One Dimensional Heat Equation

1. A rod of length *l* has its ends *A* and *B* are kept at $0^\circ C$ and $100^\circ C$ until steady state condition prevail. If the temperature at *B* is reduced suddenly to $0^\circ C$ and kept so while that of *A* is maintained. Find the temperature $u(x,t)$ at a distance x from *A* and at time *t*.

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$

<u>To find steady state solution</u> $u(x,0) = u(x)$

In steady state *t=0* then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$ $----(1)$

The boundary conditions are i) $u(0) = 0^\circ C$ ii) $u(l) = 100^\circ C$

Applying condition (i) in (1)

$$(1) \Rightarrow u(0) = 0 + B \Rightarrow \boxed{B = 0}$$

Sub B in (1)

$$u(x) = Ax \; ----(2)$$

Applying condition (ii) in (2)

$$u(l) = Al \Rightarrow 100 = Al \quad \Rightarrow \boxed{A = \frac{100}{l}}$$

Sub A in (2)

$$\boxed{u(x) = \frac{100x}{l}}$$

The boundary and initial conditions are

i) $u(0,t) = 0^\circ C$

ii) $u(l,t) = 100^\circ C$

iii) $u(x,0) = f(x) = \frac{100x}{l}, \ 0 \le x \le l$

The correct solution is

$$u(x,t) = \left(A\cos px + B\sin px\right)Ce^{-\alpha^2 p^2 t} \ -----(1)$$

Apply condition (i) in (1)

$$u(0,t) = \left(A\cos 0 + \cancel{B\sin 0}\right)Ce^{-\alpha^2 p^2 t}$$

$$0 = ACe^{-\alpha^2 p^2 t}$$

Here $C \neq 0, \ e^{-\alpha^2 p^2 t} \neq 0 \ \therefore \boxed{A = 0}$

Sub A in (1)

$$u(x,t)=\left(B\sin px\right)Ce^{-\alpha^2 p^2 t} \quad -----(2)$$

Apply condition (ii) in (2)

$$u(l,t)=\left(B\sin pl\right)Ce^{-\alpha^2 p^2 t}$$

$$0=\left(B\sin pl\right)Ce^{-\alpha^2 p^2 t}$$

Here $B\neq 0, C\neq 0, e^{-\alpha^2 p^2 t}\neq 0 \ \therefore \boxed{\sin pl=0}$

$$\sin pl=\sin n\pi \ \Rightarrow pl=n\pi \ \Rightarrow \boxed{p=\frac{n\pi}{l}}$$

Sub p in (2)

$$u(x,t)=\left(B\sin\frac{n\pi x}{l}\right)Ce^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

$$u(x,t)=b_1\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}} \qquad BC=b_1(\text{say})$$

The most general solution is

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}} \quad -----(3)$$

Apply condition (iii) in (3)

$$u(x,0)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}e^{-0}$$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \qquad \because e^{-0}=1$$

This is Fourier sine series of *f(x)* in *(0,l)*

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}dx$$

$$= \frac{200}{l^2}\int_0^l x\sin\frac{n\pi x}{l}dx$$

$$= \frac{200}{l^2}\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left(\frac{-l}{n\pi}\right)\left[x\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-200}{ln\pi}\left[l\cos n\pi - 0\right]$$

$$= \frac{-200(-1)^n}{n\pi}$$

$$\boxed{b_n = \frac{200}{n\pi}(-1)^{n+1}}$$

Sub b_n in (3)

$$u(x,t) = \sum_{n=1}^{\infty}\frac{200}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

$$\boxed{u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}}$$

This is the required temperature.

2. The ends A and B of a rod *l* cm long have their temperatures kept at $30^\circ C$ and $80^\circ C$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60^\circ C$ and that of A is increased to $40^\circ C$. Find the steady state temperature distribution in the rod after time t.

