

Improper Integrals

The Integral $I = \int_a^b f(x)dx$ is said to be proper or definite only when the limits a and b are finite and the integrand $f(x)$ is continuous in the interval $[a, b]$

Types of Improper Integrals

There are two types of improper integrals

1. With infinite limits of integration
2. The integrand is discontinuous.

Type I (Infinite limits of integration)

1. $\int_a^\infty f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx$
2. $\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx$
3. $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$, 'a' is a real number.

Provided both the limits on right side exist.

Type II (Discontinuous of the integrand)

1. If f is discontinuous at b , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$$

2. If f is discontinuous at a , then

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx$$

3. If f is discontinuous at c , in $[a, b]$ then

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \lim_{t \rightarrow c^-} \int_a^t f(x)dx + \lim_{t \rightarrow c^+} \int_t^b f(x)dx \end{aligned}$$

Provided both the integral's on right exists.

Note:

The improper integral is said to be convergent if the limit exists and is divergent if the limit does not exist.

Example:

Determine whether the integral $\int_1^\infty \frac{1}{x} dx$ is convergent or divergent.

Solution:

The given integral is $\int_1^{\infty} \frac{1}{x} dx$

an improper integral, since upper limit of integration is infinite then,

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t \rightarrow \infty} [\log x]_1^t \\ &= \lim_{t \rightarrow \infty} [\log t - \log 1] \\ &= \lim_{t \rightarrow \infty} [\log t - 0] = \infty\end{aligned}$$

The given integral is divergent and it diverges to ∞ .

Example:

Determine whether the integral $\int_0^{\infty} \frac{1}{1+x^2} dx$ is convergent or divergent.

Solution:

The given integral is $\int_0^{\infty} \frac{1}{1+x^2} dx$ an improper integral, since upper limit of integration is infinite then,

$$\begin{aligned}\int_0^{\infty} \frac{1}{1+x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1+x^2} dx \\ &= \lim_{t \rightarrow \infty} [\tan^{-1} x]_0^t \\ &= \lim_{t \rightarrow \infty} [\tan^{-1} t - \tan^{-1} 0] \\ &= \lim_{t \rightarrow \infty} \tan^{-1} t \\ &= \tan^{-1} \infty = \frac{\pi}{2}\end{aligned}$$

The given integral is convergent.

Example:

For what values of p the integral $\int_1^{\infty} \frac{1}{x^p} dx$ convergent?

Solution:

$$\begin{aligned}\text{If } p \neq 1, \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx \\ &= \lim_{t \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right]\end{aligned}$$

$$= \lim_{t \rightarrow \infty} \frac{1}{p-1} \left[1 - \frac{1}{t^{p-1}} \right]$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1, \text{converges} \\ \infty, & p \leq 1, \text{diverges} \end{cases}$$

Example:

Evaluate $\int_1^{\infty} \frac{\log x}{x} dx$

Solution:

Take $I = \int \frac{\log x}{x} dx$

Put $u = \log x$ $dv = \frac{1}{x} dx$ $du = \frac{1}{x} dx$ $v = \log x$

$$I = \int \frac{\log x}{x} dx = (\log x)^2 - \int \log x \left(\frac{1}{x} \right) dx$$

$$I = (\log x)^2 - I \Rightarrow 2I = (\log x)^2 \Rightarrow I = \frac{1}{2} (\log x)^2$$

$$\begin{aligned} \int_1^{\infty} \frac{\log x}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{\log x}{x} dx = \lim_{t \rightarrow \infty} \left(\frac{1}{2} (\log x)^2 \right)_1^t \\ &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} (\log t)^2 - \frac{1}{2} (\log 1)^2 \right] \\ &= \lim_{t \rightarrow \infty} \left[\frac{1}{2} (\log t)^2 \right] = \infty \quad [\log 1 = 0, \log \infty = \infty] \end{aligned}$$

The given integral is divergent.

Example:

Evaluate $\int_{-\infty}^{\infty} x e^{-x^2} dx$

Solution:

Consider $\int x e^{-x^2} dx$

Put $u = x^2$, $du = 2x dx$

$$\begin{aligned} \int x e^{-x^2} dx &= \int e^{-u} \frac{du}{2} = \frac{1}{2} \left[\frac{e^{-u}}{-1} \right] \\ &= -\frac{1}{2} e^{-u} = -\frac{1}{2} e^{-x^2} \dots (1) \end{aligned}$$

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^0 x e^{-x^2} dx + \int_0^{\infty} x e^{-x^2} dx \dots (2)$$

