UNIT I

Mechanics

1.2 Kinetic energy of the system of particles:

Definition:

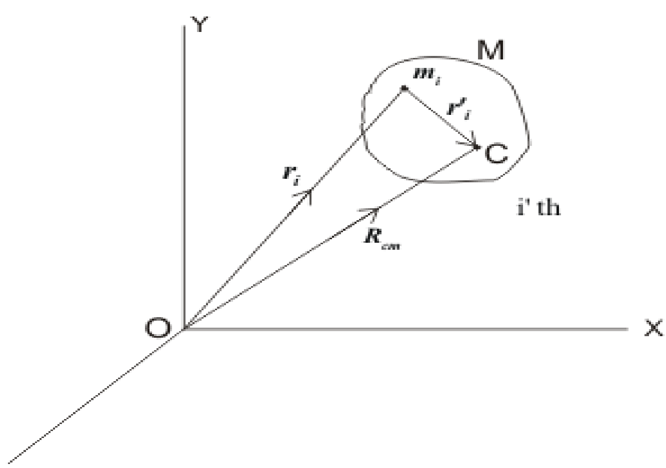
The total kinetic energy of the system is equal to the **sum** of the kinetic energy of a particle of mass M moving with velocity of the center of mass and the kinetic energy of motion of the individual particles relative to the center of mass.

- Let there are 'n' number of particles in a 'n' particle system and these particles possess some motion. The motion of the 'i'th particle of this system would depend on the external force $\mathbf{F}_i$ acting on it. Let at any time if the velocity of 'i' th particle be $\mathbf{v}_i$ then its kinetic energy would be

$$E_{Ki} = \frac{1}{2} m v_i^2$$

$$E_{Ki} = \frac{1}{2} m v_i \times v_i$$

- Let $\mathbf{r}_i$ be the position vector of the i'th particle w.r.t. O and $\mathbf{r'}_i$ be the position vector of the center of mass w.r.t. $\mathbf{r}_i$,as shown below in the figure , then



$$\frac{d\mathbf{r}_i}{dt} = \frac{d\mathbf{r'}_i}{dt} + \frac{d\mathbf{R}_{cm}}{dt}$$

or,

$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{V}_{cm} \qquad (3)$$

where v_i is the velocity of i'th particle w.r.t. center of mass and V_{cm} is the velocity of center of mass of system of particle. Putting equation (3) in (1) we get,

$$E_{Ki} = \frac{1}{2}m_i\left[(\boldsymbol{v}_i' + \boldsymbol{V}_{cm}).(\boldsymbol{v}_i' + \boldsymbol{V}_{cm})\right] = \frac{1}{2}m_i\left[(\boldsymbol{v}_i^{2\prime} + 2\boldsymbol{v}'_i.\boldsymbol{V}_{cm} + \boldsymbol{V}_{cm}^2)\right]$$

$$E_{Ki} = \frac{1}{2}m_i\boldsymbol{v}_i'^2 + m_i\boldsymbol{v}'_i.\boldsymbol{V}_{cm} + \frac{1}{2}m_i\boldsymbol{V}_{cm}^2 \qquad (4)$$

Sum of Kinetic energy of all the particles can be obtained from equation (4)

$$E_K = \sum_{i=1}^{n} E_{Ki} = \sum_{i=1}^{n}\left[\frac{1}{2}m_i\boldsymbol{v}_i'^2 + m_i\boldsymbol{v}'_i.\boldsymbol{V}_{cm} + \frac{1}{2}m_i\boldsymbol{V}_{cm}^2\right]$$

$$E_K = \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{v}_i'^2 + \sum_{i=1}^{n}m_i\boldsymbol{v}'_i.\boldsymbol{V}_{cm} + \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{V}_{cm}^2$$

$$E_K = \frac{1}{2}\boldsymbol{V}_{cm}^2\sum_{i=1}^{n}m_i + \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm}\sum_{i=1}^{n}m_i\boldsymbol{v}'_i$$

$$E_K = \frac{1}{2}\boldsymbol{V}_{cm}^2 M + \sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{v}_i'^2 + \boldsymbol{V}_{cm}\frac{d}{dt}\sum_{i=1}^{n}m_i\boldsymbol{r}'_i \qquad (5)$$

