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DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT: TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

SUBJECT CODE : MA6351

REGULATION : 2013

UNIT – I : PARTIAL DIFFERENTIAL EQUATION

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

Notations: If $z = f(x, y)$ then

$$p = \frac{\partial z}{\partial x};\ q = \frac{\partial z}{\partial y};\ r = \frac{\partial^2 z}{\partial x^2};\ s = \frac{\partial^2 z}{\partial x \partial y};\ t = \frac{\partial^2 z}{\partial y^2}$$

Formation of PDE by eliminating arbitrary constants:

Let the given equation be $z = f(x, y, a, b) - - - - - (1)$

Step 1: Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = p = f'(x, y, a, b) - - - - - (2)$$

Step 2: Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = q = f'(x, y, a, b) - - - - - (3)$$

Step 3: Eliminate a & b from (1) using (2) & (3)

1. **Obtain partial differential equation by eliminating arbitrary constant 'a' and 'b' from** $z = (x - a)^2 + (y - b)^2$

Solution:

Given $z = (x - a)^2 + (y - b)^2 - - - - (1)$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = 2(x - a) + 0$$

$p = 2(x - a) - - - - (2)$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = 0 + 2(y - b)$$

$q = 2(y - b) - - - - (3)$

Eliminate a & b from (1) using (2) & (3)

$(2) \Rightarrow (x - a) = \frac{p}{2} - - - - (4)$

$(3) \Rightarrow y - b = \frac{q}{2} - - - - (5)$

Sub (4) & (5) in (1)

$$(1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

The required the PDE is

$$p^2 + q^2 = 4z$$

2. **Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from** $z = (x^2 + a)(y^2 + b)$.

Solution:

Given $z = (x^2 + a)(y^2 + b) - - - - (1)$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) - - - - (2)$$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) - - - - (3)$$

Eliminate a & b from (1) using (2) & (3)

	$(2) \Rightarrow (y^2 + b) = \frac{p}{2x} ----(4)$ $(3) \Rightarrow x^2 + b = \frac{q}{2y} ----(5)$ Sub (4) & (5) in (1) $(1) \Rightarrow z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$ The required the PDE is $4xyz = pq$
3.	**Find the PDE of all planes having equal intercepts on the x and y axis.** **Solution:** The intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ Given that equal intercepts on the x & y axis $\Rightarrow a = b$ $\therefore \frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 ------(1)$ Diff Partially w.r.t x $\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c}p ----(2)$ Diff Partially w.r.t y $0 + \frac{1}{a}(1) + \frac{1}{c}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c}q ----(3)$ From (2) & (3) $\frac{-1}{c}p = \frac{-1}{c}q$ The required the PDE is $p = q$
4.	**Obtain the partial differential equation by eliminating arbitrary constants 'a' and ' b' from** $(x-a)^2 + (y-b)^2 + z^2 = r^2$ **Solution:** $(x-a)^2 + (y-b)^2 + z^2 = 1 ----(1)$ - (1) Diff Partially w.r.t x $2(x-a)(1-0) + 0 + 2z\frac{\partial z}{\partial x} = 0$ $\Rightarrow 2(x-a) + 2zp = 0 ------(2)$ Diff Partially w.r.t y $0 + 2(y-b)(1-0) + 2z\frac{\partial z}{\partial y} = 0$ $\Rightarrow 2(y-b) + 2zq = 0 ------(3)$ Eliminate a & b from (1) using (2) & (3) $(2) \Rightarrow x - a = -zp ------(4)$ $(3) \Rightarrow y - b = -zq ------(5)$ Sub (4) & (5) in (1) $(-zp)^2 + (-zq)^2 + z^2 = 1$ The required PDE is $z^2\left(p^2 + q^2 + 1\right) = 1$
Formation of PDE by eliminating arbitrary functions:	
1.	**Eliminate the arbitrary function *f* from** $z = f\left(\frac{y}{x}\right)$ **and form the PDE.** **Solution:**

$z = f\left(\dfrac{y}{x}\right) \;----(1)$

Diff Partially w.r.t x

$\dfrac{\partial z}{\partial x} = p = f'\left(\dfrac{y}{x}\right) \times \left(\dfrac{-y}{x^2}\right) \quad \Rightarrow \quad f'\left(\dfrac{y}{x}\right) = \dfrac{-px^2}{y} \;----(2)$

Diff Partially w.r.t x

$\dfrac{\partial z}{\partial y} = q = f'\left(\dfrac{y}{x}\right) \times \left(\dfrac{1}{x}\right) \;----(3)$

From (1) & (2) $\dfrac{p}{q} = \dfrac{-y}{x} \Rightarrow px + qy = 0$

2. **Form the partial differential equation by eliminating *f* from** $z = x^2 + 2f\left(\dfrac{1}{y} + \log x\right)$ **.**

Solution:

Given $z = x^2 + 2f\left(\dfrac{1}{y} + \log x\right) \;-----(1)$

Differentiate (1) partially w.r. t x

$\dfrac{\partial z}{\partial x} = 2x + 2f'\left(\dfrac{1}{y} + \log x\right)\left(0 + \dfrac{1}{x}\right)$

$p = 2x + 2f'\left(\dfrac{1}{y} + \log x\right)\left(\dfrac{1}{x}\right) \Rightarrow f'\left(\dfrac{1}{y} + \log x\right) = \left(p - 2x\right)\dfrac{x}{2} \;----(2)$

$\dfrac{\partial z}{\partial y} = 2f'\left(\dfrac{1}{y} + \log x\right)\left(\dfrac{-1}{y^2} + 0\right)$

$q = \dfrac{-2}{y^2} f'\left(\dfrac{1}{y} + \log x\right) \Rightarrow f'\left(\dfrac{1}{y} + \log x\right) = \dfrac{-qy^2}{2} \;-----(3)$

Eliminating f' from (2) & (3)

$\left(p - 2x\right)\dfrac{x}{2} = \dfrac{-qy^2}{2} \quad \Rightarrow \left(px - 2x^2\right) = -qy^2$

$\Rightarrow px + qy^2 = 2x^2$

Formation of PDE by eliminating *f* from $f(u, v) = 0$-------(1)

Method 1:

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

Method 2:

The required PDE is $Pp + Qq = R$

Where

$P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}$; $Q = \begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}$; $R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$

1. **Form the PDE from** $\phi\left(ax + by + cz, x^2 + y^2 + z^2\right) = 0$

Solution:

Given $\phi\left(ax + by + cz, x^2 + y^2 + z^2\right) = 0$

This is of the form $f(u, v) = 0$ where $u = ax + by + cz$ & $v = x^2 + y^2 + z^2$

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

$$\begin{vmatrix} p & q & -1 \\ a & b & c \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$\Rightarrow p(2bz - 2cy) - q(2az - 2cx) + 1(2az - 2cx) = 0$

$\div 2 \Rightarrow (bz - cy)\, p + (cx - az)q + (az - cx) = 0$

2. **Form the PDE from $\phi\left(x^2 + y^2 + z^2 , xyz\right) = 0$**

Solution:

Given $\phi\left(x^2 + y^2 + z^2 , xyz\right) = 0$

This is of the form $f(u, v) = 0$ where $u = x^2 + y^2 + z^2$ & $v = xyz$

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

$$\begin{vmatrix} p & q & -1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix} = 0$$

$\Rightarrow p(2xy^2 - 2xz^2) - q(2x^2y - 2yz^2) + 1(2x^2z - 2y^2z) = 0$

$\div 2 \Rightarrow \quad x(y^2 - z^2)p + y(z^2 - x^2) + z(x^2 - y^2) = 0$

3. **Form the PDE from $\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$**

Solution:

Given $\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$

This is of the form $f(u, v) = 0$ where $u = \frac{y}{x}$ & $v = x^2 + y^2 + z^2$

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

$$\begin{vmatrix} p & q & -1 \\ \frac{-y}{x^2} & \frac{1}{x} & 0 \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$$\Rightarrow p\left(\frac{2z}{x} - 0\right) - q\left(\frac{-2yz}{x^2} - 0\right) + 1\left(\frac{-2y^2}{x^2} - \frac{2x}{x}\right) = 0$$

$$\Rightarrow \frac{2zp}{x} + \frac{2yzq}{x^2} - 2\left(\frac{y^2}{x^2} + 1\right) = 0$$

$$\div 2 \Rightarrow \frac{zp}{x} + \frac{yzq}{x^2} - \left(\frac{y^2 + x^2}{x^2}\right) = 0$$

	$\Rightarrow \dfrac{xzp + yzq - \left(y^2 + x^2\right)}{x^2} = 0$ $xzp + yzq - \left(y^2 + x^2\right) = 0$

Solutions of standard types of First order PDE's:

Different solutions of PDE:

Complete Integral (or) Complete Solution:

If the number of arbitrary constants is equal to number of independent variables, then the solution is called Complete integral.

Singular Integral (or) Singular Solution:

Consider a PDE of first order as $f(x, y, z, p, q) = 0 - - - - - (1)$

It's complete integral may be, $f(x, y, z, a, b) = 0 - - - - - (2)$

Diff (2) partially with respect to a & b respectively,

$\dfrac{\partial g}{\partial a} = 0 - - - - - - (3)$

$\dfrac{\partial g}{\partial b} = 0 - - - - - - (4)$

Eliminating a & b from (3) & (4) will get the Singular integral.

General Integral (or) Complete solution:

A Solution which contains number of arbitrary functions is equal to the order of the given PDE.
(or) A solution which contains the maximum possible number of arbitrary functions.

Type I:

Equations of the form $f(p, q) = 0 - - - - - (1)$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c - - - - (2)$

Let $p = a$ & $q = b$ in (1)

$f(a, b) = 0$ and represent $b = \phi(a)$

$\therefore (1) \Rightarrow z = ax + \phi(a)\, y + c \; - - - - (3)$

To Find Singular Integral:

Diff (3) partially with respect to c

$0 = 1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c = g(a)$ in (3)

$(3) \Rightarrow z = ax + \phi(a)\, y + g(a) - - - - (4)$

Diff (4) partially with respect to a

$0 = x(1) + \phi'(a)\, y + g'(a) - - - - (5)$

Eliminating a from (4) & (5) we get general integral.

