

3.1. INNER PRODUCT

Definition: Let V be a vector space over a field F , An inner product on V is a function from $V \times V \rightarrow F$ that assigns, to every ordered pair of vectors x and y in V , a scalar in F , denoted by $\langle x, y \rangle$ such that for all $x, y, z \in V$ and scalar $\alpha \in F$ the following axioms hold:

$$\begin{aligned} I_1: \langle x, x \rangle &> 0 \text{ if } x \neq 0 \\ I_2: \langle x + z, y \rangle &= \langle x, y \rangle + \langle z, y \rangle \\ I_3: \langle \alpha x, y \rangle &= \alpha \langle x, y \rangle \end{aligned}$$

$$I_4: \overline{\langle x, y \rangle} = \langle y, x \rangle, \text{ where the bar denotes the complex conjugation.}$$

Note:

For real numbers i.e., $F = \mathbb{R}$, the complex conjugate of a number is itself. Then I_3 reduces to

$$\langle x, y \rangle = \langle y, x \rangle$$

Properties of inner product:

If V is an inner product space, then for $x, y, z \in V$ and scalar $a \in F$ the following statements are true.

- (i) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$
- (ii) $\langle x, x \rangle = 0$ if and only if $x = 0$
- (iii) $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$
- (iv) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- (v) $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$.

Proof:

$$\begin{aligned} \text{(i) } \langle 0, x \rangle &= \langle 0 + 0, x \rangle \\ &= \langle 0, x \rangle + \langle 0, x \rangle = 0 \\ \therefore \langle x, 0 \rangle &= \langle \bar{0}, x \rangle = \bar{0} = 0 \end{aligned}$$

- (ii) $\langle x, x \rangle = 0$ if and only if $x = 0$

Let $x = 0$. Then $\langle x, x \rangle = \langle 0, 0 \rangle = 0$

We know that $\langle x, x \rangle > 0$ if $x \neq 0$

Obviously $\langle x, x \rangle = 0$ if and only if $x = 0$.

(iii)

$$\begin{aligned}
 (x, ay) &= \overline{\langle ay, x \rangle} \\
 &= \overline{\alpha(y, x)} \\
 &= \bar{\alpha}(y, x) \\
 &= \bar{\alpha}(x, y) \\
 \therefore \overline{\langle ax, y \rangle} &= \bar{\alpha}(x, y) = \overline{\langle y + z, x \rangle} \\
 \text{(iv) } (x, y + z) &= \overline{\langle \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} + \langle x, z \rangle} \\
 &= \therefore (x, y + z) = \langle x, y \rangle + \langle x, z \rangle
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv) } (x, y + z) &= \overline{\langle y + z, x \rangle} \\
 &= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle \therefore (x, y + z) = \langle x, y \rangle + \langle x, z \rangle
 \end{aligned}$$

(v) Assume $\langle x, y \rangle = \langle x, z \rangle \dots (1)$, for all $x \in V$

$$\begin{aligned}
 \text{Consider } (x, y - z) &= \langle x, y \rangle - \langle x, z \rangle \\
 &= \langle x, y \rangle - \langle x, y \rangle \text{ [From (iv)]} \\
 &= 0 \dots (2)
 \end{aligned}$$

Take $x = y - z$, we get,

$$\begin{aligned}
 \langle y - z, y - z \rangle &= 0 \\
 \Rightarrow y - z &= 0 \\
 \Rightarrow y &= z
 \end{aligned}$$

If $x \neq y$, then from (2), we get

Either $x = 0$ or $y - z = 0$

$$\therefore y = z$$

Definition: Inner product space

A vector space endowed with a specific inner product is called product space.

Standard inner product of F^n

$$\begin{aligned}
 \text{Let } x, y \in F^n. \text{ Then } x &= (a_1, a_2, \dots, a_n) \text{ and } y = (b_1, b_2, \dots, b_n). \text{ The inner} \\
 \langle x, y \rangle &= a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n
 \end{aligned}$$

is called standard inner product on F^n .

Standard inner product of R^n

Let $x, y \in R^n$. Then $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$. The inner $\langle x, y \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$ is called standard inner product on R^n .

