f_x}{J} \right)_{x_0, y_0}$

* 1st iteration starts with $x_1 = x_0 + h_0$; $y_1 = y_0 + k_0$
* 2nd iteration starts with $x_2 = x_1 + h_1$; $y_2 = y_1 + k_1$

  where $h_1 = \left(\dfrac{g f_y - f g_y}{J} \right)_{x_1, y_1}$; $k_1 = \left(\dfrac{f g_x - g f_x}{J} \right)_{x_1, y_1}$

PROBLEMS :

(1) Solve the equations $x^2 + y^2 = 16$; $x^2 - y^2 = 4$. Given that the starting approximation of the solution is $(2\sqrt{2}, 2\sqrt{2})$

SOLUTION :

Step 1 : $f = x^2 + y^2 - 16$

$g = x^2 - y^2 - 4$

$f(x) = 2x$; $f(y) = 2y$

$g(x) = 2x$; $g(y) = -2y$

$J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -8xy$

Step 2 :

$$J(x_0, y_0) = -8 \times 2\sqrt{2} \times 2\sqrt{2} = \underline{-64}$$

$$h_0 = \left(\frac{gf_y - fg_y}{J}\right)_{x_0, y_0} = \left[\frac{(x^2-y^2-4)(2y) - (x^2+y^2-16)(-2y)}{-8xy}\right]_{2\sqrt{2}, 2}$$

$$h_0 = \frac{2(-11.812) - 0}{-64} = \underline{0.354}$$

$$k_0 = \left(\frac{fg_x - gf_x}{J}\right)_{2\sqrt{2}, 2\sqrt{2}} = \left[\frac{(x^2+y^2-16)(2x) - (x^2-y^2-4)(2x)}{-8xy}\right]_{2\sqrt{2},}$$

$$k_0 = \underline{-0.354}$$

1st iteration is $x_1 = x_0 + h_0 = 2\sqrt{2} + 0.354 = \underline{3.182}$

$y_1 = y_0 + k_0 = 2\sqrt{2} - 0.354 \quad \underline{= 2.474}$

Step 3:

$$J_\phi = J(x_1, y_1) = -8 \times 3.182 \times 2.474 = \underline{-62.978}$$

$$h_1 = \left(\frac{gf_y - fg_y}{J}\right)_{x_1, y_1} = \left[\frac{(x^2-y^2-4)(2y) - (x^2+y^2-16)(-2y)}{-8xy}\right]_{3.182, 2.474}$$

$$= \frac{0.0220087 + 1.2162184}{-62.978}$$

$$\boxed{h_1 = -0.01966}$$

$$k_1 = \left(\frac{fg_x - gf_x}{J}\right)_{x_1, y_1} = \left[\frac{(x^2+y^2-16)(2x) - (x^2-y^2-4)(2x)}{-8xy}\right]_{3.182, 2.474}$$

$$= \frac{1.56427 - 0.028307}{-62.978} = -0.0244$$

$$\boxed{k_1 = -0.0244}$$

$x_2 = x_1 + h_1$

$\Rightarrow x_2 = 3.182 - 0.01966$

$$\boxed{x_2 = 3.16234}$$

$y_2 = y_1 + k_1$

$y_2 = 2.474 - 0.0244$

$$\boxed{y_2 = 2.4496}$$

Step 4 :

$$J_2 = [-8xy]_{3.16234,\ 2.4496} = -61.972$$

$$\boxed{J_2 = -61.972}$$

$$h_2 = \left[\frac{g f_y - f g_y}{J}\right]_{3.16234,\ 2.4496}$$

$$h_2 = \left[\frac{(x^2 - y^2 - 4)(2y) - (x^2 + y^2 - 16)(-2y)}{-8xy}\right]_{3.16234,\ 2.4496}$$

$$h_2 = \frac{-7.1472 \times 10^{-4} - 4.577986 \times 10^{-8}}{-61.972}$$

$$\boxed{h_2 = 8.54 \times 10^{-5}}$$

$$k_2 = \left[\frac{f g_x - g f_x}{J}\right]_{3.16234,\ 2.4496}$$

$$k_2 = \left[\frac{(x^2 + y^2 - 16)(2x) - (x^2 - y^2 - 4)(2x)}{-8xy}\right]_{3.16234,\ 2.4496}$$

$$= \frac{5.91 \times 10^{-8} + 9.227 \times 10^{-4}}{-61.972}$$

$$\boxed{k_2 = -1.1 \times 10^{-4}}$$

$$\Rightarrow x_3 = x_2 + h_2$$
$$= 3.16254 + 8.54 \times 10^{-5}$$
$$\boxed{x_3 = 3.16234}$$

$$y_3 = y_2 + k_2$$
$$y_3 = 2.4496 - 1.1 \times 10^{-4}$$
$$\boxed{y_3 = 2.4496}$$

SOLUTION :

S.NO	x	y
1.	$2\sqrt{2}$	$2\sqrt{2}$
2.	3.182	2.474
3.	3.16234	2.449
4.	3.16234	2.449

$\therefore$ $\boxed{x = 3.16234}$ $\boxed{y = 2.449}$

(2) Solve by Newton Raphson method

$$x^2 + y^2 - 11 = 0 \quad \text{—— (1)}$$
$$y^2 + x - 7 = 0 \quad \text{—— (2)}$$
$$x_0 = 3.5 \; ; \; y_0 = -1.5$$

SOLUTION :