1. **Find the complete integral of** $p + q = pq$

Solution:

Given $p + q = pq \; - - - - (1)$

This of the form $f(p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c - - - - (2)$

Let $p = a$ & $q = b$ in (1)

$(1) \Rightarrow a + b = ab \Rightarrow a + b - ab = 0 \Rightarrow b = \dfrac{a}{a-1}$

Sub b in (2)

$z = ax + \left(\dfrac{a}{a-1}\right) y + c$

This is the required complete integral.

2. **Find the complete integral of** $p + q = 1$

Solution:

Given $p + q = 1 ----(1)$

This of the form $f(p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c ----(2)$

Let $p = a$ & $q = b$ in (1)

$(1) \Rightarrow a + b = 1 \Rightarrow b = 1 - a$

Sub b in (2)

$z = ax + (1 - a)y + c$

This is the required complete integral.

3. **Solve** $\sqrt{p} + \sqrt{q} = 1$

Solution:

Given $\sqrt{p} + \sqrt{q} = 1 ----(1)$

This of the form $f(p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c ----(2)$

Let $p = a$ & $q = b$ in (1)

$(1) \Rightarrow \sqrt{a} + \sqrt{b} = 1 \Rightarrow \sqrt{b} = 1 - \sqrt{a} \Rightarrow b = \left(1 - \sqrt{a}\right)^2$

Sub b in (2)

$z = ax + \left(1 - \sqrt{a}\right)^2 y + c \quad ----(3)$

This is the required complete integral.

To Find Singular Integral:

Diff (3) partially with respect to c

$0 = 1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c = f(a)$ in (3)

$(3) \Rightarrow \quad z = ax + \left(1 - \sqrt{a}\right)^2 y + f(a) \quad ----(4)$

Diff. (4) partially with respect to a

$0 = x(1) + 2\left(1 - \sqrt{a}\right)\frac{1}{2\sqrt{a}} y + f'(a) \quad ----(5)$

Eliminate a from (4) & (5) we get the general integral.

Type II:

Equations of the form $z = px + qy + f(p, q) ------(1)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$\therefore (1) \Rightarrow z = ax + by + f(\mathrm{a}, \mathrm{b}) ------(2)$

To Find Singular Integral:

Diff (2) partially with respect to a

$0 = x(1) + 0 + f'(a, b) ----(3)$

Diff (2) partially with respect to b

$0 = 0 + y(1) + f'(a, b) ----(4)$

Eliminating a & b from (2) using (3) &(4), we get the singular integral.

To find General integral:

Put $b = \phi(a)$ in (2)

$(2) \Rightarrow z = ax + \phi(a)y + g(\mathrm{a}) ------(5)$

Diff (4) partially with respect to a

$0 = x(1) + \phi'(a)\, y + g'(a) ----(6)$ Eliminating a from (5) & (6) we get General integral.	
1.	**Solve** $z = px + qy + p^2 q^2$ **Solution:** Given $z = px + qy + p^2 q^2 ------ (1)$ Equations of the form $z = px + qy + f(p, q)$ To find Complete Integral: Put $p = a$ & $q = b$ in (1) $\therefore (1) \Rightarrow z = ax + by + a^2b^2 ------(2)$ This is the required complete integral **To Find Singular Integral:** Diff (2) partially with respect to a $0 = x(1) + 0 + 2ab^2 \Rightarrow x + 2ab^2 = 0 \Rightarrow x = -2ab^2 ----(3)$ Diff (2) partially with respect to b $0 = 0 + y(1) + 0 + 2a^2b \Rightarrow y + 2a^2b = 0 \Rightarrow y = -2a^2b ----(4)$ Eliminating a & b from (2) using (3) &(4) $(3) \Rightarrow \frac{x}{b} = -2ab ----(5)$ $(4) \Rightarrow \frac{y}{a} = -2ab ----(6)$ From (5) & (6) $\frac{x}{b} = \frac{y}{a} = k$ (say) $\Rightarrow \frac{x}{b} = k$ & $\frac{y}{a} = k$ $\Rightarrow b = \frac{x}{k}$ & $a = \frac{y}{k} ----(7)$ Sub a & b in (2) $(2) \Rightarrow z = \frac{y}{k}x + \frac{x}{k}y + \frac{y^2}{k^2}\frac{x^2}{k^2}$ $z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2y^2}{k^4}$ $z = \frac{2xy}{k} + \frac{x^2y^2}{k^4} ----(8)$ <u>To find k</u> Sub (7) in (3) (or) (4) $(3) \Rightarrow x = -2\frac{y}{k}\frac{x^2}{k^2} \Rightarrow x = \frac{-2x^2y}{k^3} \Rightarrow k^3 = -2xy$ $(8) \Rightarrow z = \frac{2xy}{k} + \frac{x^2y^2}{k(-2xy)} \Rightarrow z = \frac{2xy}{k} - \frac{xy}{2k}$ $z = \frac{4xy - xy}{2k} \Rightarrow z = \frac{3xy}{2k} \Rightarrow z^3 = \frac{27x^3y^3}{8k^3} \Rightarrow z^3 = \frac{27x^3y^3}{8(-2xy)}$ $16z^3 = -27x^2y^2$ This is the required singular integral. **To find General integral:** Put $b = \phi(a)$ in (2) $(2) \Rightarrow z = ax + f(a)y + a^2\left[f(a)\right]^2 ------(9)$

Diff (9) partially with respect to a

$$0 = x(1) + f'(a)\, y + 2 f(a) f'(a) ----(10)$$

Eliminating a from (9) & (10) we get General integral.

2. **Find the singular integral of** $z = px + qy + p^2 + pq + q^2$

Solution:

Given $z = px + qy + p^2 + pq + q^2 -----(1)$

Equations of the form $z = px + qy + f(p, q)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow \quad z = ax + by + a^2 + ab + b^2 \quad ------(2)$$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + 2a + b + 0 \Rightarrow 2a + b = -x ----(3)$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + 0 + a + 2b \Rightarrow a + 2b = -y ----(4)$$

Eliminating a & b from (2) using (3) &(4)

$$(4)\times 2 \Rightarrow 2a + 4b = -2y -----(5)$$

$$(3) - (5) \Rightarrow -3b = -x + 2y \Rightarrow \boxed{b = \frac{x - 2y}{3}}$$

Sub the value of b in (3)

$$2a + b = -x \Rightarrow 2a = -x - b \Rightarrow 2a = -x - \left(\frac{x - 2y}{3}\right)$$

$$2a = \frac{-3x - x + 2y}{3} \Rightarrow 6a = -4x + 2y \Rightarrow \boxed{a = \frac{y - 2x}{3}}$$

Sub the value of a & b in (2)

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 - 4xy + 4y^2}{9} + \frac{xy - 2y^2 - 2x^2 + 4xy}{9} + \frac{x^2 - 4xy + 4y^2}{9}$$

$$z = \frac{3xy - 3x^2 + 3xy - 6y^2 + y^2 - 4xy + 4y^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy + 4y^2}{9}$$

$$9z = -4x^2 + y^2 + xy$$

3. **Solve** $z = px + qy + \sqrt{p^2 + q^2 + 1}$

Solution:

Given

$$z = px + qy + \sqrt{p^2 + q^2 + 1} -----(1)$$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$(1) \Rightarrow \boxed{z = ax + by + \sqrt{a^2 + b^2 + 1}} ----(2)$$

This is required complete integral.

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + \frac{1}{2\sqrt{a^2 + b^2 + 1}}(2a) \Rightarrow \boxed{x = \frac{-a}{\sqrt{a^2 + b^2 + 1}}} ----(3)$$

Diff (2) partially with respect to b

	$0 = 0 + y(1) + \frac{1}{2\sqrt{a^2+b^2+1}}(2b) \Rightarrow \boxed{y = \frac{-b}{\sqrt{a^2+b^2+1}}} ----(4)$ Eliminating a & b from (2) using (3) &(4) $(3)^2 + (4)^2 \Rightarrow x^2 + y^2 = \left(\frac{-a}{\sqrt{a^2+b^2+1}}\right)^2 + \left(\frac{-b}{\sqrt{a^2+b^2+1}}\right)^2$ $x^2 + y^2 = \frac{a^2}{a^2+b^2+1} + \frac{b^2}{a^2+b^2+1}$ $x^2 + y^2 = \frac{a^2+b^2}{a^2+b^2+1}$ $1-(x^2+y^2) = 1 - \frac{a^2+b^2}{a^2+b^2+1}$ $1-x^2-y^2 = \frac{a^2+b^2+1-a^2-b^2}{a^2+b^2+1}$ $1-x^2-y^2 = \frac{1}{a^2+b^2+1}$ $\Rightarrow 1+a^2+b^2 = \frac{1}{1-x^2-y^2}$ Taking square root on both sides $\Rightarrow \boxed{\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}} ----(5)$ Sub (5) in (2) and (3) $(3) \Rightarrow x = \frac{-a}{\frac{1}{\sqrt{1-x^2-y^2}}} \Rightarrow x = -a\sqrt{1-x^2-y^2} \Rightarrow \boxed{a = \frac{-x}{\sqrt{1-x^2-y^2}}}$ $(4) \Rightarrow y = \frac{-b}{\frac{1}{\sqrt{1-x^2-y^2}}} \Rightarrow y = -b\sqrt{1-x^2-y^2} \Rightarrow \boxed{b = \frac{-y}{\sqrt{1-x^2-y^2}}}$ Sub (5), a & b in (2) $(2) \Rightarrow z = \left(\frac{-x}{\sqrt{1-x^2-y^2}}\right)x + \left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)y + \frac{1}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \frac{-x^2}{\sqrt{1-x^2-y^2}} + \frac{-y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \sqrt{1-x^2-y^2}$ Squaring on both sides $z^2 = 1-x^2-y^2 \Rightarrow x^2+y^2+z^2 = 1$
4.	**Find the singular integral of** $z = px + qy + p^2 - q^2$ **Solution:** Given $z = px + qy + p^2 - q^2 ------(1)$ Equations of the form $z = px + qy + f(p, q)$ To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$\therefore (1) \Rightarrow z = ax + by + a^2 - b^2 ------(2)$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$0 = x(1) + 0 + 2a + 0 \Rightarrow a = \frac{-x}{2} ----(3)$

Diff (2) partially with respect to b

$0 = 0 + y(1) + 0 - 2b \Rightarrow b = \frac{y}{2} ----(4)$

Sub a & b in (2)

$\therefore (2) \Rightarrow z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$

$\therefore (2) \Rightarrow z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$

$\therefore (2) \Rightarrow z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4} \Rightarrow 4z = -x^2 + y^2$

This is the required singular integral.