3.1.1. PROBLEMS UNDER INNER PRODUCT SPACE

1. Let $x = (a_1, a_2, \dots, a_n), y = (b_1, b_2, \dots, b_n) \in F^n$. Define inner product $\langle x, y \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n$. Verify F^n is an inner space.

Sol: Let $x, y, z \in V$ and $\alpha \in F$.

Let $x = (a_1, a_2, \dots, a_n); y = (b_1, b_2, \dots, b_n)$ and $z = (c_1, c_2, \dots, c_n)$

Given $\langle x, y \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n$

$I_1: \langle x, x \rangle > 0$ if $x \neq 0$

$$\begin{aligned} \langle x, x \rangle &= a_1 \bar{a}_1 + a_2 \bar{a}_2 + \dots + a_n \bar{a}_n \\ &= |a_1|^2 + |a_2|^2 + \dots + |a_n|^2 > 0 \quad [\because a_i \neq 0 \text{ for some } i] \end{aligned}$$

$\therefore \langle x, x \rangle > 0$ if $x \neq 0$

$$\begin{aligned} \langle x, x \rangle &= a_1 \bar{a}_1 + a_2 \bar{a}_2 + \dots + a_n \bar{a}_n \\ &= |a_1|^2 + |a_2|^2 + \dots + |a_n|^2 > 0 \end{aligned}$$

$\therefore \langle x, x \rangle > 0$ if $x \neq 0$

$I_2: \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

$$x + z = (a_1, a_2, \dots, a_n) + (c_1, c_2, \dots, c_n) = (a_1 + c_1, a_2 + c_2, \dots, a_n + c_n)$$

$$\begin{aligned} \langle x + z, y \rangle &= (a_1 + c_1) \bar{b}_1 + (a_2 + c_2) \bar{b}_2 + \dots + (a_n + c_n) \bar{b}_n = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n + c_1 \bar{b}_1 + c_2 \bar{b}_2 + \dots + c_n \bar{b}_n \\ &= \langle x, y \rangle + \langle z, y \rangle \end{aligned}$$

$$\begin{aligned} &= a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n + c_1 \bar{b}_1 + c_2 \bar{b}_2 + \dots + c_n \bar{b}_n \\ &= \langle x, y \rangle + \langle z, y \rangle \end{aligned}$$

$$\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$$

$I_3: \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

We have $x = (a_1, a_2, \dots, a_n)$.

$$\begin{aligned} \therefore \alpha x &= (\alpha a_1, \alpha a_2, \dots, \alpha a_n) \quad \langle \alpha x, y \rangle = \alpha a_1 \bar{b}_1 + \alpha a_2 \bar{b}_2 + \dots + \alpha a_n \bar{b}_n \\ &= \alpha (a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_n \bar{b}_n) = \alpha \langle x, y \rangle \therefore \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \end{aligned}$$

$$I_4: \langle x, y \rangle = \langle y, x \rangle \langle x, y \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \cdots + a_n \bar{b}_n \langle \overline{\langle x, y \rangle} \rangle$$

$$= \overline{a_1 \bar{b}_1 + a_2 \bar{b}_2 + \cdots + a_n \bar{b}_n} = \overline{a_1} b_1 + \overline{a_2} b_2 + \cdots + \overline{a_n} b_n$$

$$= b_1 \bar{a}_1 + b_2 \bar{a}_2 + \cdots + b_n \bar{a}_n = \langle y, x \rangle \because \overline{\langle x, y \rangle} = \langle y, x \rangle$$

2. Consider the vector space R^n . Prove that R^n is an inner product space with inner product $\langle x, y \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

where $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$.

Sol: Let $x, y, z \in V$ and $\alpha \in F$.

Let $x = (a_1, a_2, \dots, a_n)$; $y = (b_1, b_2, \dots, b_n)$ and $z = (c_1, c_2, \dots, c_n)$

Given $\langle x, y \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

$I_1: \langle x, x \rangle > 0$ if $x \neq 0$

$$\langle x, x \rangle = a_1 a_1 + a_2 a_2 + \cdots + a_n a_n$$

$$= a_1^2 + a_2^2 + \cdots + a_n^2 > 0 [\because a_i \neq 0 \text{ for some } i]$$