Step 1: $f(x) = 2x$ | $g_x = 1$

$f_y = 2y$ | $g_y = 2y$

$$J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} 2x & \frac{1}{2y} \\ 1 & 2y \end{vmatrix} = 4xy - \cancel{2y} = 4xy - 1$$

$$J = 2y\cancel{(2x-1)}(4xy-1)$$

$$\underline{J = 4xy - 1}$$

step 2: $J_0 = 4 \times 3.5 \times (-1.5) - 1$

$$\boxed{J_0 = -22}$$

$$k_0 = \left(\frac{f g_x - g f_x}{J}\right)_{3.5,\,-1.5}$$

$$= \left[\frac{(x^2+y-11)(1) - (x+y^2-7)(2x)}{(4xy-1)}\right]_{3.5,\,-1.5}$$

$$= \frac{-0.25 + 8.75}{-22} = \frac{8.50}{-22} = -0.3864$$

$k_0 = \cancel{0.4091}$ $\boxed{k_0 = -0.3864}$

$$h_0 = \left(\frac{g f_y - f g_y}{J}\right)_{3.5,\,-1.5}$$

$$= \left[\frac{(x+y^2-7)(1) - (x^2+y-11)(2y)}{(4xy-1)}\right]_{3.5,\,-1.5}$$

$$h_0 = \frac{-1.25 + (0.75)}{-22} = 0.0909$$

$$\boxed{h_0 = 0.0909}$$

$x_1 = x_0 + h_0$

$x_1 = 3.5 + 0.0909$

$\boxed{x_1 = 3.5909}$

$y_1 = y_0 + k_0$

$y_1 = -1.5 + \cancel{0.4091}\ 0.3864$

$\boxed{y_1 = -1.886}$

<u>step 3:</u> $J_1 = 4 \times 3.5909 \times (-1.886) - 1 = \underline{-28.1}$

$$h_1 = \left(\frac{gf_y - fg_y}{J}\right)_{3.5909,\,-1.886}$$

$$h_1 = \left[\frac{(x+y^2-7)(1) - (x^2+y-11)(2y)}{4xy-1}\right]_{3.5909,\,-1.886}$$

$$h_1 = \frac{0.147896 + 0.03229}{-28.1} = \underline{-6.41 \times 10^{-8}}$$

$$k_1 = \left(\frac{fg_x - gf_x}{J}\right)_{3.5909,\,-1.886}$$

$$k_1 = \left[\frac{(x^2+y-11)(1) - (x+y^2-7)(2x)}{4xy-1}\right]_{3.5909,\,-1.886}$$

$$k_1 = \frac{8.5628 \times 10^{-3} - 1.062159}{-28.1}$$

$$\underline{k_1 = 0.03749}$$

$$x_2 = x_1 + h_1$$
$$= 3.5909 - 6.41 \times 10^{-3}$$
$$\boxed{x_2 = 3.5845}$$

$$y_2 = y_1 + k_1$$
$$= -1.886 + 0.03749$$
$$\boxed{y_2 = -1.8485}$$

<u>Step 4:</u> $J_2 = 4 \times (3.5845) \times (-1.8485) - 1 = -27.504$

$$\underline{J_2 = -27.504}$$

$$h_2 = \left(\frac{gf_y - fg_y}{J}\right)_{3.5845,\,-1.8485}$$

$$= \left[\frac{(y^2+x-7)(1) - (x^2+y-11)(2y)}{(4xy-1)}\right]_{3.5845,\,-1.8485} = \frac{1.45225 \times 10^{-3} + 5.18 \times 10^{-4}}{-27.504}$$

$$\underline{h_2 = -3.4 \times 10^{-5}}$$

$$k_2 = \left[\frac{fg_x - gf_x}{J}\right]_{3.5845,\,-1.8485}$$

$$= \left[\frac{(x^2+y-11)(1) - (x+y^2-7)(2x)}{4xy-1}\right]_{3.5845,\,-1.8485}$$

$$= \frac{1.4025 \times 10^{-4} - 0.0104}{-27.504}.$$

$$\underline{k_2 = 3.73 \times 10^{-4}}$$

$\Rightarrow x_3 = x_2 + h_2$

$x_3 = 3.5845 - 3.4 \times 10^{-5}$

$x_3 = 3.584$

$y_3 = y_2 + k_2$

$y_3 = -1.8485 + 3.73 \times 10^{-4}$

$y_3 = -1.848$

Solution:

S.No	x	y
1.	3.5	-1.5
2.	3.5909	-1.886
3.	3.584	-1.848
4.	3.584	-1.848

$\therefore$ $\boxed{x = 3.584}$ $\boxed{y = -1.848}$.

METHODS TO SOLVE Solving Simultaneous Linear Algebraic eqn

EQUATIONS

A, DIRECT METHODS:

* GAUSS ELIMINATION METHOD
* GAUSS JORDAN METHOD

B, ITERATIVE METHODS:

* GAUSS - JACOBI METHOD
* GAUSS - SEIDAL METHOD

Gauss Elimination Method:

All the methods of solving a system of simultaneous linear equations are known as direct methods which involve a certain amount of fixed computation.