Type III:

Equations of the form $f(z, p, q) = 0 ----- (1)$

In this type x & y do not appear explicitly.

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay) ------ (2)$

Let $x + ay = u$

$(2) \Rightarrow z = f(u) ------ (3)$

By total derivative,

$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \Rightarrow p = \frac{dz}{dx}(1) \qquad \because u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1$

$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \Rightarrow q = a\frac{dz}{du} \qquad \because u = x + ay \Rightarrow \frac{\partial u}{\partial y} = a$

Substitute the value of p & q in (1)

$(1) \Rightarrow f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$

This may be solve by method of separation of variables

Other solutions can obtain as usual.

1. **Solve** $p(1 + q) = qz$ **.**

Solution:

Given $p(1 + q) = qz ----(1)$

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay) ------ (2)$

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

Substitute the value of p & q in (1)

$(1) \Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$

$$1+a\frac{dz}{du}=a\,z$$

$$\frac{dz}{du}=a\,z-1$$

$$\frac{dz}{a\,z-1}=du$$

Integrating on both sides

$$\int\frac{dz}{a\,z-1}=\int du$$

$$u=\log(a\,z-1)+c \qquad \because \int\frac{f'(x)}{f(x)}dx=\log f(x)$$

$$x+ay=\log(a\,z-1)+c$$

This is the required complete integral.
Other solutions can be obtained as usual.

2. **Solve** $z^2 = 1 + p^2 + q^2$

Solution:

Given $z^2 = 1 + p^2 + q^2 - - - -(1)$

This is of the form $f\,(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f\,(x + ay) - - - - - - (2)$

Let $x + ay = u \Rightarrow z = f\,(u)$

Then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

Substitute the value of p & q in (1)

$$(1) \Rightarrow z^2=\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2+1$$

$$\Rightarrow\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2=z^2-1$$

$$\Rightarrow(1+a^2)\left(\frac{dz}{du}\right)^2=z^2-1$$

$$\Rightarrow\left(\frac{dz}{du}\right)^2=\frac{z^2-1}{1+a^2}$$

Taking square root on both sides

$$\frac{dz}{du}=\frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}$$

$$\Rightarrow\frac{dz}{\sqrt{z^2-1}}=\frac{du}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\cosh^{-1}z=\frac{1}{\sqrt{a^2-1}}u+c \qquad \because\int\frac{dz}{\sqrt{x^2-1}}=\cosh^{-1}x$$

$$\boxed{\cosh^{-1}z=\frac{1}{\sqrt{a^2-1}}(x+ay)+c} \qquad \because u=x+ay$$

This is the required complete integral.
Other solutions can be obtained as usual.

3. **Solve** $p\left(1-q^2\right)=q(1-z)$

Solution:

Given $p\left(1-q^2\right)=q(1-z)$ ----(1)

This is of the form $f(z,p,q)=0$

To find Complete Integral:

Let the complete solution of (1) is $z=f(x+ay)$ ------ (2)

Let $x+ay=u \Rightarrow z=f(u)$

Then $p=\dfrac{dz}{du}$ & $q=a\dfrac{dz}{du}$

Substitute the value of p & q in (1)

$$(1)\Rightarrow \frac{dz}{du}\left(1-\left(a\frac{dz}{du}\right)^2\right)=a\frac{dz}{du}(1-z)$$

$$1-a+az=a^2\left(\frac{dz}{du}\right)^2$$

Taking square root on both sides

$$a\frac{dz}{du}=\sqrt{1-a+az}$$

$$\frac{a\,dz}{\sqrt{1-a+az}}=du$$

Integrating on both sides

$$a\left[\frac{2\sqrt{1-a+az}}{a}\right]=u+c \qquad \because \int\frac{1}{\sqrt{ax}}dx=\frac{1}{a}\left(2\sqrt{x}\right)$$

$$\boxed{2\sqrt{1-a+az}=x+ay+c} \qquad \because u=x+ay$$

This is the required complete integral.

Other solutions can be obtained as usual.

Type IV:

Equations of the form $f_1(x,p)=f_2(y,q)$ ------(1)

To find Complete Integral:

Let $f_1(x,p)=f_2(y,q)=a$ (say)

$\therefore f_1(x,p)=a$; $f_2(y,q)=a$

From the above we get $p=f_1(x,a)$; $q=f_2(y,b)$

Substitute the value of p & q in $z=\int p\,dx+\int q\,dy$

Integrating we get complete integral

Other solutions can obtain as usual.

1. **Solve** $p^2+q^2=x^2+y^2$

Solution:

Given $p^2+q^2=x^2+y^2$

$p^2-x^2=y^2-q^2$ ----- (1)

This is of the form $f_1(x,p)=f_2(y,q)$

To find Complete Integral:

Let $p^2-x^2=y^2-q^2=a^2$ (say)

$\therefore p^2-x^2=a^2$; $y^2-q^2=a^2$

$\therefore p^2=a^2+x^2$; $q^2=y^2-a^2$

$\boxed{p=\sqrt{a^2+x^2}}$; $\boxed{q=\sqrt{y^2-a^2}}$

Substitute the value of p & q in

$$z = \int \sqrt{x^2 + a^2}\, dx + \int \sqrt{y^2 - a^2}\, dy$$

$$z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + c$$

$$\because \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right);\quad \int \sqrt{y^2 - a^2}\, dy = \frac{y}{2}\sqrt{y^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)$$

Integrating we get complete integral

Other solutions can obtain as usual.

2. **Find the complete integral of** $p^2 y(1 + x^2) = qx^2$

Solution:

Given $p^2 y(1 + x^2) = qx^2$

$$\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y} - - - - - (1)$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y} = a$ (say)

$$\frac{p^2(1 + x^2)}{x^2} = a\ ;\ \frac{q}{y} = a$$

$$p^2 = a\frac{x^2}{1 + x^2}\quad ;\ q = ay$$

$$\boxed{p = \frac{\sqrt{a}\,x}{\sqrt{1 + x^2}}}\quad ;\ q = ay$$

Substitute the value of p & q in

$$z = \int \frac{\sqrt{a}\,x}{\sqrt{1 + x^2}}\, dx + \int ay\, dy$$

let $1 + x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2}$

$$z = \sqrt{a}\int \frac{1}{\sqrt{t}}\frac{dt}{2} + a\frac{y^2}{2}$$

$$z = \frac{\sqrt{a}}{2}\, 2\sqrt{t} + \frac{ay^2}{2} \qquad \because \int \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}$$

$$\boxed{z = \sqrt{a}\sqrt{1 + x^2} + \frac{ay^2}{2} + c}$$

3. **Find the complete integral of** $p + q = \sin x + \sin y$

Solution:

Given $p + q = \sin x + \sin y$

$$p - \sin x = \sin y - q - - - - - (1)$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $p - \sin x = \sin y - q = a$ (say)

$\therefore p - \sin x = a$; $\sin y - q = a$

$\therefore p = \sin x + a$; $q = \sin y - a$

Substitute the value of p & q in

$$z = \int (\sin x + a)\, dx + \int (\sin y - a)\, dy$$
$$z = \cos x + ax + \cos y - ay + c$$
$$z = \cos x + \cos y - a(x - y) + c$$

This is the required complete integral

Type V:

Equations reducible to standard form (Identification:

1. **Solve** $z^2(p^2 + q^2) = x^2 + y^2$

Solution:

Given $z^2(p^2 + q^2) = x^2 + y^2$

This is of the form equations reducible to standard form

$$z^2 p^2 + z^2 q^2 = x^2 + y^2$$
$$(zp)^2 + (zq)^2 = x^2 + y^2$$
$$\left(\frac{z\partial z}{\partial x}\right)^2 + \left(\frac{z\partial z}{\partial y}\right)^2 = x^2 + y^2 -----(1) \qquad \because p = \frac{\partial z}{\partial x};\ q = \frac{\partial z}{\partial y}$$

Let $z\partial z = \partial Z$; $\partial x = \partial X$; $\partial y = \partial Y$

Integrating $\boxed{\frac{z^2}{2} = Z \ ;\ x = X \ ;\ y = Y}$

$$(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = X^2 + Y^2$$
$$P^2 + Q^2 = X^2 + Y^2 \qquad \because P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$$

This is of the form $F_1(X, P) = F_2(Y, Q)$

To find Complete Integral:

Let $P^2 - X^2 = Y^2 - Q^2 = a^2$ (say)

$\therefore P^2 - X^2 = a^2$; $Y^2 - Q^2 = a^2$

$\therefore P^2 = a^2 + X^2$; $Q^2 = Y^2 - a^2$

$$\boxed{P = \sqrt{a^2 + X^2} \ ;\ Q = \sqrt{Y^2 - a^2}}$$

Substitute the value of p & q in

$$z = \int \sqrt{X^2 + a^2}\, dX + \int \sqrt{Y^2 - a^2}\, dY$$
$$Z = \frac{X}{2}\sqrt{X^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{X}{a}\right) + \frac{Y}{2}\sqrt{Y^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{Y}{a}\right) + c$$
$$\frac{z^2}{2} = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + c$$
$$z^2 = x\sqrt{x^2 + a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2 - a^2} + a^2\cosh^{-1}\left(\frac{y}{a}\right) + 2c$$
$$\because \int\sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right);\ \int\sqrt{y^2 - a^2}\, dy = \frac{y}{2}\sqrt{y^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)$$

This is the required complete integral

Other solutions can obtain as usual.