$\therefore \langle x, x \rangle > 0$ if $x \neq 0$

$$I_2: \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle x + z = (a_1, a_2, \dots, a_n) + (c_1, c_2, \dots, c_n)$$

$$= (a_1 + c_1, a_2 + c_2, \dots, a_n + c_n) \langle x + z, y \rangle$$

$$= (a_1 + c_1) b_1 + (a_2 + c_2) b_2 + \cdots + (a_n + c_n) b_n$$

$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n + c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

$$= \langle x, y \rangle + \langle z, y \rangle \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle I_3: \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

We have $x = (a_1, a_2, \dots, a_n)$.

$$\alpha x = (\alpha a_1, \alpha a_2, \dots, \alpha a_n) \langle \alpha x, y \rangle = \alpha a_1 b_1 + \alpha a_2 b_2 + \cdots + \alpha a_n b_n = \alpha (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) = \alpha \langle x, y \rangle$$

$$\therefore \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

$$I_4: \overline{\langle x, y \rangle} = \langle y, x \rangle$$

$$\langle x, y \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\overline{\langle x, y \rangle} = \overline{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}$$

$$= \overline{a_1 b_1} + \overline{a_2 b_2} + \cdots + \overline{a_n b_n}$$

$$= \overline{a_1} \bar{b}_1 + \overline{a_2} \bar{b}_2 + \cdots + \overline{a_n} \bar{b}_n$$

$$= b_1 \bar{a}_1 + b_2 \bar{a}_2 + \cdots + b_n \bar{a}_n$$

$$= \langle y, x \rangle$$

$$\therefore \overline{\langle x, y \rangle} = \langle y, x \rangle$$

Hence R^n is an inner product space.

3. Prove that R^2 is an inner product space with an inner product defined by $\langle x, y \rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$ where $x = (a_1, a_2)$; $y = (b_1, b_2)$.

Sol; Let $x, y, z \in R^2$ and $\alpha \in F$

Let $x = (a_1, a_2)$; $y = (b_1, b_2)$ and $z = (c_1, c_2)$

Given $\langle x, y \rangle = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$

I_1 ; $\langle x, x \rangle > 0$ if $x \neq 0$

$$\begin{aligned}\langle x, x \rangle &= a_1a_1 - a_2a_1 - a_1a_2 + 2a_2a_2 = a_1^2 - 2a_1a_2 + 2a_2^2 \\ &= a_1^2 - 2a_1a_2 + a_2^2 + a_2^2\end{aligned}$$

$$= (a_1 - a_2)^2 + a_2^2 > 0 [\because a_1 \neq 0 \text{ or } a_2 \neq 0]$$

$\therefore \langle x, x \rangle > 0$ if $x \neq 0$

$$\begin{aligned}I_2; \langle x + z, y \rangle &= \langle x, y \rangle + \langle z, y \rangle \\ &= (a_1, a_2) + (c_1, c_2) \\ &= (a_1 + c_1, a_2 + c_2) \langle x + z, y \rangle \\ &= (a_1 + c_1)b_1 - (a_2 + c_2)b_1 - (a_1 + c_1)b_2 + 2(a_2 + c_2)b_2 \\ &= a_1b_1 + c_1b_1 - a_2b_1 - c_2b_1 - a_1b_2 - c_1b_2 + 2a_2b_2 + 2c_2b_2 \\ &= a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2 + c_1b_1 - c_2b_1 - c_1b_2 + 2c_2b_2 \\ &= \langle x, y \rangle + \langle z, y \rangle \therefore \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle\end{aligned}$$

$$I_3: \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

We have $x = (a_1, a_2)$

$$\therefore \alpha x = (\alpha a_1, \alpha a_2)$$

$$\begin{aligned}\langle \alpha x, y \rangle &= \alpha a_1b_1 - \alpha a_2b_1 - \alpha a_1b_2 + 2\alpha a_2b_2 \\ &= \alpha(a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2) \\ &= \alpha \langle x, y \rangle\end{aligned}$$

$$\therefore \langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

$$I_4: \overline{\langle x, y \rangle} = \langle y, x \rangle$$

$$\begin{aligned}\frac{\langle x, y \rangle}{\langle x, y \rangle} &= \frac{a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2}{a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2} \\ &= \frac{a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2}{b_1a_1 - b_2a_2 - a_2b_1 + 2b_2a_2} \\ &= \langle y, x \rangle\end{aligned}$$

$$\therefore \overline{\langle x, y \rangle} = \langle y, x \rangle$$

4. Let V be the set of all real functions defined on the clo interval $[0, 1]$. The inner product on V is defined by $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt$ Prove that $V(R)$ is an inner product space.