PROBLEMS:

① Solve the system of linear equations

$$2x + y + 4z = 12 \quad \text{------} \to (1)$$
$$8x - 3y + 2z = 20 \quad \text{------} \to (2)$$
$$4x + 11y - z = 33 \quad \text{------} \to (3)$$

Solution:

Step 1: Augmented matrix of the given system is

$$\begin{array}{cccccc} & & & & & \\ & \left[\begin{array}{cccc} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & +11 & -1 & 33 \end{array}\right] & & & & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \\ & C_1 \quad C_2 \quad C_3 \quad C_4 & & & & \end{array}$$

STEP 2 :

Eliminating x from (2) and (3) equation.

$$M_1 = -\frac{a_{21}}{a_{11}} = -\frac{8}{2} = -4 \; ; \quad M_2 = -\frac{a_{31}}{a_{11}} = -\frac{4}{2} = -2$$

$$M_1 = -4 \quad ; \quad M_2 = -2$$

$$\text{Equn } (2) = M_1 \times (1) + (2)$$

$$(3) = M_2 \times (1) + (3)$$

STEP 3 :

$$\begin{array}{c} \\ -4 \\ -2 \end{array}\begin{bmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & +2 & 20 \\ 4 & 11 & -1 & 33 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{bmatrix}$$

STEP 4 :

Eliminating y from (3) equation,

$$M_3 = -\frac{a_{32}}{a_{22}} = \frac{-9}{-7} = \frac{9}{7} \Rightarrow M_3 = 9/7$$

$$M_3 \times (2) + (3) = (3)$$

$$\frac{9}{7}\begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -27 & -27 \end{bmatrix}$$

STEP 5 :

Applying backward substitution to find the values of x, y and z,

$$\left[\begin{array}{cccc} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -27 & -27 \end{array}\right] \Rightarrow \begin{array}{l} -27z = -27 \longrightarrow (a) \\ -7y - 14z = -28 \longrightarrow (b) \\ 2x + y + 4z = 12 \longrightarrow (c) \end{array}$$

$M_1 \times 1 + ②$

$M_2 \times 1 + ③$ -

$M_3 \times 2 + 3 \Rightarrow 3$

$\Rightarrow \boxed{z = 1}$

(b) $\Rightarrow$ $-7y - 14 = -28$

$-7y = -14$

$\boxed{y = 2}$

(c) $\Rightarrow$ $2x + 2 + 4 = 12$

$2x = 6$

$\boxed{x = 3}$

SOLUTION :

$$\boxed{\begin{array}{l} x = 3 \\ y = 2 \\ z = 1 \end{array}}$$

---

(2) Solve the system of linear equations

$$x + 2y + z = 5 \longrightarrow (1)$$

a_{11} a_{12} a_{13}

$$2x + 3y + 3z = 10 \longrightarrow (2)$$

a_{21}

$$3x - y + 2z = 13 \longrightarrow (3)$$

STEP 1 :

Augmented matrix is given by

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{bmatrix} \begin{matrix} \longrightarrow (1) \\ \longrightarrow (2) \\ \longrightarrow (3) \end{matrix}$$

STEP 2 : To eliminate x from equation (2) and (3),

$$M_1 = -\frac{a_{21}}{a_{11}} = \underline{-2}$$

$$M_2 = -\frac{a_{31}}{a_{11}} = \underline{-3}$$

$$M_1 \times (1) + (2) = (2) \quad ; \quad M_2 \times (1) + (3) = (3)$$

STEP 3 :

$$\begin{array}{c} \\ -2 \\ -3 \end{array} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{bmatrix} \begin{matrix} \longrightarrow (1) \\ \longrightarrow (2) \\ \longrightarrow (3) \end{matrix}$$

STEP 4 : To eliminate y from equation (3)

$$M_3 = -\frac{a_{32}}{a_{22}} = \frac{7}{-1} = \underline{-7}$$

$$M_3 \times (2) + (3) = (3)$$

$$\Rightarrow \begin{array}{c} \\ -7 \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{bmatrix}$$

STEP 5 : By applying backward substitution to get the values of x, y, z,

$-8z = -24$	$-y + 3 = 4$	$x + 2y - 3 = 3$
$\boxed{z = 3}$	$\boxed{y = -1}$	$\boxed{x = 2}$

$x + 2y + z = 3$
$x + 2(-1)$

M.D.K

SOLUTION:

$$\boxed{\begin{array}{l} x = 2 \\ y = -1 \\ z = 3 \end{array}}$$

GAUSS JORDAN METHOD:

(1) Solve $x + y + 3z = 6 \longrightarrow (1)$

$x + 3y + z = 8 \longrightarrow (2)$

$2x + y + z = 5 \longrightarrow (3)$

SOLUTION:

STEP 1: Augmented matrix is given by $\begin{bmatrix} 1 & 1 & 3 & 6 \\ 1 & 3 & 1 & 8 \\ 2 & 1 & 1 & 5 \end{bmatrix}$

STEP 2: To eliminate co-eff of x and y from (2) and (3)

$$M_1 = -\frac{a_{21}}{a_{11}} = -1 \; ; \quad M_2 = -\frac{a_{31}}{a_{11}} = -2$$

$$M_1 \times (1) + (2) = (2) \; ; \quad M_2 \times (1) + (3) = (3)$$

$$\Rightarrow \begin{array}{c} \\ -1 \\ -2 \end{array} \begin{bmatrix} 1 & 1 & 3 & 6 \\ 1 & 3 & 1 & 8 \\ 2 & 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 2 & -2 & 2 \\ 0 & -1 & -5 & -7 \end{bmatrix}$$

