

UNITV

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

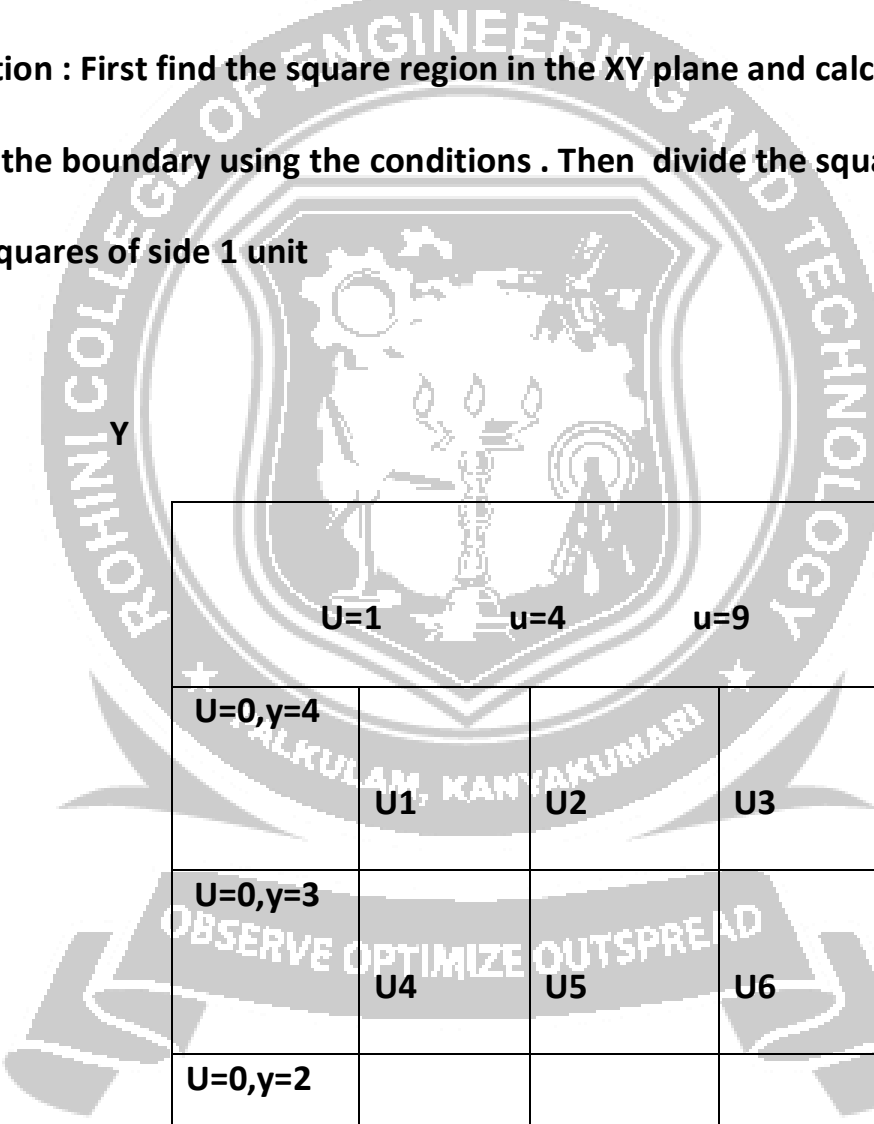
PROBLEMS BASED ON TWO DIMENSIONAL LAPLACE EQUATIONS

Solve the Laplace'S equation over the square mesh of side 4 units satisfying the boundary conditions :

$$u(0, y) = 0, 0 \leq y \leq 4 ; u(4, y) = 12 + y, 0 \leq y \leq 4$$

$$u(x, 0) = 3x, 0 \leq x \leq 4 ; u(x, 4) = x^2, 0 \leq x \leq 4$$

Solution : First find the square region in the XY plane and calculate the numerical values of u on the boundary using the conditions . Then divide the square mesh into a network of subsquares of side 1 unit

				X=4 U=16, y=4
U=0, y=4	U1	U2	U3	U=15
U=0, y=3	U4	U5	U6	U=14
U=0, y=2	U7	U8	U9	U=13
U=0, y=0	X=1, u=3	X=2, u=6	x=3, u=9	X=4 u=12

Let $u_1, u_2, u_3, \dots$ be the values of u at the internal grid points Here u_5 is the centre point