Sol:

Let $f, g, h \in V$ and $\alpha \in F$.

Hence R^2 is an inner product space with the given inner product.

$$\text{Given } \langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

$$I_1: \langle f, f \rangle > 0 \text{ if } f \neq 0$$

$$\langle f, f \rangle = \int_{-1}^1 f(t)f(t)dt$$

$$= \int_{-1}^1 [f(t)]^2 dt > 0$$

$$\therefore \langle f, f \rangle > 0 \text{ if } f \neq 0$$

$$I_2: \langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$$

$$\langle f + h, g \rangle = \int_{-1}^1 [f(t) + h(t)]g(t)dt$$

$$= \int_{-1}^1 f(t)g(t)dt + \int_{-1}^1 h(t)g(t)dt$$

$$= \langle f, g \rangle + \langle h, g \rangle$$

$$\therefore \langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$$

$$I_3: \langle \alpha f, g \rangle = \alpha \langle f, g \rangle$$

$$\langle \alpha f, g \rangle = \int_{-1}^1 (\alpha f)(t)g(t)dt$$

$$\begin{aligned}
 &= \alpha \int_{-1}^1 f(t)g(t)dt \\
 &= \alpha \langle f, g \rangle \\
 \therefore \langle \alpha f, g \rangle &= \alpha \langle f, g \rangle \\
 I_4: \overline{\langle f, g \rangle} &= \langle g, f \rangle \\
 \langle f, g \rangle &= \int_{-1}^1 f(t)g(t)dt \\
 \overline{\langle f, g \rangle} &= \int_{-1}^1 f(t)g(t)dt \\
 &= \int_{-1}^1 f(t)g(t)dt \\
 &= \int_{-1}^1 g(t)f(t)dt \\
 &= \langle g, f \rangle \\
 \therefore \overline{\langle f, g \rangle} &= \langle g, f \rangle
 \end{aligned}$$

Therefore $V(R)$ is an inner product space.

5. Let H be the vector space of all continuous complex value functions on $[0, 1]$. Show that V is a complex inner product space with is product

$$\langle f, g \rangle = \frac{1}{3\pi} \int_0^1 f(t) \overline{g(t)} dt.$$

Sol:

Let $f, g, h \in V$ and $a \in F$.

$$\text{Given } \langle f, g \rangle = \frac{1}{2\pi} \int_0^1 f(t)g(t)dt$$

$$I_1: \langle f, f \rangle > 0 \text{ if } f \neq 0$$

$$\langle f, f \rangle > 0 \text{ for } f \neq 0$$

$$\langle f, f \rangle = \frac{1}{2\pi} \int_0^1 f(t) \overline{f(t)} dt$$

$$= \frac{1}{2\pi} \int_0^1 |f(t)|^2 dt > 0$$

$$\therefore \langle f, f \rangle > 0 \text{ if } f \neq 0$$

$$\begin{aligned}
 I_2: (f + h, g) &= \langle f, g \rangle + \langle h, g \rangle \\
 &= \frac{1}{2\pi} \int_0^1 (f + h)(t) \overline{g(t)} dt \\
 &= \frac{1}{2\pi} \int_0^1 [f(t) + h(t)] \overline{g(t)} dt \\
 &= \frac{1}{2\pi} \int_0^1 f(t) \overline{g(t)} dt + \frac{1}{2\pi} \int_0^1 h(t) \overline{g(t)} dt = \langle f, g \rangle + \langle h, g \rangle \\
 \therefore \langle f + h, g \rangle &= \langle f, g \rangle + \langle h, g \rangle \\
 I_3: \langle \alpha f, g \rangle &= \alpha \langle f, g \rangle \\
 &= \frac{1}{2\pi} \int_0^1 (\alpha f)(t) \overline{g(t)} dt = \alpha \frac{1}{2\pi} \int_0^1 f(t) \overline{g(t)} dt = \alpha \langle f, g \rangle \\
 \therefore \langle \alpha f, g \rangle &= \alpha \langle f, g \rangle \\
 I_4: \overline{(f, g)} &= (g, f) \\
 &= \frac{1}{2\pi} \int_0^1 f(t) \overline{g(t)} dt \overline{(f, g)} = \frac{1}{2\pi} \int_0^1 \overline{f(t) \overline{g(t)}} dt
 \end{aligned}$$

Therefore $V(C)$ is an inner product space.