Take $\int_{-\infty}^0 x e^{-x^2} dx = \lim_{t \rightarrow -\infty} \int_t^0 x e^{-x^2} dx = \lim_{t \rightarrow -\infty} \left[-\frac{1}{2} e^{-x^2} \right]_t^0$ by (1)

$$= \lim_{t \rightarrow -\infty} \left[-\frac{1}{2} + \frac{1}{2} e^{-t^2} \right] = \frac{-1}{2}$$

$$\begin{aligned}
 \text{Take } \int_0^{\infty} x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} dx = \lim_{t \rightarrow \infty} \left[\frac{-1}{2} e^{-x^2} \right]_0^t \text{ by (1)} \\
 &= \lim_{t \rightarrow \infty} \left[\frac{-1}{2} e^{-t^2} + \frac{1}{2} \right] = \frac{1}{2} \\
 \therefore (2) \Rightarrow \int_{-\infty}^{\infty} x e^{-x^2} dx &= \frac{-1}{2} + \frac{1}{2} = 0
 \end{aligned}$$

Example:

Evaluate $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$

Solution:

Consider $\int \frac{1}{(x-2)^{3/2}} dx \dots (1)$

Put $u = x - 2 \Rightarrow du = dx$

$$\begin{aligned}
 (1) \Rightarrow \int \frac{1}{(x-2)^{3/2}} dx &= \int \frac{1}{u^{3/2}} du = \int u^{-3/2} du = \frac{u^{-3/2+1}}{-3/2+1} = \frac{u^{-1/2}}{-1/2} \\
 &= \frac{-2}{\sqrt{u}} = \frac{-2}{\sqrt{x-2}} \\
 \int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx &= \lim_{t \rightarrow \infty} \left[\int_3^t \frac{1}{(x-2)^{3/2}} dx \right] = \lim_{t \rightarrow \infty} \left[\frac{-2}{\sqrt{x-2}} \right]_3^t \\
 &= \lim_{t \rightarrow \infty} \left[\left(\frac{-2}{\sqrt{t-2}} \right) - \left(\frac{-2}{\sqrt{1}} \right) \right] \\
 &= \lim_{t \rightarrow \infty} \left(\frac{-2}{\sqrt{t-2}} \right) + 2 = 0 + 2 = 2 \text{ (finite)}
 \end{aligned}$$

The given integral $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$ is convergent.

Example:

Evaluate $\int_0^2 \frac{1}{\sqrt{x}} dx$

Solution:

Here, infinite discontinuity occurs at $x=0$

$$\begin{aligned}
 \therefore \int_0^2 \frac{1}{\sqrt{x}} dx &= \lim_{t \rightarrow 0^+} \int_t^2 x^{-1/2} dx \\
 &= \lim_{t \rightarrow 0^+} \left[\frac{x^{1/2}}{1/2} \right]_t^2 \\
 &= \lim_{t \rightarrow 0^+} [2\sqrt{x}]_t^2
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{t \rightarrow 0^+} [2\sqrt{2} - 2\sqrt{t}] \\
 &= 2\sqrt{2} \text{ (finite)}
 \end{aligned}$$

The given integral $\int_0^2 \frac{1}{\sqrt{x}} dx$ is convergent.

Example:

Evaluate $\int_0^3 \frac{1}{x-1} dx$

Solution:

Here, infinite discontinuity occurs at $x = 1$

$$\therefore \int_0^3 \frac{1}{x-1} dx = \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx$$

$$\begin{aligned}
 \text{Take } \int_0^1 \frac{1}{x-1} dx &= \lim_{t \rightarrow 1^-} \int_0^t \frac{1}{x-1} dx = \lim_{t \rightarrow 1^-} [\log(x-1)]_0^t \\
 &= \lim_{t \rightarrow 1^-} \log(t-1) = -\infty
 \end{aligned}$$

$$\int_0^1 \frac{1}{x-1} dx \text{ is divergent.}$$

$$\Rightarrow \int_1^3 \frac{1}{x-1} dx \text{ is also divergent.}$$

The given integral $\int_0^3 \frac{1}{x-1} dx$ is divergent

Example:

Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}} dx$

Solution:

The infinite discontinuity occurs at $x = 2$

$$\begin{aligned}
 \therefore \int_2^5 \frac{1}{\sqrt{x-2}} dx &= \lim_{t \rightarrow 2^+} \int_2^t \frac{1}{\sqrt{x-2}} dx \\
 &= \lim_{t \rightarrow 2^+} [2\sqrt{x-2}]_2^t \\
 &= \lim_{t \rightarrow 2^+} (2\sqrt{3} - 2\sqrt{t-2}) \\
 &= 2\sqrt{3} \text{ (finite)}
 \end{aligned}$$

The given integral $\int_2^5 \frac{1}{\sqrt{x-2}} dx$ is convergent.