STEP 3: To eliminate co-eff of y from (3)

$$M_3 = -\frac{a_{32}}{a_{22}} = \frac{+1}{2} = 1/2$$

$$M_3 \times (2) + (3) = (3)$$

$$\Rightarrow \begin{array}{c} \\ \\ \frac{1}{2} \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 2 & -2 & 2 \\ 0 & -1 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -6 & -6 \end{bmatrix}$$

STEP 4 : To eliminate the co-eff of y from (1)

$$M_4 = -\frac{a_{12}}{a_{22}} = -\frac{1}{2}$$

$M_4 \times (2) + (1) = (1)$

$$\Rightarrow -\frac{1}{2}\begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -6 & -6 \end{bmatrix}$$

STEP 5 : By backward substitution, to get the value of x, y & z

$-6z = -6$ $\boxed{z = 1}$ | $2y - 2 = 2$ $\boxed{y = 2}$ | $x + 4 = 5$ $\boxed{x = 1}$

SOLUTION : $\boxed{\begin{matrix} x = 1 \\ y = 2 \\ z = 1 \end{matrix}}$

---

(4) Solve the system of linear equations

$$p + q + r + (-s) = 2 \longrightarrow (1)$$
$$7p + q + 3r + s = 12 \longrightarrow (2)$$
$$8p - q + r - 3s = 5 \longrightarrow (3)$$
$$10p + 5q + 3r + 2s = 20 \longrightarrow (4)$$

SOLUTION :

Step 1 : Augmented matrix is given by $\begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 7 & 1 & 3 & 1 & 12 \\ 8 & -1 & 1 & -3 & 5 \\ 10 & 5 & 3 & 2 & 20 \end{bmatrix}$

Step 2 : To eliminate the co-efficient of p from ②, ③, ④,

$$M_1 = -\frac{a_{21}}{a_{11}} = \underline{-7}$$

$$M_2 = -\frac{a_{31}}{a_{11}} = \underline{-8}$$

$$M_3 = -\frac{a_{41}}{a_{11}} = \underline{-10}$$

$M_1 \times ① + ② = ②'$; $M_2 \times ① + ③ = ③'$; $M_3 \times ① + ④ = ④'$

$$\begin{array}{c} \\ -7 \\ -8 \\ -10 \end{array}\begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 7 & 1 & 3 & 1 & 12 \\ 8 & -1 & 1 & -3 & 5 \\ 10 & 5 & 3 & 2 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & -6 & -4 & +8 & -2 \\ 0 & -9 & -7 & +5 & -11 \\ 0 & -5 & -7 & +2 & 0 \end{bmatrix}$$

Step 3 : To eliminate the co-efficient of q from ③ and ④

$$M_4 = -\frac{a_{42}}{a_{22}} = \frac{5}{-6} = \underline{\frac{-5}{6}} \; ; \; M_5 = -\frac{a_{32}}{a_{22}} = 9/-6 = \underline{-9/6}$$

$M_4 \times ② + ④ = ④'$; $M_5 \times ② + ③ = ③'$

$$\Rightarrow \begin{array}{c} \\ \\ -\frac{5}{6} \\ -\frac{9}{6} \end{array}\begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & -6 & -4 & 8 & -2 \\ 0 & 0 & -1 & -7 & -8 \\ 0 & 0 & -11/3 & 16/3 & 5/3 \end{bmatrix}$$

Step 4 : To eliminate the co-eff of r from ④, $M_6 = -\frac{a_{43}}{a_{33}} = \underline{-11/3}$

$M_6 \times ③ + ④ = ④'$

$$\Rightarrow -11/3 \begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & -6 & -4 & 8 & -2 \\ 0 & 0 & -1 & -7 & -8 \\ 0 & 0 & -11/3 & 16/3 & 5/3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & -6 & -4 & 8 & -2 \\ 0 & 0 & -1 & -7 & -8 \\ 0 & 0 & 0 & 31 & 31 \end{bmatrix}$$

Step 5 : By backward substitution, to get the values of p, q, r, s

$3 \mid s = 31$ $\boxed{s = 1}$ | $-r - 7 = -8$ $\boxed{r = 1}$ | $-6q - 4 + 8 = -2$ $\boxed{q = 1}$

$p + 1 + 1 - 1 = 2$

$\boxed{p = 1}$

SOLUTION :

$$\boxed{\begin{array}{l} p = 1 \\ q = 1 \\ r = 1 \\ s = 1 \end{array}}$$

To Find The Inverse Of A Matrix By Gauss Jordan Method

(1) Solve $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

SOLUTION : let $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Step 1 : $A = \begin{pmatrix} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{pmatrix}$

Reducing this to Gauss - Jordan form

$M_1 = -\frac{a_{21}}{a_{11}} = -1$; $M_2 = \frac{-a_{31}}{a_{11}} = -1$

$M_1 \times ① + ② = ②$; $M_2 \times ① + ③ = ③$

$$\Rightarrow \begin{array}{c} \\ -1 \\ -1 \end{array} \left(\begin{array}{cccccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cccccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

<u>Step 2 :</u> Eliminating the co-efficient of y in ①,

$$M_3 = -\frac{a_{12}}{a_{22}} = \underline{-3}$$

$M_3 \times ② + ① = ①$

$$\Rightarrow \begin{array}{c} -3 \\ \\ \\ \end{array} \left(\begin{array}{cccccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) = \left(\begin{array}{cccccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