3.1.2. NORM OF A VECTOR

Definition

Let V be an inner product space and let $x \in V$ then norm or length of x is $\|x\|$ and is defined by $\|x\| = \sqrt{\langle x, x \rangle}$

9. Find the norm of the following vectors in $V_3(R)$ with, inner product:

(a) $(1, 1, 1)$, (b) $(1, 2, 3)$, (c) $(3, -4, 0)$, (d) $(4x + 5y)$ where $x = (1, -1, 0)$ and $y = (1, 2, 3)$

Sol:

Let $x = (a_1, a_2, a_3)$; $y = (b_1, b_2, b_3) \in V_3(R)$

The standard inner product space is

$$\begin{aligned}
 \langle x, y \rangle &= \langle x, y \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \\
 \therefore \langle x, x \rangle &= a_1^2 + a_2^2 + a_3^2
 \end{aligned}$$

(a) Let $x = (1, 1, 1)$

$$\begin{aligned}
 \|x\|^2 &= \langle x, x \rangle \\
 &= 1^2 + 1^2 + 1^2 \\
 &= 3 \\
 \Rightarrow \|x\| &= \sqrt{3}
 \end{aligned}$$

(b) Let $x = (1, 2, 3)$

$$\begin{aligned}
 \|x\|^2 &= \langle x, x \rangle \\
 &= 1^2 + 2^2 + 3^2 \\
 &= 14 \\
 \Rightarrow \|x\| &= \sqrt{14}
 \end{aligned}$$

(c) Let $x = (3, -4, 0)$

$$\begin{aligned}
 \|x\|^2 &= 3^2 + (-4)^2 + 0^2 \\
 &= 9 + 16 \\
 &= 25 \\
 \Rightarrow \|x\| &= 5
 \end{aligned}$$

(d) Let $u = 4x + 5y$

$$\begin{aligned}
 &= 4(1, -1, 0) + 5(1, 2, 3) \\
 &= (4, -4, 0) + (5, 10, 15) \\
 &= (9, 6, 15) \\
 \|u\|^2 &= \langle u, u \rangle \\
 &= 9^2 + 6^2 + 15^2 \\
 &= 342 \\
 \Rightarrow \|u\| &= \sqrt{342}
 \end{aligned}$$

10. Find the norm of the following vectors in Euclidean space R^3 with standard inner product (a) $u = (2, 1, -1)$, (b) $v = \left(\frac{1}{2}, \frac{2}{3}, -\frac{1}{4}\right)$

Sol:

(a) Let $u = (2, 1, -1)$

$$\begin{aligned}
 \|u\|^2 &= 2^2 + 1^2 + (-1)^2 \\
 &= 6
 \end{aligned}$$

$$\|u\| = \sqrt{6}$$

(b) Let $v = \left(\frac{1}{2}, \frac{2}{3}, -\frac{1}{4}\right)$

$$\|v\|^2 = 6^2 + 8^2 + (-3)^2$$

$$=109$$

$$\|v\| = \sqrt{109}$$

11. Find the norm of the following vectors in F^3 with standard inner product: $x = (1 + i, 2, i), y = (3i, 2 + 3i, 4)$. Find (a) $\|x\|$, (b) $\|y\|$, (c) $\|x + y\|$, (d) $\langle x, y \rangle$