2. **Solve** $p^2 + x^2y^2q^2 = x^2z^2$

Solution:

Given $p^2 + x^2y^2q^2 = x^2z^2$

This is of the form equations reducible to standard form

$\div x$ on bothsides $\frac{p^2}{x^2} + y^2 q^2 = z^2$

$\left(\frac{p}{x}\right)^2 + (yq)^2 = z^2$

$\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y\partial z}{\partial y}\right)^2 = z^2 \qquad \because p = \frac{\partial z}{\partial x};\ q = \frac{\partial z}{\partial y}$

$\left(\frac{\partial z}{x\partial x}\right)^2 + \left(\frac{\partial z}{\frac{\partial y}{y}}\right)^2 = z^2 \quad ------(1) \qquad \because p = \frac{\partial z}{\partial x};\ q = \frac{\partial z}{\partial y}$

Let $\partial z = \partial Z \ ;\ x\partial x = \partial X \ ;\ \frac{\partial y}{y} = \partial Y$

Integrating $\boxed{z = Z \ ;\ \frac{x^2}{2} = X \ ;\ \log y = Y}$

$(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = Z^2$

$\boxed{P^2 + Q^2 = Z^2} \ ----(2) \qquad \because P = \frac{\partial Z}{\partial X};\ Q = \frac{\partial Z}{\partial Y}$

This is of the form $F(Z, P, Q) = 0$

To find Complete Integral:

Let the complete solution of (2) is $Z = F(X + aY)$

Let $X + aY = U \ \Rightarrow Z = F(U)$

Then $P = \frac{dZ}{dU}$ & $Q = a\frac{dZ}{dU}$

Substitute the value of P & Q in (2)

$(2) \Rightarrow \left(\frac{dZ}{dU}\right)^2 + \left(a\frac{dZ}{dU}\right)^2 = Z^2$

$\left(\frac{dZ}{dU}\right)^2 = \frac{Z^2}{(1+a^2)}$

Taking square root on both sides

$\frac{dZ}{dU} = \frac{Z}{\sqrt{1+a^2}}$

$\frac{dZ}{Z} = \frac{dU}{\sqrt{1+a^2}}$

Integrating on both sides

$\log Z = \frac{1}{\sqrt{1+a^2}} U$

$\log Z = \frac{1}{\sqrt{1+a^2}} (X + aY)$

This is complete integral of (2)

$\boxed{\log z = \frac{1}{\sqrt{1+a^2}}\left(\frac{x^2}{2} + a\log y\right) + c}$

This is the required complete integral .

	Other solutions can be obtained as usual.
3.	**Find the complete integral of** $z^2\left(x^2p^2+q^2\right)=1$ **Solution:** Given $z^2\left(x^2p^2+q^2\right)=1$ This is of the form equations reducible to standard form $x^2p^2+q^2=\frac{1}{z^2}$ $(xp)^2+q^2=\frac{1}{z^2}$ $\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{z^2}\qquad \because p=\frac{\partial z}{\partial x};q=\frac{\partial z}{\partial y}$ $\left(\frac{\partial z}{\frac{\partial x}{x}}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{z^2}------(1)$ Let $\partial z=\partial Z$; $\frac{\partial x}{x}=\partial X$; $\partial y=\partial Y$ Integrating $z=Z$; $\log x=X$; $y=Y$ $(1)\Rightarrow\left(\frac{\partial Z}{\partial X}\right)^2+\left(\frac{\partial Z}{\partial Y}\right)^2=\frac{1}{Z^2}$ $\boxed{P^2+Q^2=\frac{1}{Z^2}}\;----(2)\qquad\because P=\frac{\partial Z}{\partial X};Q=\frac{\partial Z}{\partial Y}$ This is of the form $F(Z,P,Q)=0$ **To find Complete Integral:** Let the complete solution of (2) is $Z=F(X+aY)$ Let $X+aY=U\Rightarrow Z=F(U)$ This is of the form $F(Z,P,Q)=0$ Then $P=\frac{dZ}{dU}$ & $Q=a\frac{dZ}{dU}$ Substitute the value of P & Q in (2) $(2)\Rightarrow\left(\frac{dZ}{dU}\right)^2+\left(a\frac{dZ}{dU}\right)^2=\frac{1}{Z^2}$ $\left(\frac{dZ}{dU}\right)^2=\frac{1}{Z^2(1+a^2)}$ Taking square root on both sides $\frac{dZ}{dU}=\frac{1}{Z\sqrt{1+a^2}}$ $ZdZ=\frac{dU}{\sqrt{1+a^2}}$ Integrating on both sides $\frac{Z^2}{2}=\frac{1}{\sqrt{1+a^2}}U$ $\frac{Z^2}{2}=\frac{1}{\sqrt{1+a^2}}(X+aY)$

	This is the complete integral of (2) $\frac{z^2}{2} = \frac{1}{\sqrt{1+a^2}}\left(\log x + ay\right) + c$ This is the required complete integral . Other solutions can be obtained as usual.
4.	Solve $x^2p^2 + y^2q^2 = z^2$
5.	Solve $x^4p^2 + y^2zq = 2z^2$

Lagrange's Linear Differential Equations:

Equations of the form $Pp + Qq = R$ (or) $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$

Where P,Q,R are functions of x, y, z or constants.

Procedure :

1. Write the auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
2. Solve the auxiliary equation by using
   a) Method of grouping
   b) Method of multipliers

a) Method of grouping: In the auxiliary equation, if the variables can be separated in any pair of equations, then we get a solution of the form $u(x, y, z) = c_1$ & $v(x, y, z) = c_2$

$\therefore$ The general solution is $\phi(u, v) = 0$

b) Method of Multipliers:

i) Choose any three multipliers l, m, n which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$ then $ldx + mdy + ndz = 0$

Integrating this we get $u(x, y, z) = c_1$

ii) Choose another any three multipliers l', m', n' which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

If it is possible to choose l', m', n' such that $l'P + m'Q + n'R = 0$ then $l'dx + m'dy + n'dz = 0$

Integrating this we get $v(x, y, z) = c_2$

$\therefore$ The general solution is $\phi(u, v) = 0$

1.	**Solve** $x(y - z)p + y(z - x)q = z(x - y)$ **Solution:** Given $x(y - z)p + y(z - x)q = z(x - y)$ This is of the form $Pp + Qq = R$ Where $P = x(y - z)$; $Q = y(z - x)$; $R = z(x - y)$ The auxiliary equation be $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $\frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}$ ----(1) i) Choose the multipliers as (1,1,1) $(1) \Rightarrow \frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}$

$$= \frac{dx + dy + dz}{xy - xz + yz - xy + xz - yz} = \frac{dx + dy + dz}{0}$$

$\therefore dx + dy + dz = 0$

Integrating $\int dx + \int dy + \int dz = 0$

$x + y + z = c_1$

ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$(1) \Rightarrow \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} = \frac{dx + dy + dz}{0}$$

$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

Integrating $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$\log x + \log y + \log z = \log c_2$

$\log xyz = \log c_2 \quad \Rightarrow xyz = c_2$

$\therefore$The general solution is $\boxed{\phi\left(x + y + z, xyz\right) = 0}$

2. **Solve** $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

Solution:

Given $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

This is of the form $Pp + Qq = R$

Where $P = x(z^2 - y^2)$; $Q = y(x^2 - z^2)$; $R = z(y^2 - x^2)$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \quad ----(1)$$

i) Choose the multipliers as (x,y,z)

$$(1) \Rightarrow \frac{xdx}{x^2(z^2 - y^2)} = \frac{ydy}{y^2(x^2 - z^2)} = \frac{zdz}{z^2(y^2 - x^2)}$$

$$= \frac{xdx + ydy + zdz}{x^2z^2 - x^2y^2 + x^2y^2 - y^2z^2 + y^2z^2 - x^2z^2} = \frac{xdx + ydy + zdz}{0}$$

$\therefore xdx + ydy + zdz = 0$

Integrating $\int xdx + \int ydy + \int zdz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1^2}{2}$$

$x^2 + y^2 + z^2 = c_1^2$

ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

	$(1) \Rightarrow \dfrac{\frac{dx}{x}}{\frac{x}{x}(z^2-y^2)} = \dfrac{\frac{dy}{y}}{\frac{y}{y}(x^2-z^2)} = \dfrac{\frac{dz}{z}}{\frac{z}{z}(y^2-x^2)}$ $= \dfrac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{z^2-y^2+x^2-z^2+y^2-x^2} = \dfrac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$ $\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$ Integrating $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$ $\log x + \log y + \log z = \log c_2$ $\log xyz = \log c_2 \quad \Rightarrow xyz = c_2$ $\therefore$The general solution is $\phi\left(x^2+y^2+z^2, xyz\right) = 0$
3.	**Solve** $(x^2-y^2-z^2)\,p + 2xyq = 2zx$ **Solution:** Given $(x^2-y^2-z^2)\,p + 2xyq = 2zx$ This is of the form $Pp + Qq = R$ Where $P = (x^2-y^2-z^2)$; $Q = 2xy$; $R = 2zx$ The auxiliary equation be $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} ----(1)$ i) by method of grouping, from last two ratios $(1) \Rightarrow \frac{dy}{2xy} = \frac{dz}{2zx}$ $\frac{dy}{y} = \frac{dz}{z}$ Integrating $\int \frac{dy}{y} = \int \frac{dz}{z}$ $\log y = \log z + \log c_1$ $\log y - \log z = \log c_1$ $\log \frac{y}{z} = \log c_1$ $\boxed{\frac{y}{z} = c_1}$ ii) Choose the multipliers as (x,y,z) $(1) \Rightarrow \dfrac{xdx}{x\left(x^2-y^2-z^2\right)} = \dfrac{ydy}{y\left(2xy\right)} = \dfrac{zdz}{z\left(2zx\right)}$ $= \dfrac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \dfrac{xdx+ydy+zdz}{x^3+xy^2+xz^2}$