<u>Step 3 :</u> Eliminating the co-eff of constant term in ③,

$$M_4 = -\frac{a_{13}}{a_{33}} = \underline{-3}$$

$M_4 \times ③ + ① = Ⓘ$

$$\Rightarrow \begin{array}{c} -3 \\ \\ \\ \end{array} \left(\begin{array}{cccccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \quad A^{-1}$$

<u>Step 4 :</u> Inverse of the given matrix is

$$A^{-1} = \left(\begin{array}{ccc} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right)$$

(2). Find the inverse of $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{pmatrix}$

Solution: $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & -3 \end{pmatrix}$

Step 1: $A = \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \rightarrow (1) \\ \rightarrow (2) \\ \rightarrow (3) \end{array}$

Step 2: To eliminate a_{21} and a_{31},

$$M_1 = -\frac{a_{21}}{a_{11}} = -2 \; ; \quad M_2 = -\frac{a_{31}}{a_{11}} = -4$$

$$M_1 \times (1) + (2) = (2') \; ; \quad M_2 \times (1) + (3) = (3')$$

$$\Rightarrow \begin{array}{c} \\ -2 \\ -4 \end{array} \begin{pmatrix} 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 & 1 & \\ 0 & 3 & -1 & 1 & -2 & \\ 0 & 6 & -7 & 0 & -4 & \end{pmatrix}$$

Step 3: To eliminate a_{32} and a_{12}, $M_3 = -\frac{a_{32}}{a_{22}} = -\frac{6}{3} = -2$

$$M_3 \times (2') + (3') = (3'')$$

$$\Rightarrow \begin{array}{c} \\ \\ -2 \end{array} \begin{pmatrix} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 6 & -7 & 0 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{pmatrix}$$

$$M_4 = -\frac{a_{12}}{a_{22}} = \frac{1}{3} \; ; \quad M_4 \times (2') + (1) = (1')$$

$$\frac{1}{3}\begin{pmatrix} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2/3 & 1/3 & 1/3 & 0 \\ 0 & 3 & -1/1 & 1 & -2 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{pmatrix}$$

Step 4 : To eliminate a_{13}, $M_5 = -\frac{2/3}{-5} = \frac{2}{15} = \frac{2}{15}$

$M_5 \times (3) + (1) = (1)$

$$\frac{2}{15}\begin{pmatrix} 1 & 0 & 2/3 & 1/3 & 1/3 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1/15 & 1/3 & 2/15 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{pmatrix}$$

Step 5 : To eliminate a_{22} and a_{33}, multiply (2) by $\frac{1}{3}$ & (3) by $-\frac{1}{5}$

$(2) \times \frac{1}{3}$ & $(3) \times -\frac{1}{5}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1/15 & 1/3 & 2/15 \\ 0 & 1 & 0 & 7/15 & -2/3 & -1/15 \\ 0 & 0 & 1 & 2/5 & 0 & -1/5 \end{pmatrix}$$

Step 6 : The inverse of the given matrix is

$$\begin{pmatrix} 1/15 & 1/3 & 2/15 \\ 7/15 & -2/3 & -1/15 \\ 2/5 & 0 & -1/15 \end{pmatrix}$$

---

(3) Find the inverse of $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

Solution : $A = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$

Step 1 : $A = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \dashrightarrow (1) \\ \dashrightarrow (2) \\ \dashrightarrow (3) \end{array}$

Step 2 : To eliminate a_{31}, $M_1 = -\frac{a_{31}}{a_{11}} = -3$

$M_1 \times (1) + (3) = (3)$

$$\Rightarrow -3 \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{pmatrix}$$

Step 3 : To eliminate a_{32}, $M_2 = -\frac{a_{32}}{a_{22}} = \frac{5}{2}$

$M_2 \times (2) + (3) = (3)$

$$\Rightarrow \frac{5}{2} \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{pmatrix}$$

Step 4 : To eliminate a_{12}, $M_3 = -\frac{a_{12}}{a_{22}} = -2$

$M_3 \times (2) + (1) = (1)$

$$\Rightarrow -2 \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{pmatrix}$$

Step 5 : To eliminate a_{13}, $M_4 = -\frac{a_{13}}{a_{33}} = \frac{1}{2}$

$M_4 \times (3) + (1) = (1)$

$$\Rightarrow \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{pmatrix}$$

Step 6 : To eliminate, a_{23}, $M_5 = -\frac{a_{23}}{a_{33}} = -\frac{2}{2} = -1$

$M_5 \times (3) + (2) = (2)$

$$\Rightarrow -1 \begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{pmatrix}$$

Step 7 : To make $a_{33} = 1$, Multiply equ (3) by $1/2$

$$\Rightarrow A^{-1} \begin{pmatrix} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{pmatrix}$$

Solution : Inverse of given matrix is $\begin{pmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{pmatrix}$

ITERATIVE METHODS :

* ⋆ GAUSS JACOBI METHOD [USING PREVIOUS SET OF VALUES]
* ⋆ GAUSS SEIDAL METHOD [USING CURRENT VALUES]

CONDITION : The system should be diagnolly dominant.

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(1) Solve the system by Gauss Jacobi & Gauss Seidal method

$10x - 5y - 2z = 3$ ------- (1)

$4x - 10y + 3z = -3$ ------- (2)

$x + 6y + 10z = -3$ ------- (3)

Step 1: $|10| > |-5| + |-2|$

$|10| > |4| + |3|$

$|10| > |1| + |6|$

The condition is checked and the system is diagnolly dominant.

