

## CAYLEY HAMILTON THEOREM

### Cayley Hamilton Theorem

**Statement:** Every square matrix satisfies its own characteristic equation.

### Uses of Cayley Hamilton Theorem:

To calculate (i) the positive integral power of A and  
(ii) the inverse of a non-singular square matrix A.

**Example:** Show that the matrix  $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$  satisfies its own characteristic equation.

**Solution:**

$$\text{Let } A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

The characteristic equation of the given matrix is  $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_1 = 0$$

Where  $S_1 =$  sum of the main diagonal elements.

$$= 1 + 1 = 2$$

$$S_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 + 4 = 5$$

$\therefore$  The characteristic equation is  $\lambda^2 - 2\lambda + 5 = 0$

**To prove:**  $A^2 - 2A + 5I = 0$

$$A^2 = AA = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - 2 \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} + 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Therefore, the given matrix satisfies its own characteristic equation.

**Example:** Verify Cayley – Hamilton theorem find  $A^4$  and  $A^{-1}$  when  $A = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

**Solution:**

The characteristic equation of A is  $|A - \lambda I| = 0$

i.e.,  $\lambda^3 - S_1\lambda^2 + S_2\lambda = 0$  where

$$S_1 = \text{sum of its leading diagonal elements} = 2 + 2 + 2 = 6$$

$S_2$  = sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4 - 1) + (4 - 2) + (4 - 1) = 3 + 2 + 3 = 8$$

$$S_3 = |A| = 2(4 - 1) + 1(-2 + 1) + 2(1 - 2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6 - 1 - 2 = 3$$

∴ The characteristic equation of A is  $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$i.e., \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By Cayley-Hamilton theorem

[Every square matrix satisfies its own characteristic equation]

$$(i.e.) A^3 - 6A^2 + 8A - 3I = 0 \quad \dots (1)$$

**Verification:**

$$A^2 = A \times A = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$+ 8 \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

**To find  $A^4$ :**

$$(1) \Rightarrow A^3 - 6A^2 - 8A + 3I \quad \dots (2)$$

Multiply A on both sides, we get

$$A^4 = 6A^3 - 8A^2 + 3A = 6[6A^2 - 8A + 3I] - 8A^2 + 3A \text{ by (2)}$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$A^4 = 28A^2 - 45A + 18I \quad \dots (3)$$

$$\begin{aligned}
 (1) \Rightarrow A^4 &= 28 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} \\
 &= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}
 \end{aligned}$$

**To find  $A^{-1}$ :**

$$(1) \times A^{-1} \Rightarrow A^2 - 6A + 8I - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 6A + 8I$$

$$\begin{aligned}
 3A^{-1} &= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} \\
 3A^{-1} &= \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}
 \end{aligned}$$

**Example: Find  $A^{-1}$  if  $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ , using Cayley- Hamilton theorem.**

**Solution:**

The characteristic equation of A is  $|A - \lambda I| = 0$

(i. e.)  $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$  where

$S_1$  = sum of its leading diagonal elements

$$= 1 + 2 + (-1) = 2$$

$S_2$  = sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= (-2 + 1) + (-1 - 8) + (2 + 3)$$

$$= (-1) + (-9) + 5 = -5$$

$$S_3 = |A| = \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1(-2 + 1) + 1(-3 + 2) + (3 - 4)$$

$$= 1(-1) + 1(-1) + 4(-1)$$

$$= -1 - 1 - 4 = -6$$

∴ The Characteristic equation is  $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley Hamilton Theorem we get

[Every square matrix satisfies its own characteristic equation]

$$\therefore A^3 - 2A^2 - 5A + 6I = 0 \quad \dots(1)$$

**To find  $A^{-1}$**

$$(1) \times A^{-1} \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$$

$$A^2 - 2A - 5I + 6A^{-1} = 0$$

$$6A^{-1} = -A^2 + 2A + 5I$$

$$A^{-1} = \frac{1}{6}[-A^2 + 2A + 5I] \quad \dots(2)$$

$$A^2 = A \times A$$

$$= \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + 2 \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

$$\text{From (2)} \Rightarrow A^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

**Example: Use Cayley – Hamilton theorem to find the value of the matrix given by**

(i)  $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

(ii)  $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$  if the matrix  $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

**Solution:**





$$= (1 - 1) + (1 - 3) + (1 - 0)$$

$$= 0 - 2 + 1 = -1$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1(1 - 1) - 0(2 + 1) + 3(-2 - 1)$$

$$= 0 - 0 + 3(-3) = -9$$

∴ The characteristic equation A is  $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley - Hamilton Theorem every square matrix satisfies its own Characteristic equation

$$\therefore A^3 - 3A^2 - A + 9I = 0$$

$$A^{-1} = \frac{-1}{9} [A^2 - 3A - I] \quad \dots (1)$$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-3A = \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix}$$

$$(1) \Rightarrow A^{-1} = \frac{-1}{9} \left[ \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

$$= \frac{-1}{9} \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

**Example: Verify Cayley- Hamilton for the matrix**  $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

**Solution :**

$$\text{Given } A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

The characteristic equation A is  $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \dots (1) \quad \text{where}$$

$$S_1 = \text{Sum of the main diagonal elements} \\ = 1 + 2 + 1 = 4$$

$$S_2 = \text{Sum of the minors of its leading diagonal elements} \\ = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} \\ = (2 - 6) + (1 - 7) + (2 - 12) \\ = -4 - 6 - 10 = -20$$

$$S_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} \\ = 1(2 - 6) - 3(4 - 3) + 7(8 - 2) \\ = -4 - 3(1) + 7(6) \\ = -4 - 3 + 42 = 35$$

$$\therefore (1) \Rightarrow \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

By Cayley-Hamilton theorem

$$(2) \Rightarrow A^3 - 4A^2 - 20A - 35I = 0$$

To find  $A^2$  and  $A^3$ :

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix} \\ = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \\ A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \\ = \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix} \\ = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$A^3 - 4A^2 - 20A - 35I$$



$$\begin{aligned}
 &= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4 \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20 \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + \begin{bmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{bmatrix} + \begin{bmatrix} -20 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{bmatrix} + \\
 &\quad \begin{bmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

∴ The given matrix A satisfies its own characteristic equation.

Hence, Cayley Hamilton theorem is verified.