	$= \dfrac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$ From 3^{rd} and last ratio $\dfrac{dz}{(2zx)} = \dfrac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$ $\dfrac{dz}{z} = \dfrac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$ Integrating $\int \dfrac{dz}{z} = \int \dfrac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$ $\log z = \log\left(x^2 + y^2 + z^2\right) \quad \because \int \dfrac{f'(x)}{f(x)} dx = \log[f(x)]$ $\log z = \log\left(x^2 + y^2 + z^2\right) + \log c_2$ $\log z - \log\left(x^2 + y^2 + z^2\right) = \log c_2$ $\log \dfrac{z}{x^2 + y^2 + z^2} = \log c_2$ $\boxed{\dfrac{z}{x^2 + y^2 + z^2} = c_2}$ The general solution is $\phi\left(\dfrac{y}{z}, \dfrac{z}{x^2 + y^2 + z^2}\right) = 0$
4.	**Solve** $(3z - 4y)p + (4x - 2z)q = 2y - 3x$ **Hint:** The multipliers are (x, y, z) & $(2, 3, 4)$ The general solution is $\phi\left(x^2 + y^2 + z^2, 2x + 3y + 4z\right) = 0$
5.	**Solve** $(y - xz)p + (yz - x)q = (x + y)(x - y)$ **Hint:** The multipliers are (x, y, z) & $(y, x, 1)$ The general solution is $\phi\left(x^2 + y^2 + z^2, xy + z\right) = 0$
6.	**Solve** $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$ **Hint:** The multipliers are $\left(\dfrac{1}{x}, \dfrac{-1}{y}, \dfrac{1}{z}\right)$ & $(x, -y, -1)$ The general solution is $\phi\left(\dfrac{xz}{y}, x^2 - y^2 - 2z\right) = 0$

Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = f(x, y) \quad ----- \quad (1)$$

Let the differential operator $D = \dfrac{\partial}{\partial x}$ & $D' = \dfrac{\partial}{\partial y}$

$(1) \Rightarrow \left(D^2 + a_1 DD' + a_2 D'^2\right) z = f(x, y) \quad ----- \quad (2)$

The general solution of equation (2) is

z = complementary function+ Particular Integral=C.F+P.I

To find complementary Function:

1. Write the Auxiliary equation by putting $D = m, D' = 1, z = 1, \& \ RHS = 0$

   $(2) \Rightarrow m^2 + a_1 m + a_2 = 0$

2. Solve the auxiliary equation, we get the roots of m. Say the roots are m_1, m_2
3. Comparing the roots of m and write the complementary function.

   Case 1: The Roots are real and distinct : say $m_1 \neq m_2$

   $C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

   Case 2: The Roots are real and equal : say $m_1 = m_2 = m$

   $C.F = f_1(y + mx) + x f_2(y + mx)$

Note: If the roots are $m = \alpha \pm i\beta$

then $C.F = f_1[y + (\alpha + i\beta)] + f_2[y + (\alpha - i\beta)]$

To find Particular Integral :

Type : I

If $RHS = e^{ax+by}$ then

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

Rule: $D = a \ \& \ D' = b$

$$P.I = \frac{1}{f(a,b)} e^{ax+by}, \text{ Provided Denominator} \neq 0$$

If Denominator = 0 , then 1) multiply the numerator by x 2) differentiating denominator partially w.r.to D

$$P.I = \frac{x}{f'(D,D')} e^{ax+by}$$

$$P.I = \frac{1}{f'(a,b)} e^{ax+by}, \text{ Provided Denominator} \neq 0 \because \text{ Replace } D = a \ \& \ D' = b$$

Continuing this process until we get $Dr \neq 0$.

Type : II

If $RHS = \sin(ax + by)$ (or) $RHS = \cos(ax + by)$ then

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by)$$

Rule: $D^2 = -(a^2); DD' = (-ab); D'^2 = -(b^2)$

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by), \text{ Provided Dr} \neq 0$$

Note 1: After substitutions the denominator will be in terms of $D \ \& \ D'$. Multiply and divide by D so that the denominator will have $D^2 \ \& \ DD'$ terms.

Note 2: After substitutions the denominator will be in terms of D & constant terms,

For eg. $P.I = \frac{1}{D-5} \sin(x - 2y)$

Take conjugate of denominator with constant term and multiplied with both numerator and denominator.

$$P.I = \frac{1}{D-5} \times \frac{D+5}{D+5} \sin(x - 2y) = \frac{1}{D^2 - 25} \sin(x - 2y)$$

Then apply the rule as usual.

Type : III

If $RHS = x^m y^n$ (polynomial type) **then**

$P.I = \frac{1}{f(D, D)} x^m y^n$, we bring this into a standard binomial format, by taking out highest power term of D.

$(i.e)\ P.I = [1 \pm f(D, D')]^{-1} x^m y^n$

This will be expanded by using the formulae

$(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + ...$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...$

Note:

$\frac{1}{D} = \int dx$, $D' = \frac{\partial}{\partial y}$

Type : IV (Exponential shifted rule)

If $RHS = e^{ax+by}\cos(ax+by)$ (or) $RHS = e^{ax+by}\sin(ax+by)$ (or) $RHS = e^{ax+by} x^m y^n$ then

$$P.I = \frac{1}{f(D, D')} e^{ax+by}\sin(ax+by)$$

$$P.I = e^{ax+by} \frac{1}{f(D+a, D'+b)}\sin(ax+by)$$

Here after apply the rule as we discussed in Type II&III

Type : V

If $RHS = y\cos x$ (or) $RHS = y\sin x$ then

Case 1:

$$P.I = \frac{1}{D - mD'} y\cos x$$

$$P.I = \frac{1}{D - mD'}\int (c - mx)\cos x\,dx \qquad \because Rule : y = c - mx$$

Case 2:

$$P.I = \frac{1}{D + mD'} y\cos x$$

$$P.I = \frac{1}{D + mD'}\int (c + mx)\cos x\,dx \qquad \because Rule : y = c - mx$$

Note: After integration we have to replace $c - mx = y$

1. **Solve** $\left(D^2 - DD' - 20D'^2\right)z = e^{5x+y} + \sin(4x - y)$

Solution:

Given $\left(D^2 - DD' - 20D'^2\right)z = e^{5x+y} + \sin(4x - y)$

To find C.F

The auxiliary equation is

$m^2 - m - 20 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$

$(m-5)(m+4) = 0$

$m = 5, -4$

$\therefore C.F = f_1(y - 4x) + f_2(y + 5x)$ $\because$ The roots are real and distinct $\Rightarrow C.F = f_1(y + mx) + xf_2(y + mx)$

To find P.I

$$P.I = \frac{1}{D^2 - DD' - 20D'^2}\left[e^{5x+y} + \sin(4x - y)\right]$$

$P.I = P.I_1 + P.I_2$

$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$ Rule: replace $D = 5 \,\&\, D' = 1$ Type:1

$= \frac{1}{25 - 5 - 20} e^{5x+y} = \frac{1}{0} e^{5x+y}$ $\because$ Introducing x in Nr.Diff Dr.partially w.r.to D

$= \frac{x}{2D - D'} e^{5x+y}$ Rule: replace $D = 5 \,\&\, D' = 1$

$= \frac{x}{10 - 1} e^{5x+y}$

$P.I_1 = \frac{x}{9} e^{5x+y}$

$P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x - y)$ here $a = 4 \,\&\, b = -1$ (Type:2)

Rule: replace $D^2 = -(a^2) = -16$; $D'^2 = -(b^2) = -1 \,\&\, DD' = -(ab) = 4$

$P.I_2 = \frac{1}{-16 - 4 - 20(-1)} \sin(4x - y) = \frac{1}{0} \sin(4x - y)$

$= \frac{x}{2D - D'} \sin(4x - y)$ $\because$ Introducing x in Nr.Diff Dr.partially w.r.to D

$= \frac{x}{2D - D'} \times \frac{D}{D} \sin(4x - y)$

$= \frac{xD}{2D^2 - DD'} \sin(4x - y)$

$= \frac{xD}{2(-16) - 4} \sin(4x - y)$

$= \frac{xD\left(\sin(4x - y)\right)}{-32 - 4}$

$P.I_2 = \frac{4x\cos(4x + 3)}{-36} = \frac{-1}{9} x\cos(4x + 3)$ $\because \frac{d}{dx}(\sin nx) = n\cos nx$

$P.I_2 \ \frac{-1}{9} x\cos(4x + 3)$

$P.I = \frac{x}{9} e^{5x+y} - \frac{1}{9} x\cos(4x + 3) = \frac{x}{9}\left[e^{5x+y} - \cos(4x + 3)\right] 9$

The general solution is

$z = C.F + P.I$

$z = f_1(y - 4x) + f_2(y + 5x) + \frac{x}{9}\left[e^{5x+y} - \cos(4x + 3)\right]$

2.

Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4\sin(x + y)$

Solution: same as previous problem

Hint:

Given $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4\sin(x + y)$

$\left(D^3 - 2D^2D'\right) z = e^{x+2y} + 4\sin(x + y)$ $\because D = \frac{\partial}{\partial x} \,\&\, D' = \frac{\partial}{\partial y}$

$m = 0, 0, 2$

$C.F = f_1(y) + x f_2(y) + f_2(y + 2x)$

$$P.I = \frac{-1}{3}e^{x+2y} - 4\cos(x+2y)$$

3. **Solve** $\left(D^2 + DD' - 6D'^2\right)z = x^2y + e^{3x+y}$

Solution:

Given $\left(D^2 + DD' - 6D'^2\right)z = x^2y + e^{3x+y}$

To find C.F

The auxiliary equation is

$m^2 + m - 6 = 0$ $\qquad \because$ replace $D = m, D' = 1, z = 0$

$(m-2)(m+3) = 0$

$m = 2, -3$

$\therefore C.F = f_1(y-3x) + f_2(y+2x)$ $\qquad \because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+m_1x) + f_2(y+m_2x)$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2}\left(x^2y + e^{3x+y}\right)$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 + DD' - 6D'^2}x^2y \qquad \text{(Type:3)}$$

$$= \frac{1}{D^2\left[1 + \left(\frac{DD' - 6D'^2}{D^2}\right)\right]}x^2y$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1}x^2y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right) + \ldots\right]x^2y \qquad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D} + \frac{6D'^2}{D^2}\right]x^2y \qquad \because D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2}\left[x^2y - \frac{D'(x^2y)}{D} + \frac{6D'^2(x^2y)}{D^2}\right]$$

$$= \frac{1}{D^2}\left[x^2y - \frac{x^2}{D}\right]$$

$$= \frac{1}{D^2}\left[x^2y - \int x^2\,dx\right] \qquad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D^2}\left[x^2y - \frac{x^3}{3}\right]$$

$$= \frac{1}{D}\int\left[x^2y + \frac{x^3}{3}\right]dx$$

$$= \int\left[\frac{x^3y}{3} - \frac{x^4}{12}\right]dx$$

$$\boxed{P.I_1 = \frac{x^4y}{12} - \frac{x^5}{60}}$$

$$P.I_2 = \frac{1}{D^2 + DD' - 6D'^2}e^{3x+y} \qquad \text{Rule: replace } D = a = 3 \;\&\; D' = b = 1 \qquad \text{Type:1}$$

$$= \frac{1}{9+3-6(1)} e^{3x+y} = \frac{1}{6} e^{3x+y}$$

$$P.I_2 = \frac{1}{6} e^{3x+y}$$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-3x) + f_2(y+2x) + \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

4.