Example:

Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}} dx$

Solution:

Here infinite discontinuity occurs at $x = 1$

$$1) \int_0^3 \frac{1}{(x-1)^{2/3}} dx = \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^3 \frac{1}{(x-1)^{2/3}} dx \quad \dots(1)$$

$$\begin{aligned} \text{Take } \int_0^1 \frac{1}{(x-1)^{2/3}} dx &= \lim_{t \rightarrow 1^-} \int_0^t \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{t \rightarrow 1^-} \left[3(x-1)^{1/3} \right]_0^t \\ &= \lim_{t \rightarrow 1^-} \left[3(t-1)^{1/3} + 3 \right] \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{Take } \int_1^3 \frac{1}{(x-1)^{2/3}} dx &= \lim_{t \rightarrow 1^+} \int_t^3 \frac{1}{(x-1)^{2/3}} dx \\ &= \lim_{t \rightarrow 1^+} \left[3(x-1)^{1/3} \right]_t^3 \\ &= \lim_{t \rightarrow 1^+} \left[3 \left[2^{1/3} - (t-1)^{1/3} \right] \right] \\ &= 3 \left(2^{1/3} \right) \end{aligned}$$

$$\begin{aligned} (1) \Rightarrow \int_0^3 \frac{1}{(x-1)^{2/3}} dx &= 3 + 3 \left(2^{1/3} \right) \\ &= 3 \left[1 + 2^{1/3} \right] \end{aligned}$$

Comparison test for improper integrals

Let $\int_a^b f(x) dx$ be an improper integral.

- i) If there exists a $g(x)$ such that $|f(x)| \leq g(x)$ for all x in $[a, b]$ and $\int_a^b g(x) dx$ converges then $\int_a^b f(x) dx$ also converges.
- ii) If there exists function $g(x)$ such that $f(x) \geq |g(x)|$ for all x in $[a, b]$ and $\int_a^b g(x) dx$ diverges then $\int_a^b f(x) dx$ also diverges.

Limit form of comparison Tests.

Let $f(x) > 0$ and $g(x) > 0$ and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = k$ where $k \neq 0$

Then, the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge or diverge together.

If $k = 0$, only the convergence of $\int_a^\infty g(x)dx$ implies that of $\int_a^\infty f(x)dx$

Absolute Convergence

The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f(x)|dx$ is convergent.

Note:

- 1) The same definition holds for $\int_a^\infty f(x)dx$ also
- 2) When the improper integral changes sign within the limits of the integration, then the above test is applied.

Example:

Discuss the convergence of $\int_1^\infty \frac{x \tan^{-1} x}{\sqrt{4+x^3}} dx$

Solution:

$$\text{Let } f(x) = \frac{x \tan^{-1} x}{\sqrt{4+x^3}} = \frac{\tan^{-1} x}{\sqrt{x} \sqrt{1+4x^{-3}}} \text{ and } g(x) = \frac{1}{\sqrt{x}}$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{\sqrt{1+4x^{-3}}} = \frac{\pi}{2}$$

Hence, by comparison test 2, the integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together, Now $\int_1^\infty g(x)dx$ is divergent.

$\therefore \int_1^\infty f(x)dx$ is also divergent.

Example :

Discuss the convergence of $\int_1^\infty \frac{\sin x}{x^4} dx$

Solution:

$$\left| \int_1^\infty \frac{\sin x}{x^4} dx \right| \leq \int_1^\infty \left| \frac{\sin x}{x^4} \right| dx \leq \int_1^\infty \frac{dx}{x^4}$$

$$\Rightarrow \text{convergent}$$

$\int_1^\infty \frac{\sin x}{x^4} dx$ is absolutely convergent and hence convergent.

Example:

Test the convergence of $\int_0^\infty e^{-x^2} dx$

Solution:

The given integral $\int_0^\infty e^{-x^2} dx$ is an improper integral of first kind and the integral can be written as $\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$

The first integral in the right hand side $\int_0^1 e^{-x^2} dx$ is proper integral. So it is enough to check the second one.

We have that,

$$\begin{aligned}
 x &\geq 1 \\
 x^2 &\geq x \\
 -x^2 &\leq -x \\
 e^{-x^2} &\leq e^{-x} \\
 \int_1^\infty e^{-x^2} dx &\leq \int_1^\infty e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_1^b \\
 &= \lim_{b \rightarrow \infty} [e^{-1} - e^{-b}] \\
 &= [e^{-1} - 0] = \frac{1}{e}
 \end{aligned}$$

Hence by comparison test the given integral is convergent.