Sol: Let $x, y, z \in F^3$

$$\text{Let } x = (a_1, a_2, a_3); y = (b_1, b_2, b_3)$$

$$\langle x, y \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2 + a_3 \bar{b}_3$$

$$\langle x, x \rangle = |a_1|^2 + |a_2|^2 + |a_3|^2$$

$$(a) \|x\|^2 = \langle x, x \rangle$$

Then

Sol:

$$= |1 + i|^2 + |2|^2 + |i|^2$$

$$= 1^2 + 1^2 + 2^2 + 1^2$$

$$= 7$$

$$\|x\| = \sqrt{7}$$

$$(b) \|y\|^2 = \langle y, y \rangle$$

$$= |3i|^2 + |2 + 3i|^2 + |4|^2$$

$$= 3^2 + 2^2 + 3^2 + 4^2$$

$$= 9 + 4 + 9 + 16$$

$$= 38$$

$$\|y\| = \sqrt{38}$$

$$(c) x + y = (1 + i, 2, i) + (3i, 2 + 3i, 4)$$

$$= (1 + 4i, 4 + 3i, 4 + i)$$

$$\|x + y\|^2 = |1 + 4i|^2 + |4 + 3i|^2 + |4 + i|^2$$

$$= 1^2 + 4^2 + 4^2 + 3^2 + 4^2 + 1^2$$

$$= 59$$

$$\|x + y\| = \sqrt{59}$$

$$(d) \langle x, y \rangle = \langle (1 + i, 2, i), (3i, 2 + 3i, 4) \rangle$$

$$= (1 + i)(\bar{3i}) + 2(2 + 3i) + i4$$

$$= (1 + i)(-3i) + 2(2 - 3i) + 4i$$

$$= -3i + 3 + 4 - 6i + 4i$$

$$= 7 - 5i$$

12. Let V be an vector space of polynomials with the inner product given by

$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Let $f(t) = t + 2$ and $g(t) = t^2 - 2t - 3$ find (i)

$\langle f, g \rangle$ (ii) $\| f \|$.

Sol:

$$\begin{aligned} \text{Let } \langle f, g \rangle &= \int_0^1 f(t)g(t)dt \\ \text{(i)} \quad &= \int_0^1 (t+2)(t^2-2t-3)dt \\ &= \int_0^1 (t^3-2t^2-3t+2t^2-4t-6)dt \end{aligned}$$

$$\begin{aligned} &= \int_0^1 (t^3-7t-6)dt \\ &= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t \right]_0^1 \\ &= \frac{1}{4} - \frac{7}{2} - 6 \\ &= -\frac{37}{4} \end{aligned}$$

ii)

$$\begin{aligned} \| f \|^2 &= \langle f, f \rangle \\ &= \int_0^1 [f(t)]^2 dt \\ &= \int_0^1 (t+2)^2 dt \\ &= \int_0^1 (t^2+4t+4)dt \\ &= \left[\frac{t^3}{3} + \frac{4t^2}{2} + 4t \right]_0^1 \\ &= \frac{1}{3} + 2 + 4 \\ &= \frac{19}{3} \\ \| f \| &= \frac{\sqrt{19}}{\sqrt{3}} \end{aligned}$$

13. For any non-zero vector, $x \in V$. prove that $y = \frac{x}{\|x\|}$ is a vector such that

$$\|y\| = 1.$$

Sol: Consider

$$\begin{aligned}\langle y, y \rangle &= \left\langle \frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \\ &= \frac{1}{\|x\|} \cdot \frac{1}{\|x\|} \langle x, x \rangle \\ \langle y, y \rangle &= \frac{1}{\|x\|^2} \|x\|^2 \\ \|y\|^2 &= 1 \\ \|y\| &= 1\end{aligned}$$

Theorem 3.1: In an inner product space V ,

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|$

Proof:

(i)

$$\begin{aligned}\|x\| &= \sqrt{\langle x, x \rangle} \\ \|x\|^2 &= \langle x, x \rangle \geq 0 \\ \|x\|^2 &\geq 0 \\ \|x\| &\geq 0\end{aligned}$$

Also $\langle x, x \rangle \geq 0$ if and only if $x = 0$

Therefore $\|x\|^2 = 0$ if and only if $x = 0$

(ii)

$$\begin{aligned}\|\alpha x\|^2 &= \langle \alpha x, \alpha x \rangle \\ &= \alpha \langle x, \alpha x \rangle \\ &= \alpha \bar{\alpha} \langle x, x \rangle \\ &= |\alpha|^2 \|x\|^2 \\ \|\alpha x\| &= |\alpha| \|x\|\end{aligned}$$

Theorem 3.2: [Schwarz's inequality]

For any two vectors x and y in an inner product space V ,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Proof:

If $x = 0$, then $\|x\| = 0$.

$$\therefore \|x\| \|y\| = 0 \dots (1)$$

Also $\langle x, y \rangle = \langle 0, y \rangle = 0$

$$\therefore |\langle x, y \rangle| = 0 \dots (2).$$

From (1) and (2)

$$|\langle x, y \rangle| = \|x\| \|y\|$$

So the result is true.