Solve $\left(D^2 + 2DD' + D'^2\right) z = x^2 y + e^{x-y}$

Solution: same as previous problem

Hint:

$$m = -1, -1$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2} e^{x-y}$$

5.

Solve $\left(D^2 - 6DD' + 5D'^2\right) z = xy + e^x \sinh y$

Solution:

Given $\left(D^2 - 6DD' + 5D'^2\right) z = xy + e^x \sinh y$

$$= xy + e^x \left(\frac{e^y - e^{-y}}{2}\right) \qquad \because \boxed{\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}}, \text{ here } \theta = y$$

$$= xy + \left(\frac{e^x e^y - e^x e^{-y}}{2}\right)$$

$$\left(D^2 - 6DD' + 5D'^2\right) z = xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \qquad \because e^a e^b = e^{a+b}$$

To find C.F

The auxiliary equation is

$$m^2 - 6m + 5 = 0 \qquad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m-1)(m-5) = 0$$

$$m = 1, 2$$

$$\therefore C.F = f_1(y+x) + f_2(y+5x) \qquad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y+m_1 x) + x f_2(y+m_2 x)$$

To find P.I

$$P.I = \frac{1}{D^2 - 6DD' + 5D'^2}\left(xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2}\right)$$

$$P.I = P.I_1 + P.I_2 - P.I_3 - - - - - (1)$$

To find P.I$_1$

$$P.I_1 = \frac{1}{D^2 - 6DD' + 5D'^2} xy \qquad \text{(Type:3)}$$

$$= \frac{1}{D^2\left[1 + \left(\frac{-6DD' + 5D'^2}{D^2}\right)\right]} xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{-6D'}{D}+\frac{5D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{1}{D^2}\left[1-\left(\frac{-6D'}{D}+\frac{5D'^2}{D^2}\right)+\dots\right] xy \qquad \because (1+x)^{-1} = 1-x+x^2-x^3+\dots$$

$$= \frac{1}{D^2}\left[1+\frac{6D'}{D}-\frac{5D'^2}{D^2}+\dots\right] xy$$

$$= \frac{1}{D^2}\left[xy+\frac{6x}{D}-\frac{5(0)}{D^2}+\dots\right] xy \qquad \because D'(y)=\frac{\partial}{\partial y}(y)=1 \;\&\; D'^2(y)=\frac{\partial^2}{\partial y^2}(y)=0$$

$$= \frac{1}{D^2}\left[xy+6\int x\,dx\right] = \frac{1}{D^2}\left[xy+\frac{6x^2}{2}\right] \qquad \because \frac{1}{D}=\int dx$$

$$= \frac{1}{D}\int\left(xy+3x^2\right)dx = \frac{1}{D}\left[\frac{x^2y}{2}+\frac{3x^3}{3}\right] = \int\left[\frac{x^2y}{2}+x^3\right]dx$$

$$\boxed{P.I_1 = \frac{x^3y}{6}+\frac{x^4}{4}}$$

To find P.I₂

$$P.I_2 = \frac{1}{D^2-6DD'+5D'^2}\,\frac{e^{x+y}}{2} \qquad \text{Type:1}$$

$$= \left(\frac{1}{2}\right)\frac{1}{1-6+5}e^{x+y} = \left(\frac{1}{2}\right)\frac{1}{0}e^{x+y} \qquad \text{Rule: replace } D=a=1 \;\&\; D'=b=1$$

$$= \left(\frac{1}{2}\right)\frac{x}{2D-6D'+0}e^{x+y}$$

$$= \left(\frac{1}{2}\right)\frac{x}{2-6}e^{x+y}$$

$$= \left(\frac{1}{2}\right)\frac{-x}{4}e^{x+y}$$

$$P.I_2 = \frac{-x}{8}e^{x+y}$$

To find P.I₃

$$P.I_3 = \frac{1}{D^2-6DD'+5D'^2}\,\frac{e^{x-y}}{2} \qquad \text{Type:1}$$

$$= \left(\frac{1}{2}\right)\frac{1}{1+6+5}e^{x+y} \qquad \text{Rule: replace } D=a=1 \;\&\; D'=b=-1$$

$$P.I_3 = \frac{1}{24}e^{x+y}$$

$$(1) \Rightarrow P.I = P.I_1+P.I_2-P.I_3 = \boxed{\frac{x^3y}{6}+\frac{x^4}{4}-\frac{x}{8}e^{x+y}-\frac{1}{24}e^{x+y}}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x)+f_2(y+5x)+\frac{x^3y}{6}+\frac{x^4}{4}-\frac{x}{8}e^{x+y}-\frac{1}{24}e^{x+y}$$

6. **Solve** $\left(D^2+2DD'+D'^2\right)z = \sinh(x+y)+e^{x+2y}$

Solution: same as previous problem

Hint:

$$\left(D^2+2DD'+D'^2\right)z=\sinh(x+y)+e^{x+2y}$$

$$\left(D^2+2DD'+D'^2\right)z=\frac{e^{x+y}-e^{(x-y)}}{2}+e^{x+2y}=\frac{e^{x+y}}{2}-\frac{e^{-x-y}}{2}+e^{x+2y}$$

$$\left(D^2+2DD'+D'^2\right)z=\frac{e^{x+y}}{2}-\frac{e^{-x-y}}{2}+e^{x+2y}$$

$m=-1,-1$

$C.F=f_1(y-x)+xf_2(y-x)$

$$P.I=\frac{e^{x+y}}{8}-\frac{e^{-x-y}}{8}-\frac{e^{x+2y}}{9}$$

7. Solve $\left(D^2-4DD'+4D'^2\right)z=e^{x+2y}$

Solution:

Hint: $m=2,\,2$

$C.F=f_1(y+x)+f_2(y+5x)$

$$P.I=\frac{x^2}{2}e^{2x+y}$$

8. Solve $\left(2D^2-5DD'+2D'^2\right)z=5\sin(2x+y)$

Solution:

Given $\left(2D^2-5DD'+2D'^2\right)z=5\sin(2x+y)$

To find C.F

The auxiliary equation is

$2m^2-5m+2=0$ $\quad\because$ replace $D=m, D'=1, z=0$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\qquad \text{here } a=2,\ b=-5,\ c=2$$

$$=\frac{-(-5)\pm\sqrt{25-4(2)(2)}}{2(2)}$$

$$=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm\sqrt{9}}{4}$$

$$=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm\sqrt{9}}{4}=\frac{5\pm 3}{4}$$

$$m=\frac{5+3}{4},\ m=\frac{5-3}{4}\Rightarrow \boxed{m=2,\ m=\frac{1}{2}}$$

$\therefore C.F=f_1(y+2x)+f_2\left(y+\frac{x}{2}\right)$ $\quad\because$ The roots are real and distinct $\Rightarrow C.F=f_1(y+m_1x)+f_2(y+m_2x)$

To find P.I

$$P.I=\frac{1}{2D^2-5DD'+2D'^2}5\sin(2x+y)\qquad \text{Type:2}$$

$$P.I=\frac{1}{2D^2-5DD'+2D'^2}5\sin(2x+y)\qquad \text{here } a=2\ \&\ b=1$$

Rule: replace $D^2=-(a^2)=-4;\ D'^2=-(b^2)=-1\ \&\ DD'=-(ab)=-2$

$$P.I=\frac{1}{2(-4)-5(-2)+2(-1)}5\sin(2x+y)$$

$$P.I=\frac{1}{-8+10-2}5\sin(2x+y)=\frac{1}{0}5\sin(2x+y)$$

$$= \frac{x}{4D - 5D' + 0} 5\sin(2x + y) \qquad \{\text{Introducing } x \text{ in Nr.\& Diff } D\text{r.partially w.r.to } D\}$$

$$= \frac{x}{4D - 5D'} \times \frac{D}{D} 5\sin(2x + y)$$

$$= 5\frac{xD}{4D^2 - 5DD'} \sin(2x + y)$$

$$= 5\frac{xD}{4(-4) - 5(-2)} \sin(2x + y) = 5\frac{xD\left[\sin(2x + y)\right]}{-16 + 10}$$

$$= \frac{5}{-6} x \left[2\cos(2x + y)\right]$$

$$P.I = \frac{-5}{3} x \cos(2x + y)$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y + 2x) + f_2\left(y + \frac{x}{2}\right) - \frac{5}{3} x \cos(2x + y)$$

9.