- Now last term in above equation which is

$$\sum_{i=1}^{n}m_i\boldsymbol{r}'_i = 0$$

would vanish as it defines the null vector because

$$\sum_{i=1}^{n}m_i\boldsymbol{r}'_i = \sum_{i=1}^{n}m_i(\boldsymbol{r}_i - \boldsymbol{R}_{cm}) = M\boldsymbol{R}_{cm} - M\boldsymbol{R}_{cm} = 0$$

Therefore kinetic energy of the system of particles is

$$E_K = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}\sum_{i=1}^{n}m_i v_i'^2 = E_{Kcm} + E'_K \qquad (6)$$

where,

$$E_{Kcm} = \frac{1}{2}V_{cm}^2 M$$

$$E'_K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i'^2$$

is the kinetic energy obtained as if all the mass were concentrated at the center of mass

- Hence it is clear from equation 6 that kinetic energy of the system of particles consists of two parts: the kinetic energy obtained as if all the mass were concentrated at the centre of mass plus the kinetic energy of motion about the centre of mass.
- If there were no external force acting on the particle system then the velocity of the centre of mass of the system will remain constant and Kinetic Energy of the system would also remain constant.

Rotation of rigid bodies:

Introduction:

According to Newton's first law of motion, "Every body continues to be in a state of rest or in a state of uniform motion along a straight line, unless it is compelled to change that state by some external force"

This law consists of two parts i.e., (1) A body at rest will not move on its own unless an external force acts on it and (2) A body which is in uniform linear motion will not change either its speed or direction of motion on its own, without the help of an external force.

The inability of a body to change its state on its own, without the help of external force is termed as inertia. Inertia is a fundamental property of the matter. The more is the mass of the body, the more will be inertia.

Rigid body

If the distance between any two points in a body is not altered by applying a force, however large the force may be the body is said to be a rigid body.

A rigid body may be defined as that body which does not undergo any change in its shape or size due to the application of force. Actually, no body is a perfect rigid body. When the changes in the body are negligible, it can be considered as a rigid body.

Rotational motion:

Each body is made of large number of tiny particles. In the case of linear motion, all the particles present in the body will have same linear velocity.

When the body rotates about a fixed line (axis of rotation), its motion is known as rotatory motion.

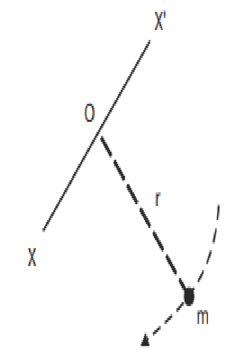
The axis of rotation may lie within the body or outside the body. If all the particles of a body move in a circular path about the axis of rotation, the rigid body is said to have pure rotational motion.

When a rigid body is in pure rotational motion, **all particles in the body rotate through the same angle during the same time interval**. Thus, all particles have the same angular velocity and the same angular acceleration.

Suppose a particle of mass ‘m’ is at a distance r from an axis, the product mr^2 is called the moment of inertia of the particle about that axis.

Moment of Inertia of a particle:

The moment of inertia of a particle about an axis is equal to the product of the mass of the particle and square of its distance from the axis.

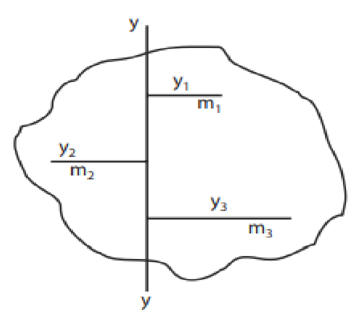


Consider a particle of mass ‘m’ is placed at a distance ‘r’ from the fixed axis. Then, the moment of inertia of the particle about the axis = mr^2. The S.I. unit for moment of inertia is kg m^2.