Step 2: From (1), $x = \frac{1}{10}(3 + 5y + 2z)$

From (2), $y = \frac{1}{10}(4x + 3z + 3)$

From (3), $z = \frac{1}{10}(-x - 6y - 3)$

GAUSS JACOBI METHOD

Step 3:

I	x	y	z
0	0	0	0
1	0.3	0.3	-0.3
2	0.39	0.33	-0.51
3	0.363	0.303	-0.537
4	0.3441	0.2841	-0.5181
5	0.33843	0.28221	-0.50487
6	0.340131	0.283911	-0.503169
7	0.3413217	0.2851017	-0.5043597
8	0.34167891	0.28522077	-0.5051939
9	0.3416	0.285	-0.505

Solution: $x = 0.341$

$y = 0.285$

$z = -0.505$

GAUSS - SEIDAL METHOD :

using current values,

I	x	y	z
0	0	0	0
1	0.3	0.42	-0.582
2	0.3936	0.28284	-0.5091
3	0.3396072	0.28813	-0.5038
4	0.340797	0.2852	-0.5052
5	0.3415476	0.2851	-0.5051
6	0.3415	0.2851	-0.5051

solution: $\boxed{x = 0.341 \quad y = 0.285 \quad z = -0.505}$

(2) Solve $8x - 3y + 2z = 20$

$4x + 11y - z = 33$

$6x + 3y + 12z = 35$

Solution : $8x - 3y + 2z = 20 \longrightarrow (1)$

$4x + 11y - z = 33 \longrightarrow (2)$

Step 1: $6x + 3y + 12z = 35 \longrightarrow (3)$

From (1), $x = \frac{1}{8}(20 + 3y - 2z)$

From (2), $y = \frac{1}{11}(33 - 4x + z)$

From (3), $z = \frac{1}{12}(35 - 6x - 3y)$

Step 2 : Condition is checked

$|8| > |-3| + |2|$

$|11| > |4| + |-1|$

$|12| > |6| + |3|$

$\therefore$ The gn. system is diagnolly dominant.

Step 3 : GAUSS - JACOBI METHOD :

Using previous set of values,

ITERATION	x	y	z
0	0	0	0
1	2.5	3	2.9167
2	2.895825	2.35606	0.9167
3	3.1543475	2.030309	0.879739
4	3.04148	1.9340317	0.8319156
5	3.01728	1.969654	0.9124487
6	3.0105093	1.985767	0.9156
7	3.015759	1.98845	0.91497
8	3.016926	1.986539	0.91167466
9	3.017033	1.985815	0.911568916
10	3.017	1.98576	0.911696

Step 4: GAUSS SEIDAL METHOD

Using current values,

ITERATION	x	y	z
0	0	0	0
1	2.5	2.4909	~~0.582~~ 1.143939
2	2.998106	2.01377	0.914170
3	3.02662	1.982516	0.907726
4	3.01652	1.985607	0.91200898
5	3.016623	1.985956	0.9118
6	3.016	1.98594	0.912

Solution: $x = 3.016$

$y = 1.985$

$z = 0.912$

To Find Eigen Values And Eigen Vectors By Power Method:

Procedure:

* This is a iterative method to find the numerically largest eigen value of the matrix.
* In this method, we start with an arbitrary value say X_1 such that $AX_1 = \lambda_1 X_2$ where λ_1 - eigen value

  X_2 - approximate eigen vector.
* Similarly we have to find $AX_2 = \lambda_2 X_3$
* This process is continued to get the accurate eigen vector.

PROBLEMS:

① Find the largest eigen value of the matrix $\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$ by power method and compare the characteristic equation.

SOLUTION:

Step 1: Let us consider $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is non-zero eigen vector

$$AX_1 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \begin{matrix} 4+0 \\ 4+0 \end{matrix} \quad \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

4

$$= 4\begin{pmatrix} 1 \\ 1/4 \end{pmatrix} = \lambda_1 X_2$$

Step 2: $AX_2 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 4.25 \\ 1.75 \end{pmatrix} = 4.25\begin{pmatrix} 1 \\ 7/17 \end{pmatrix} = \lambda_2 X_3$

$$AX_3 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 7/17 \end{pmatrix} = \begin{pmatrix} 4.41176 \\ 2.235 \end{pmatrix} = 4.41176\begin{pmatrix} 1 \\ 0.50666 \end{pmatrix} = \lambda_3 X_4$$

$$AX_4 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.50666 \end{pmatrix} = \begin{pmatrix} 4.50666 \\ 2.51998 \end{pmatrix} = 4.50666\begin{pmatrix} 1 \\ 0.5592 \end{pmatrix} = \lambda_4 X_5$$

$$AX_5 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5592 \end{pmatrix} = \begin{pmatrix} 4.5592 \\ 2.6775001 \end{pmatrix} = 4.5592\begin{pmatrix} 1 \\ 0.587278 \end{pmatrix} = \lambda_5 X_6$$

$$AX_6 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.587278 \end{pmatrix} = \begin{pmatrix} 4.587278 \\ 2.761835 \end{pmatrix} = 4.587278\begin{pmatrix} 1 \\ 0.60206 \end{pmatrix} = \lambda_6 X_7$$

$$AX_7 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.60206 \end{pmatrix} = \begin{pmatrix} 4.60206 \\ 2.806192 \end{pmatrix} = 4.60206\begin{pmatrix} 1 \\ 0.609768 \end{pmatrix} = \lambda_7 X_8$$