Solve the equation $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right) z = \cos(2x + y)$

Solution:

Given $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right) z = \cos(2x + y)$

To find C.F

The auxiliary equation is

$$m^3 + m^2 - 4m - 4 = 0 \qquad \because \text{replace } D = m, D' = 1, z = 0$$

$$m^2(m + 1) - 4(m + 1) = 0$$

$$(m + 1)\left(m^2 - 4\right) = 0$$

$$m = -1, -2, 2$$

$$\therefore C.F = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x)$$

To find P.I

$$P.I = \frac{1}{D^3 + D^2D' - 4DD'^2 - 4D'^3} \cos(2x + y) \qquad here\ a = 2, b = 1 \ (\text{Type:2})$$

Rule: replace $D^2 = -(a^2) = -4;\ D'^2 = -(b^2) = -1 \ \&\ DD' = -(ab) = -2$

$$P.I = \frac{1}{-4D - 4D' - 4D(-1) - 4(-1)D'} \cos(2x + y)$$

$$= \frac{1}{-4D - 4D' + 4D + 4D'} \cos(2x + y) = = \frac{1}{0} \cos(2x + y)$$

$$P.I = \frac{x}{3D^2 + 2DD' - 4D'^2 - 0} \cos(2x + y) \qquad \{\text{Introducing } x \text{ in Nr.\& Diff } D\text{r.partially w.r.to } D\}$$

$$= \frac{x}{3(-4) + 2(-2) - 4(-1)} \cos(2x + y) = \frac{x}{-12 - 4 + 4} \cos(2x + y)$$

$$P.I = \frac{-x}{12} \cos(2x + y)$$

The general solution is

$$z = C.F + P.I$$

$$\boxed{z = f_1(y - x) + f(y - 2x) + f(y + 2x) - \frac{x}{12} \cos(2x + y)}$$

10. **Solve** $\left(D^3 - 7DD'^2 - 6D'^3\right)z = \cos(x+2y) + 4$

Solution:

Given $\left(D^3 - 7DD'^2 - 6D'^3\right)z = \cos(x+2y) + 4$

To find C.F

The auxiliary equation is

$m^3 - 7m - 6 = 0 \qquad \because \text{replace } D = m, D' = 1, z = 0$

m=-1	1	0	-7	6
	0	-1	1	6
	1	-1	-6	0

$m = -1,\ m^2 - m - 6 = 0$

$m = -1,\ (m-3)(m+2) = 0$

$\boxed{m = -1, -2, 3}$

$\therefore \boxed{C.F = f_1(y-x) + f_2(y-2x) + f_3(y+3x)}$

To find P.I

$P.I = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\left[\cos(x+2y) + 4\right]$

$P.I = P.I_1 + P.I_2$

$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\cos(x+2y) \qquad here\ a = 1, b = 2 \quad (\text{Type:2})$

Rule: replace $D^2 = -(a^2) = -1;\ D'^2 = -(b^2) = -4\ \&\ DD' = -(ab) = -2$

$P.I_1 = \frac{1}{(-1)D - 7D(-2) - 6(-2)D'}\cos(x+2y) = \frac{1}{-D + 14D + 12D'}\cos(x+2y)$

$= \frac{1}{13D + 12D'} \times \frac{D}{D}\cos(x+2y)$

$= \frac{D}{13D^2 + 12D'D}\cos(x+2y)$

$= \frac{D}{13(-1) + 12(-2)}\cos(x+2y) = \frac{D\cos(x+2y)}{-13-24}$

$\boxed{P.I_1 = \frac{-\sin(x+2y)}{37}}$

To find P.I₂

$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}4e^{0x+0y} \qquad \because e^0 = 1 \quad \text{Type:1}$

$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}4e^{0x+0y} = \frac{1}{0}4e^{0x+0y} \qquad \text{Rule: Replace } D = 0,\ D' = 0$

$= \frac{x}{3D^2 - 7D'^2 - 0}4e^{0x+0y} = \frac{x}{0}4e^{0x+0y}$

$= \frac{x}{6D - 0}4e^{0x+0y} = \frac{x}{0}4e^{0x+0y}$

$= \frac{x^2}{6}4$

$$P.I_2 = \frac{2x^2}{3}$$

$$P.I = \frac{-1}{37}\sin(x+2y) + \frac{2x^2}{3}$$

The general solution is

$z = C.F + P.I$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{37}\sin(x+2y) + \frac{2x^2}{3}$$

11. **Solve** $\left(D^2 + 3DD' - 4D'^2\right)z = xy + \cos(2x+y)$

Solution: same as previous problem

Hint:

$m = -4,1$

$C.F = f_1(y-4x) + xf_2(y+x)$

$P.I = \frac{-1}{6}\cos(2x+y) + \frac{x^3y}{6} - \frac{x^4}{8}$

12. **Solve** $\left(D^2 + 3DD' - 4D'^2\right)z = x + \sin y$

Hint:

$m = -4,1$

$C.F = f_1(y-4x) + xf_2(y+x)$

$P.I = \frac{1}{D^2 + 3DD' - 4D'^2}\left[x + \sin(0x+y)\right] = \dots = \frac{x^3}{6} + \frac{\sin y}{4}$

13. **Solve** $\left(D^2 - DD' - 2D'^2\right)z = \left(2x+3y\right) + e^{3x+4y}$

Hint:

$m = -1, 2$

$C.F = f_1(y-x) + f_2(y+2x)$

$P.I = \frac{5x^3}{6} + \frac{3x^2y}{2} + \frac{1}{35}e^{3x+4y}$

14. Solve $\left(D^2 - 2DD' + D'^2\right)z = (2+4x)e^{x+2y}$

Solution:

Given $\left(D^2 - 2DD' + D'^2\right)z = x^2y^2e^{x+2y}$

To find C.F

The auxiliary equation is

$m^2 - 2m + 1 = 0$ $\quad\because$ replace $D = m,\ D' = 1, z = 0$

$(m-1)(m-1) = 0$

$m = 1,1$

$\therefore$ $C.F = f_1(y+x) + xf_2(y+x)$

To find P.I

$P.I = \frac{1}{D^2 - 2DD' + D'^2}(2+4x)e^{x+2y}$ $\quad here\ a = 2, b = 1$ (Type:4)

$P.I = \frac{1}{(D-D')^2}(2+4x)\,e^{x+2y}$

Rule:replace $D = D + a = D + 1;\ D' = D' + b = D' + 2$

$P.I = e^{x+2y}\frac{1}{(D+1-D'-2)^2}(2+4x)$

$$= e^{x+2y}\frac{1}{(D-D'-1)^2}(2+4x)$$

$$= e^{x+2y}\frac{1}{[-(1-D+D')]^2}(2+4x) = e^{x+2y}\frac{1}{[1-(D-D')]^2}(2+4x)$$

$$= e^{x+2y}[1-(D-D')]^{-2}(2+4x)$$

$$= e^{x+2y}\left[1+2(D-D')+3(D-D')^2+\ldots\right](2+4x) \qquad \because (1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots$$

$$= e^{x+2y}\left[1+2D-2D'+3\left(D^2-2DD'+D'^2\right)+\ldots\right](2+4x)$$

$$= e^{x+2y}\left[1+2D-2D'+3D^2-6DD'+3D'^2\right](2+4x)$$

$= e^{x+2y}\left[1+2D+3D^2\right](2+4x)$ $\because$ there is no y term in RHS, neglect the term D'

$$= e^{x+2y}\left[(2+4x)+2D(2+4x)+3D^2(2+4x)\right]$$

$$= e^{x+2y}\left[2+4x+2(4)+0\right]$$

$$P.I = e^{x+2y}\left[4x+10\right]$$

The general solution is

$$z = C.F + P.I$$

$$\boxed{z = f_1(y+x) + x f_2(y+x) + e^{x+2y}[4x+10]}$$

15. **Solve** $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = e^{x-y}\sin(2x+3y)$

Solution:

Given $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = e^{x-y}\sin(2x+3y)$

$$\left(D^2 - D'^2\right) z = e^{x-y}\sin(2x+3y)$$

To find C.F

The auxiliary equation is

$m^2 - 1 = 0$ $\qquad\because$ replace $D = m, D' = 1, z = 0$

$m^2 = 1 \Rightarrow m = -1, 1$

$\therefore C.F = f_1(y-x) + f_2(y+x)$

To find P.I

$P.I = \dfrac{1}{D^2 - D'^2} e^{x-y}\sin(2x+3y)$ $\quad$ *here* $a = 1, b = -1$ (Type:4)

Rule:replace $D = D + a = D + 1$; $D' = D' + b = D' - 1$

$$P.I = e^{x-y}\frac{1}{(D+1)^2-(D'-1)^2}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{D^2+2D+1-D'^2+2D'-1}\sin(2x+3y)$$

$= e^{x-y}\dfrac{1}{D^2+2D-D'^2+2D'}\sin(2x+3y)$ Here $a = 2$, $b = 3$

Rule: replace $D^2 = -(a^2) = -4$; $D'^2 = -(b^2) = -9$ & $DD' = -(ab) = -6$

$$= e^{x-y}\frac{1}{-4+2D-(-9)+2D'}\sin(2x+3y) = e^{x-y}\frac{1}{2D+2D'+5}\sin(2x+3y)$$

$$= e^{x-y}\frac{1}{2D+2D'+5}\times\frac{D}{D}\sin(2x+3y)$$

$$= e^{x-y}\frac{D}{2D^2+2DD'+5D}\sin(2x+3y)$$

$$= e^{x-y}\frac{D}{-8-12+5D}\sin(2x+3y) = e^{x-y}\frac{D}{5D-20}\sin(2x+3y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constent term and multiplied with both Nr. & Dr.

$$= e^{x-y}\frac{D}{5D-20}\times\frac{5D+20}{5D+20}\sin(2x+3y)$$

$$= e^{x-y}\frac{5D^2+20D}{25D^2-400}\sin(2x+3y)$$

$$= e^{x-y}\frac{5D\sin(2x+3y)+20D\sin(2x+3y)}{25(-4)-400}$$

$$= e^{x-y}\frac{5D\cos(2x+3y)\times 2+20\cos(2x+3y)\times 2}{-100-400}$$

$$= e^{x-y}\frac{-20\sin(2x+3y)+40\cos(2x+3y)}{-500}$$

$$P.I = \frac{e^{x-y}}{25}\left[\sin(2x+3y)-2\cos(2x+3y)\right]$$

The general solution is

$z = C.F + P.I$

$$z = f_1(y-x)+f_2(y+x)+\frac{e^{x-y}}{25}\left[\sin(2x+3y)-2\cos(2x+3y)\right]$$

16. **Solve** $\left(D^2+DD'-6D'^2\right)z = y\cos x$

Solution:

Given $\left(D^2+DD'-6D'^2\right)z = y\cos x$

To find C.F

The auxiliary equation is

$m^2+m-6=0$ $\quad\because$ replace $D=m, D'=1, z=0$

$(m-2)(m+3)=0$

$m = 2, -3$

$\therefore C.F = f_1(y-3x)+f_2(y+2x)$ $\quad\because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+m_1x)+f_2(y+m_2x)$