Moment of Inertia of a rigid body:

Inertia of a body is its inability to change by itself its state of rest or of uniform motion in a straight line.

Similarly Moment of inertia of a body is its inability to change by itself its state of rest or of uniform rotatory motion about an axis.



An external torque (rotating effect of force) is necessary to change its state.

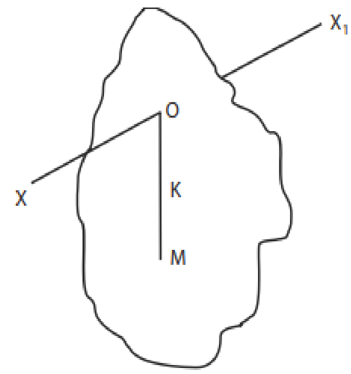
Let us consider the rotation of a rigid body about an axis. It consists of a large number of particles of masses m_1, m_2, m_3 etc., situated at distances r_1 , r_2 , r_3 etc., from the axis yy’. Then $m_1 r_1{}^2$ is known as the moment of inertia of the particle of mass m_1 about that fixed axis.

Then, the moment of inertia of the rigid body = Sum of moments of inertia of all the particles present in the body.

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots\ldots\ldots$$
$$I = \Sigma\, mr^2$$

Hence the moment of inertia of a rigid body about a fixed axis is the sum of the moment of inertia of all the particles of the rigid body.

Radius of gyration:



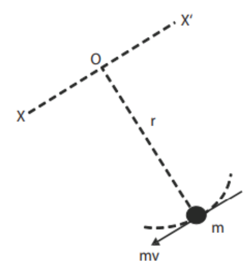
Radius of gyration is the distance between the given axis and the center of mass of the body. The center of mass of a body is point where the entire mass of the body is supposed to be concentrated.

It is denoted by 'K '.

If M is mass of the body, then moment of inertia $I = MK^2$. Hence, $mr^2 = MK^2$

Angular momentum:

The moment of linear momentum is known as angular momentum. Consider a particle of mass m is at a distance r from the axis of rotation. When a particle is in rotational motion about an axis, it has both linear velocity 'v' and angular velocity 'ω'.



Then, Angular momentum of the particle = linear momentum × perpendicular distance between the particle and the axis of rotation.

∴ Angular momentum = $mr^2\,\omega$

Where ω is the angular velocity of the particle. The SI unit for angular momentum is kg $m^2\, s^{-1}$

Law of conservation of angular momentum:

When there is no external torque acting on a rotating body, the angular momentum of that body remains a constant. This is the statement of law of conservation of angular momentum.

i.e., if I_1 and ω_1 are the initial moment of inertia and angular velocity of a rotating body and if I_2 and ω_2 are new moment of inertia and angular velocity of the body, without the help of any external torque, then according to this law,

$$I_1\, \omega_1 = I_2\, \omega_2$$

Examples:

a) Consider a person standing on a turn-table with arms extended and a pair of weights, one in each hand. The table is made to rotate by a motor and then the motor is switched off. Now, if that person pulls his arms inwards, we can see a considerable increase in the speed of rotation. This is because, in the new position, his moment of inertia I about the axis of rotation decrease, due to the decrease in the value of r. Since the angular momentum is conserved, a decrease in the value I results in an increase in the value of angular velocity ω. Therefore the person is found to rotate faster.

b) A circus acrobat, a diver or skater, all take advantage of this principle. consider a diver has just left a diving board with his arms and legs extended, with a particular angular momentum. If he now pulls his arms and legs in, his moment of inertia I decreases, leading to increase in angular velocity ω. By increasing ω, the diver can perform more number of somersaults, before entering the swimming pool.