$$AX_8 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.609768 \end{pmatrix} = \begin{pmatrix} 4.609768 \\ 2.82950 \end{pmatrix} = 4.609768\begin{pmatrix} 1 \\ 0.60376 \end{pmatrix}$$

$$AX_9 = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.61376 \end{pmatrix} = \begin{pmatrix} 4.61376 \\ 2.841288 \end{pmatrix} = 4.61376\begin{pmatrix} 1 \\ 0.615829 \end{pmatrix}$$

$$AX_{10} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.615829 \end{pmatrix} = \begin{pmatrix} 4.615829 \\ 2.847487 \end{pmatrix} = 4.615829\begin{pmatrix} 1 \\ 0.616896 \end{pmatrix}$$

$$AX_{11} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.616896 \end{pmatrix} = \begin{pmatrix} 4.616896 \\ 2.850688 \end{pmatrix} = 4.616896\begin{pmatrix} 1 \\ 0.617446 \end{pmatrix}$$

$$AX_{12} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.617446 \end{pmatrix} = \begin{pmatrix} 4.617446 \\ 2.8523 \end{pmatrix} = 4.617\begin{pmatrix} 1 \\ 0.617 \end{pmatrix}$$

$$\therefore \lambda = 4.617 \; ; \; X = \begin{pmatrix} 1 \\ 0.617 \end{pmatrix}$$

Step 3 : Characteristic equation is given by

$$\begin{vmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (4-\lambda)(3-\lambda) - 1 = 0$$

$$12 - 3\lambda - 4\lambda + \lambda^2 - 1 = 0.$$

$$\lambda^2 - 7\lambda + 11 = 0$$

$$\lambda = \frac{7 \pm \sqrt{7^2 - 4(11)}}{2} = \frac{7 \pm \sqrt{49-44}}{2}$$

$$\lambda = \frac{7 \pm \sqrt{5}}{2}$$

$$\boxed{\lambda = 4.618}$$

... of the matrix $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

SOLUTION :

Step 1 : $AX_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$AX_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix}$

$AX_3 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4286 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.57 \\ 1.8572 \\ 0 \end{pmatrix} = 3.57\begin{pmatrix} 1 \\ 0.5202 \\ 0 \end{pmatrix}$

Step 2 :

$AX_4 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.52029 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.1213 \\ 2.04044 \\ 0 \end{pmatrix} = 4.1213\begin{pmatrix} 1 \\ 0.495096 \\ 0 \end{pmatrix}$

$AX_5 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.495096 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.97057 \\ 1.9902 \\ 0 \end{pmatrix} = 3.97057\begin{pmatrix} 1 \\ 0.50123 \\ 0 \end{pmatrix}$

$AX_6 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.50123 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.0074 \\ 2.00246 \\ 0 \end{pmatrix} = 4.0074\begin{pmatrix} 1 \\ 0.4996 \\ 0 \end{pmatrix}$

$AX_7 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4996 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.998 \\ 1.9992 \\ 0 \end{pmatrix} = 3.998\begin{pmatrix} 1 \\ 0.50005 \\ 0 \end{pmatrix}$

$AX_8 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.50005 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.00 \\ 2.00 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

Step 3 :

$$\boxed{\lambda = 4 \quad X = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}}$$

Characteristic equation is given by

$$\begin{vmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) - 6(3-\lambda) + 1(0) = 0$$

$$(2-2\lambda-\lambda+\lambda^2)(3-\lambda) - 6(3-\lambda) = 0$$

$$(3-\lambda)\left[\lambda^2 - 3\lambda - 4\right] = 0$$

$$(3-\lambda)(\lambda-4)(\lambda+1) = 0$$

$\lambda = 4$.

Solution: $\lambda = 4$; $X = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

(3) Find the eigen value and eigen vector of $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

Solution:

Step 1: $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$; $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$AX_1 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix}$$

$$AX_2 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 5.2 \\ 0 \\ 2 \end{pmatrix} = 5.2\begin{pmatrix} 1 \\ 0 \\ 0.3846 \end{pmatrix}$$

$$AX_3 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.3846 \end{pmatrix} = \begin{pmatrix} 5.3846 \\ 0 \\ 2.9231 \end{pmatrix} = 5.3846\begin{pmatrix} 1 \\ 0 \\ 0.4985 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.6312 \end{pmatrix} = \begin{pmatrix} 5.6312 \\ 0 \\ 4.156 \end{pmatrix} = 5.6312\begin{pmatrix} 1 \\ 0 \\ 0.73803 \end{pmatrix}$$

$$AX_6 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.73803 \end{pmatrix} = \begin{pmatrix} 5.73803 \\ 0 \\ 4.690 \end{pmatrix} = 5.73803\begin{pmatrix} 1 \\ 0 \\ 0.8174 \end{pmatrix}$$

$$AX_7 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.8174 \end{pmatrix} = \begin{pmatrix} 5.8174 \\ 0 \\ 5.0869 \end{pmatrix} = 5.8174\begin{pmatrix} 1 \\ 0 \\ 0.8744 \end{pmatrix}$$

$$AX_8 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.8744 \end{pmatrix} = \begin{pmatrix} 5.8744 \\ 0 \\ 5.372 \end{pmatrix} = 5.8744\begin{pmatrix} 1 \\ 0 \\ 0.9145 \end{pmatrix}$$