Let $x \neq 0$. Then $\|x\| > 0$

Therefore $\frac{1}{\|x\|}$ is a positive number

Consider the vector

$$\begin{aligned} w &= y - \frac{\langle y, x \rangle}{\|x\|^2} x \\ \langle w, w \rangle &= \left\langle y - \frac{\langle y, x \rangle}{\|x\|^2} x, y - \frac{\langle y, x \rangle}{\|x\|^2} x \right\rangle \\ &= \langle y, y \rangle - \left\langle y, \left[\frac{\langle y, x \rangle}{\|x\|^2} x \right] \right\rangle - \left\langle \frac{\langle y, x \rangle}{\|x\|^2} x, y \right\rangle + \left\langle \frac{\langle y, x \rangle}{\|x\|^2} x, \frac{\langle y, x \rangle}{\|x\|^2} x \right\rangle \\ &= \|y\|^2 - \frac{\text{Inner Product}}{\|x\|^2} \langle y, x \rangle - \frac{\langle y, x \rangle}{\|x\|^2} \langle x, y \rangle + \frac{\langle y, x \rangle \langle y, x \rangle}{\|x\|^4} \langle x, x \rangle \\ &= \|y\|^2 - \frac{\langle x, y \rangle \langle x, y \rangle}{\|x\|^2} - \frac{\langle x, y \rangle}{\|x\|^2} + \frac{\langle x, y \rangle \langle x, y \rangle \|x\|^2}{\|x\|^4} \\ &= \|y\|^2 - \frac{|\langle x, y \rangle|^2}{\|x\|^2} - \frac{|\langle x, y \rangle|^2}{\|x\|^2} + \frac{|\langle x, y \rangle|^2}{\|x\|^2} [\because z\bar{z} = |z|^2] \\ \langle w, w \rangle &= \|y\|^2 - \frac{|\langle x, y \rangle|^2}{\|x\|^2} \\ \therefore \|y\|^2 - \frac{|\langle x, y \rangle|^2}{\|x\|^2} &\geq 0 \\ \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 &\geq 0 \end{aligned}$$

$$\|x\|^2 \|y\|^2 \geq |\langle x, y \rangle|^2 \implies |\langle x, y \rangle| \leq \|x\| \|y\|$$

$$\therefore |\langle x, y \rangle| \leq \|x\| \|y\|$$

Theorem 3.3: [Triangle inequality]

For any two vectors x and y in an inner product space V ,

$$\|x + y\| \leq \|x\| + \|y\|.$$

Proof:

Using the norm of vectors we have

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 \\ &= \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \quad [\because z + \bar{z} = 2\operatorname{Re}(z)] \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \quad [\because \operatorname{Re}(z) \leq |z|] \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad [\text{By Schwarz's inequality}] \\ &\leq (\|x\| + \|y\|)^2 \\ \|x + y\| &\leq \|x\| + \|y\|. \end{aligned}$$

Theorem 3.4 : [Parallelogram law]

For any two vectors x and y in an inner product space V ,

$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$. What does this equation state about parallelograms in R^2 ?

Proof:

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \langle x, y \rangle + \bar{\langle x, y \rangle} + \|y\|^2 \\ &= \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \dots (1) \end{aligned}$$

and

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 - [\langle x, y \rangle + \langle x, y \rangle] + \|y\|^2 \\ &= \|x\|^2 - 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \dots (2) \end{aligned}$$

(1) + (2) $\Rightarrow$

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \dots (3)$$

Let $OABC$ be a parallelogram with sides of length $OA = \|x\|$ and $OC = \|y\|$ in R^2 . Therefore the length of the hypotenuses of $OABC$ are $AC = \|x + y\|$ and $OB = \|x - y\|$

$$(3) \Rightarrow OB^2 + AC^2 = OA^2 + AB^2 + BC^2 + CA^2 [\because |OA| = |BC|, |AB| = |CO|]$$

Therefore sum of the squares of the two diagonals is equal to the sum of squares of four sides.