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ where $\alpha^2=\frac{k}{\rho c}$

<u>To find steady state solution 1</u> $u(x,0)=u(x)$

In steady state *t=0* then $\frac{\partial u}{\partial t}=0$

$$\alpha^2\frac{\partial^2 u}{\partial x^2}=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}=0 \Rightarrow \frac{d^2u}{dx^2}=0$$

Integrating twice we get $\boxed{u(x)=Ax+B}$ ----(1)

The boundary conditions are i) $u(0)=30^\circ C$ ii) $u(l)=80^\circ C$

Applying condition (i) in (1)

$$(1)\Rightarrow u(0)=0+B\Rightarrow \boxed{B=30}$$

Sub B in (1)

$u(x) = Ax + 30 - - - -(2)$

Applying condition (ii) in (2)

$u(l) = Al + 30 \Rightarrow 80 = Al + 30 \quad \Rightarrow \boxed{A = \frac{50}{l}}$

Sub A in (2)

$u(x) = \frac{50x}{l} + 30 = f(x)$

This $u(x)$ will be treated as the initial conditions $u(x,0) = f(x)$

<u>To find steady state solution 2</u> $u(x,0) = u(x)$

Integrating twice we get $\boxed{u_t(x) = Ax + B} - - - -(3)$

The boundary conditions are i) $u_t(0) = 40^\circ C$ ii) $u_t(l) = 60^\circ C$

Applying condition (i) in (3)

$(3) \Rightarrow u_t(0) = 0 + B \Rightarrow \boxed{B = 40}$

Sub B in (1)

$u_t(x) = Ax + 40 - - - -(4)$

Applying condition (ii) in (4)

$u_t(l) = Al + 40 \Rightarrow 60 = Al + 40 \quad \Rightarrow \boxed{A = \frac{20}{l}}$

Sub A in (2)

$$u_t(x) = \frac{20x}{l} + 40$$

This $u_t(x)$ will be treated as the transient state temperature.

The required temperature is

$$u(x,t) = u_t(x,0) + \left(A\cos px + B\sin px\right)Ce^{-\alpha^2 p^2 t}$$

$$u(x,t) = \frac{20x}{l} + 40 + \left(A\cos px + B\sin px\right)Ce^{-\alpha^2 p^2 t} ------(5)$$

The boundary and initial conditions are

i) $u(0,t) = 40^\circ C$

ii) $u(l,t) = 60^\circ C$

iii) $u(x,0) = f(x) = \frac{50x}{l} + 30,\ 0 \le x \le l$

Apply condition (i) in (5)

$$u(0,t) = 0 + 40 + \left(A\cos 0 + B\sin 0\right)Ce^{-\alpha^2 p^2 t}$$

$$40 = 0 + 40 + \left(A\cos 0 + \cancel{B\sin 0}\right)Ce^{-\alpha^2 p^2 t}$$

$$0 = ACe^{-\alpha^2 p^2 t}$$

This $C \ne 0, e^{-\alpha^2 p^2 t} \ne 0 \therefore \boxed{A = 0}$

Sub A in (5)

$$u(x,t) = \frac{20x}{l} + 40 + \left(B\sin px\right)Ce^{-\alpha^2 p^2 t} -----(6)$$

Apply condition (ii) in (6)

$$u(l,t) = 20 + 40 + \left(B\sin pl\right)Ce^{-\alpha^2 p^2 t}$$

$$60 = 20 + 40 + \left(B\sin pl\right)Ce^{-\alpha^2 p^2 t}$$

$$0 = \left(B\sin pl\right)Ce^{-\alpha^2 p^2 t}$$

$$B \neq 0, C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \ \therefore \sin pl = 0$$

$$\sin pl = \sin n\pi \ \Rightarrow \ pl = n\pi \ \Rightarrow \ \boxed{p = \frac{n\pi}{l}}$$

Sub p in (6)

$$u(x,t) = \frac{20x}{l} + 40 + \left(B\sin\frac{n\pi x}{l}\right)Ce^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$u(x,t) = \frac{20x}{l} + 40 + b_1 \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} \qquad \because BC = b_1$$

The most general solution is

$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} -----(7)$$

Apply condition (iii) in (7)

$$u(x,0) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-0}$$

$$\frac{50x}{l} + 30 = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{30x}{l} - 10 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

To find b_n:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l \left(\frac{30x}{l} - 10\right)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[\left(\frac{30x}{l} - 10\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{30}{l}\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{-l}{n\pi}\right)\left(\frac{30x}{l} - 10\right)\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-2}{n\pi}\left[(20)\cos n\pi + 10\right]$$

$$\boxed{b_n = \frac{-20}{n\pi}\left[2(-1)^n + 1\right]}$$

Sub b_n in (7)

$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} \frac{-20}{n\pi}\left[2(-1)^n + 1\right]\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$\boxed{u(x,t) = \frac{20x}{l} + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[2(-1)^n + 1\right]\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}}$$