To find P.I

$$P.I = \frac{1}{D^2+DD'-6D'^2}\,y\cos x \qquad \text{Type: 5}$$

$$= \frac{1}{\left(\cos x D+3D'\right)\left(D-2D'\right)}y$$

$$= \frac{1}{\left(D+3D'\right)}\int (c-2x)\cos x\,dx \qquad \because Rule: y = c-mx \text{ here } m=2$$

$$= \frac{1}{\left(D+3D'\right)}\left[(c-2x)(\sin x)-(-2)(-\cos x)\right] \qquad \because \int uv\,dx = uv_1 - u'v_2 + \ldots$$

$$= \frac{1}{\left(\cos x\right]D+3D')}\left[y\sin x-2\right.$$

$$= \frac{1}{\left(D+3D'\right)}\int\left[(c+3x)\sin x-2\cos x\right]dx \qquad \because Rule: y = c+mx \text{ here } m=3$$

$$= (c+3x)(-\cos x)-(3)(-\sin x)-2\sin x \qquad \because y = c+3x$$

$= -y\cos x + 3\sin x - 2\sin x$

$P.I = \sin x - y\cos x$

The general solution is

$z = C.F + P.I$

$z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y\cos x$

17. **Solve** $\left(D^2 - 5DD' + 6D'^2\right) z = y\sin x$

Solution:

Given $\left(D^2 - 5DD' + 6D'^2\right) z = y\sin x$

To find C.F

The auxiliary equation is

$m^2 - 5m + 6 = 0$ $\qquad \because$ replace $D = m, D' = 1, z = 0$

$(m-2)(m-3) = 0$

$m = 2, 3$

$\therefore C.F = f_1(y + 3x) + f_2(y + 2x)$ $\qquad \because$ The roots are real and distinct $\Rightarrow C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

To find P.I

$P.I = \dfrac{1}{D^2 - 5DD' + 6D'^2}\, y\sin x$ $\qquad$ Type: 5

$= \dfrac{1}{\left(D - 3D'\right)\left(D - 2D'\right)}\, y\sin x$

$= \dfrac{1}{\left(D - 3D'\right)} \int (c - 2x)\sin x\, dx$ $\qquad \because$ *Rule* : $y = c - mx$ here $m = 2$

$= \dfrac{1}{\left(D - 3D'\right)} \left[(c - 2x)(-\cos x) - (-2)(-\sin x)\right]$ $\qquad \because \int uv\,dx = uv_1 - u'v_2 + \ldots$

$= \dfrac{1}{\left(D - 3D'\right)} \left[-y\cos x - 2\sin x\right] = \dfrac{-1}{\left(D - 3D'\right)} \left[y\cos x + 2\sin x\right]$

$= -\int \left[(c - 3x)\cos x + 2\sin x\right] dx$ $\qquad \because$ *Rule* : $y = c - mx$ here $m = 3$

$= -\left[(c - 3x)(\sin x) - (-3)(-\cos x) + 2(-\cos x)\right]$ $\qquad \because y = c - 3x$

$= -\left[y\sin x - 3\cos x - 2\cos x\right]$

$P.I = 5\cos x - y\sin x$

The general solution is

$z = C.F + P.I$

$z = f_1(y + 3x) + f_2(y + 2x) + 5\cos x - y\sin x$

Non-Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order non-homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} + a_3 \frac{\partial z}{\partial x} + a_4 \frac{\partial z}{\partial y} = f(x, y) \quad ----- \quad (1)$$

Let the differential operator $D = \dfrac{\partial}{\partial x}$ & $D' = \dfrac{\partial}{\partial y}$

$$(1) \Rightarrow \left(D^2 + a_1 DD' + a_2 D'^2 + a_3 D + a_4 D'\right) z = f(x, y) \quad ----- (2)$$

The general solution of equation (2) is

z = complementary function+ Particular Integral=C.F+P.I

To find complementary Function:

Case : I

The given PDE will bring into the form of $(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0$

$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$

Case : II

The given PDE will bring into the form of $(D - mD' - C)^2 z = 0$

$C.F = e^{cx} f_1(y + mx) + xe^{cx} f_2(y + mx)$

Note: Particular Integral can be obtained, similar like in Homogeneous types.

1. **Solve** $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right) z = e^{3x+y} + \sin(x + 2y)$

Solution:

Given $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right) z = e^{3x+y} + \sin(x + 2y)$

To find C.F

$\left((D + D')^2 - 2(D + D')\right) z = 0$

$(D + D')(D + D' - 2)z = 0 - - - - - (1)$

This is of the form

$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 - - - - - (2)$

Comparing (1) & (2)

$m_1 = -1, C_1 = 0, m_2 = -1, C_2 = 2$.

$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$

$C.F = f_1(y - x) + e^{2x} f_2(y - x) \qquad \because e^0 = 1$

To find P.I

$$P.I = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'}\left(e^{3x+y} + \sin(x + 2y)\right)$$

$P.I = P.I_1 + P.I_2$

To find P.I$_1$

$$P.I_1 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} e^{3x+y} \qquad \text{Rule:replace } D = 3 \ \& \ D' = 1 \text{ Type:1}$$

$$= \frac{1}{9 + 6 + 1 - 6 - 2} e^{3x+y}$$

$$P.I_1 = \frac{1}{8} e^{3x+y}$$

To find P.I$_2$

$$P.I_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y) \qquad \text{here } a = 1, b = 2 \quad \text{type:2}$$

Rule: replace $D^2 = -(a^2) = -1;\ D'^2 = -(b^2) = -4 \ \& \ DD' = -(ab) = -2$

$$P.I_2 = \frac{1}{-1 + 2(-2) - 4 - 2D - 2D'} \sin(x + 2y) = \frac{1}{-1 - 4 - 4 - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{-1}{2D + 2D' + 9} \times \frac{D}{D} \sin(x + 2y)$$

$$= \frac{-D}{2D^2 + 2D'^2 + 9D} \sin(x + 2y)$$

$$= \frac{-D}{-2 - 8 + 9D} \sin(x + 2y) = \frac{-D}{9D - 10} \sin(x + 2y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constent term and multiplied with both Nr. & Dr.

$$= \frac{-D}{9D - 10} \times \frac{9D + 10}{9D + 10} \sin(x + 2y)$$

$$= \frac{-9D^2 - 10}{81D^2 - 100}\sin(x+2y)$$

$$= \frac{-9D^2\sin(x+2y) - 10D\sin(x+2y)}{-81-100}$$

$$= \frac{1}{-181}\left[-9D\cos(x+2y) - 10\cos(x+2y)\right]$$

$$P.I_2 = \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]$$

$$P.I = \frac{1}{8}e^{3x+y} + \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]$$

The general solution is

$z = C.F + P.I$

$$z = f_1(y-x) + e^{2x}f_2(y-x) + \frac{1}{8}e^{3x+y} + \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]$$

2.

Solve $\left(D^2 - D'^2 - 3D + 3D'\right)z = e^{3x+y} + 4$

Solution:

Given $\left(D^2 - D'^2 - 3D + 3D'\right)z = e^{3x+y} + 4$

To find C.F

$\left((D+D')(D-D') - 3(D-D')\right) z = 0$

$(D-D')(D+D'-3)z = 0 - - - - - (1)$

This is of the form

$(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0 - - - - - (2)$

Comparing (1) & (2)

$m_1 = 1,\ C_1 = 0,\ m_2 = -1,\ C_2 = 3.$

$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$

$C.F = f_1(y+x) + e^{3x} f_2(y-x) \qquad \because e^0 = 1$

To find P.I

$$P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'}\left(e^{3x+y} + 4\right)$$

$P.I = P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D^2 - D'^2 - 3D + 3D'}e^{3x+y} \qquad \text{here } a = 3, b = 1 \qquad \text{type : 1}$$

$$= \frac{1}{9-1-9+3}e^{3x+y} \qquad \text{Rule:Replace } D = 3, D' = 1$$

$$P.I_1 = \frac{1}{2}e^{3x+y}$$

$$P.I_2 = \frac{1}{D^2 - D'^2 - 3D + 3D'}4e^{0x+0y} \qquad \text{here } a = 0, b = 0 \qquad \text{type : 1}$$

$$= \frac{1}{0}4e^{0x+0y} \qquad \text{Rule:Replace } D = 0, D' = 0$$

Introduce x in Nr. and Diff. Dr. Partially w.r.to.D in the previous step

$$= \frac{x}{2D - 0 - 3 + 0}4e^{0x+0y}$$

$$= \frac{4x}{-3}$$

$$\boxed{P.I_2 = \frac{-4x}{3}}$$

$$P.I = \frac{1}{2}e^{3x+y} - \frac{4x}{3}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{1}{2}e^{3x+y} - \frac{4x}{3}$$

3. **Solve** $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right)z = xe^y$

Solution:

Given $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right)z = xe^y$

To find C.F

$$\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = 0$$

$$(2D + D')(D - D') + 3(2D + D')z = 0$$

$$(2D + D')(D - D' + 3)z = 0$$

$$\left(D + \frac{D'}{2}\right)(D - D' + 3)z = 0$$

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 - - - - - (2)$$

Comparing (1) & (2)

$$m_1 = \frac{-1}{2},\ C_1 = 0,\ m_2 = 1,\ C_2 = -3.$$

$$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$$

$$C.F = f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_2(y+x) \qquad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^y \qquad \text{Type}:4$$

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^{0x+y} \qquad \text{here } a = 0, b = 1$$

Rule:replace $D = D + a = D + 0 = D;\ D' = D' + b = D' + 1$

$$P.I = e^{0x+y} \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x$$

$$= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x \qquad \text{Type}:3$$

$$= e^y \frac{1}{2D^2 - DD' + 5D - D'^2 + D' + 2} x$$

$$= e^y \frac{1}{2\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right)\right]} x$$

[normally we take out highest power term of D in the homogeneous type, but it is not necessary in the non-homogeneous type]

$$= \frac{e^y}{2}\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right)\right]^{-1} x \qquad \because D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

	$= \frac{e^y}{2}\left[1 - \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right) + \ldots\right]x$ [neglect the terms D′, since $D'(x) = 0$] $= \frac{e^y}{2}\left[1 - \left(\frac{5D}{2}\right) + \ldots\right]x \quad \because D^2(x) = 0$ $= \frac{e^y}{2}\left[x - \frac{5(1)}{2}\right] \quad \because D^2(x) = 0$ $= \frac{e^y}{2}\left[\frac{2x - 5}{2}\right]$ $P.I = \frac{e^y}{4}[2x-5]$ **The general solution is** $z = C.F + P.I$ $\boxed{z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{e^y}{4}[2x-5]}$