$$AX_9 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.9145 \end{pmatrix} = \begin{pmatrix} 5.9145 \\ 0 \\ 5.5725 \end{pmatrix} = 5.9145\begin{pmatrix} 1 \\ 0 \\ 0.9422 \end{pmatrix}$$

$$AX_{10} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.9422 \end{pmatrix} = \begin{pmatrix} 5.9422 \\ 0 \\ 5.71088 \end{pmatrix} = 5.9422\begin{pmatrix} 1 \\ 0 \\ 0.96107 \end{pmatrix}$$

$$AX_{11} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.96107 \end{pmatrix} = \begin{pmatrix} 5.96107 \\ 0 \\ 5.8054 \end{pmatrix} = 5.96107\begin{pmatrix} 1 \\ 0 \\ 0.974 \end{pmatrix}$$

$$AX_{12} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.974 \end{pmatrix} = \begin{pmatrix} 5.974 \\ 0 \\ 5.8694 \end{pmatrix} = 5.974\begin{pmatrix} 1 \\ 0 \\ 0.982 \end{pmatrix}$$

$$AX_{13} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.982 \end{pmatrix} = \begin{pmatrix} 5.982 \\ 0 \\ 5.912 \end{pmatrix} \approx 5.982\begin{pmatrix} 1 \\ 0 \\ 0.988 \end{pmatrix}$$

$$AX_{14} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.988 \end{pmatrix} = \begin{pmatrix} 5.988 \\ 0 \\ 5.942 \end{pmatrix} = 5.99 \begin{pmatrix} 1 \\ 0 \\ 0.99 \end{pmatrix}$$

$$AX_{15} = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.99 \end{pmatrix} = \begin{pmatrix} 5.99 \\ 0 \\ 5.96 \end{pmatrix} = 5.99 \begin{pmatrix} 1 \\ 0 \\ 0.99 \end{pmatrix}$$

$$\underline{\lambda = 5.99}; \quad X = \begin{pmatrix} 1 \\ 0 \\ 0.99 \end{pmatrix}$$

<u>Step 8:</u> characteristic equation is given by

$$\begin{vmatrix} 5-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-2-\lambda)(5-\lambda) + 1\left(-(-2-\lambda)\right) = 0$$

$$(5-\lambda)(-2-\lambda)(5-\lambda) - (-2-\lambda) = 0$$

$$(-2-\lambda)\left[(5-\lambda)^2 - 1\right] = 0$$

$$(-2-\lambda)\left[\lambda^2 - 10\lambda + 24\right] = 0$$

$$(-2-\lambda)(\lambda-6)(\lambda-4) = 0$$

$$\underline{\lambda = 6}$$

<u>Solution:</u> $\lambda = 6$; $X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

To Find Eigen Values and Eigen Vector by Gauss Jacobi Method:

Problems:

(1) Find the eigen values and eigen vector of $\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$

Solution:

Step 1: The largest off diagnol elements are $\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$

$$a_{ik} = a_{ki} = a_{13} = a_{31} = 2.$$

Step 2: The other elements $a_{ii}, a_{ik}, a_{ki}, a_{kk}$ form the submatrix. Hence $a_{13}, a_{31}, a_{11}, a_{33} = 1$ Replace the submatrix $\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$ by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Here $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{ik}}{a_{ii}-a_{kk}}\right)$ & $a_{ii} \neq a_{kk}$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11}-a_{33}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\times 2}{1-1}\right) = \frac{1}{2}\tan^{-1}(\infty)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

$$\left\{ \begin{array}{l} \theta = \frac{\pi}{4} \text{ for } a_{ik} > 0 \\ \theta = -\frac{\pi}{4} \text{ for } a_{ik} < 0. \end{array} \right.$$

Here $\theta = \pi/4$ since $a_{ik} = a_{13} = 2 > 0$.

Step 3: Fill this matrix $\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$ by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

and remaining places with zero except on the diagnol where it is 1

Remaining elets are similar to unit matrix

$$R_1 = \begin{pmatrix} \cos \pi/4 & 0 & -\sin \pi/4 \\ 0 & 1 & 0 \\ \sin \pi/4 & 0 & \cos \pi/4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

Step 4: First Iteration is $\underline{B_1 = S_1^{-1} A S_1}$ or $S_1^T A S_1$

$$R_1^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & +\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$D_1 = R_1^T A R_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 1/\sqrt{2} + 2/\sqrt{2} & 1+1 & 2/\sqrt{2} + 1/\sqrt{2} \\ \sqrt{2} & 3 & \sqrt{2} \\ -1/\sqrt{2} + 2/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3/\sqrt{2} & 2 & 3/\sqrt{2} \\ \sqrt{2} & 3 & \sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Step 5 : $\begin{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & -1 \end{pmatrix}$

$a_{12} = a_{21} = 2 > 0 \quad \therefore \theta = \pi/4$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is replaced by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Step 6 : $B_2 = R_2^T A R_2$

$$B_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where $5, 1, -1$ are eigen values

Step 6 : Eigen vector is given by $S_1 S_2$.

$$\text{Eigen vector} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{pmatrix}$$

(2) Find the eigen value and eigen vector of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Solution :

Step 1: $a_{12} = a_{21} = 2$

$\therefore \theta = \pi/4$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Step 2 : $B_1 = S_1^T A S_1$

$$B_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 3/\sqrt{2} & 3/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Solution : Eigen values are 3, -1

Eigen vector is $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$