Rough values :

$$u_5^{(0)} = \frac{1}{4} [6 + 14 + 4 + 0] = 6 \quad (\text{SFPP})$$

$$u_1^{(0)} = \frac{1}{4} [0 + 6 + 4 + 0] = 2.5 \quad (\text{DFPP})$$

$$u_3^{(0)} = \frac{1}{4} [4 + 6 + 14 + 16] = 10 \quad (\text{DFPP})$$

$$u_7^{(0)} = \frac{1}{4} [0 + 6 + 6 + 0] = 3 \quad (\text{DFPP})$$

$$u_9^{(0)} = \frac{1}{4} [6 + 12 + 14 + 6] = 9.5 \quad (\text{DFPP})$$

$$u_2^{(0)} = \frac{1}{4} [4 + 6 + 2.5 + 10] = 5.625 \quad (\text{SFPP})$$

$$u_4^{(0)} = \frac{1}{4} [0 + 6 + 2.5 + 3] = 3.125 \quad (\text{SFPP})$$

$$u_6^{(0)} = \frac{1}{4} [6 + 14 + 10 + 9.5] = 9.875 \quad (\text{SFPP})$$

$$u_8^{(0)} = \frac{1}{4}[6 + 6 + 3 + 9.5] = 6.125 \quad (\text{SFPP})$$

Here after we should use SFPP ONLY FOR ITERATION

First iteration

$$u_1^{(1)} = \frac{1}{4}[1 + 5.625 + 3.125 + 0] = 2.4375$$

$$u_2^{(1)} = \frac{1}{4}[4 + 6 + 2.4375 + 10] = 5.6094$$

$$u_3^{(1)} = \frac{1}{4}[9 + 9.875 + 5.6094 + 15] = 9.8711$$

$$u_4^{(1)} = \frac{1}{4}[0 + 6 + 2.4375 + 3] = 2.8594$$

$$u_5^{(1)} = \frac{1}{4}[5.6094 + 2.8594 + 9.875 + 6.125] = 6.1172$$

$$u_6^{(1)} = \frac{1}{4}[9.8711 + 6.1172 + 14 + 9.5] = 9.8721$$

$$u_7^{(1)} = \frac{1}{4}[0 + 3 + 2.8594 + 6.125] = 2.9948$$

$$u_8^{(1)} = \frac{1}{4}[6.1172 + 2.9948 + 6 + 9.5] = 6.153$$

$$u_9^{(1)} = \frac{1}{4}[9.8721 + 6.153 + 13 + 9] = 9.5063$$

Second iteration

$$u_1^{(2)} = \frac{1}{4}[1 + 0 + 5.6094 + 2.8594] = 2.3672$$

$$u_2^{(2)} = \frac{1}{4}[2.3672 + 4 + 9.8711 + 6.1172] = 5.5888$$

$$u_3^{(2)} = \frac{1}{4}[9 + 15 + 5.5888 + 9.8721] = 9.8652$$

$$u_4^{(2)} = \frac{1}{4}[0 + 2.3672 + 6.1172 + 2.9948] = 2.8698$$

$$u_5^{(2)} = \frac{1}{4}[5.5888 + 2.8698 + 9.8721 + 6.153] = 6.1209$$

$$u_6^{(2)} = \frac{1}{4}[14 + 9.8652 + 6.1209 + 9.5063] = 9.8731$$

$$u_7^{(2)} = \frac{1}{4}[0 + 3 + 2.8698 + 6.153] = 3.0057$$

$$u_8^{(2)} = \frac{1}{4}[6.1209 + 3.0057 + 6 + 9.5063] = 6.1582$$

$$u_9^{(2)} = \frac{1}{4}[9.8731 + 6.1587 + 13 + 9] = 9.5078$$

Third iteration

$$u_1^{(3)} = \frac{1}{4}[0 + 2.8698 + 5.5888 + 1] = 2.3647$$

$$u_2^{(3)} = \frac{1}{4}[2.3647 + 6.1209 + 9.8652 + 4] = 5.5880$$

$$u_3^{(3)} = \frac{1}{4}[5.5880 + 9.8731 + 15 + 9] = 9.8652$$

$$u_4^{(3)} = \frac{1}{4}[0 + 3.0057 + 6.1209 + 2.3647] = 2.8728$$

$$u_5^{(3)} = \frac{1}{4}[2.8728 + 6.1582 + 9.8731 + 5.5880] = 6.1230$$

$$u_6^{(3)} = \frac{1}{4}[6.1230 + 9.5078 + 14 + 9.8652] = 9.8740$$

$$u_7^{(3)} = \frac{1}{4}[0 + 3 + 2.8728 + 6.1582] = 3.0077$$

$$u_8^{(3)} = \frac{1}{4}[3.0077 + 3.0057 + 6.123 + 9.5078] = 6.1596$$

$$u_9^{(3)} = \frac{1}{4}[9.8740 + 6.1596 + 13 + 9] = 9.5084$$

$$u_1 = 2.3647,$$

$$u_2 = 5.5880$$

$$u_3 = 9.8652$$

$$u_4 = 2.8728$$

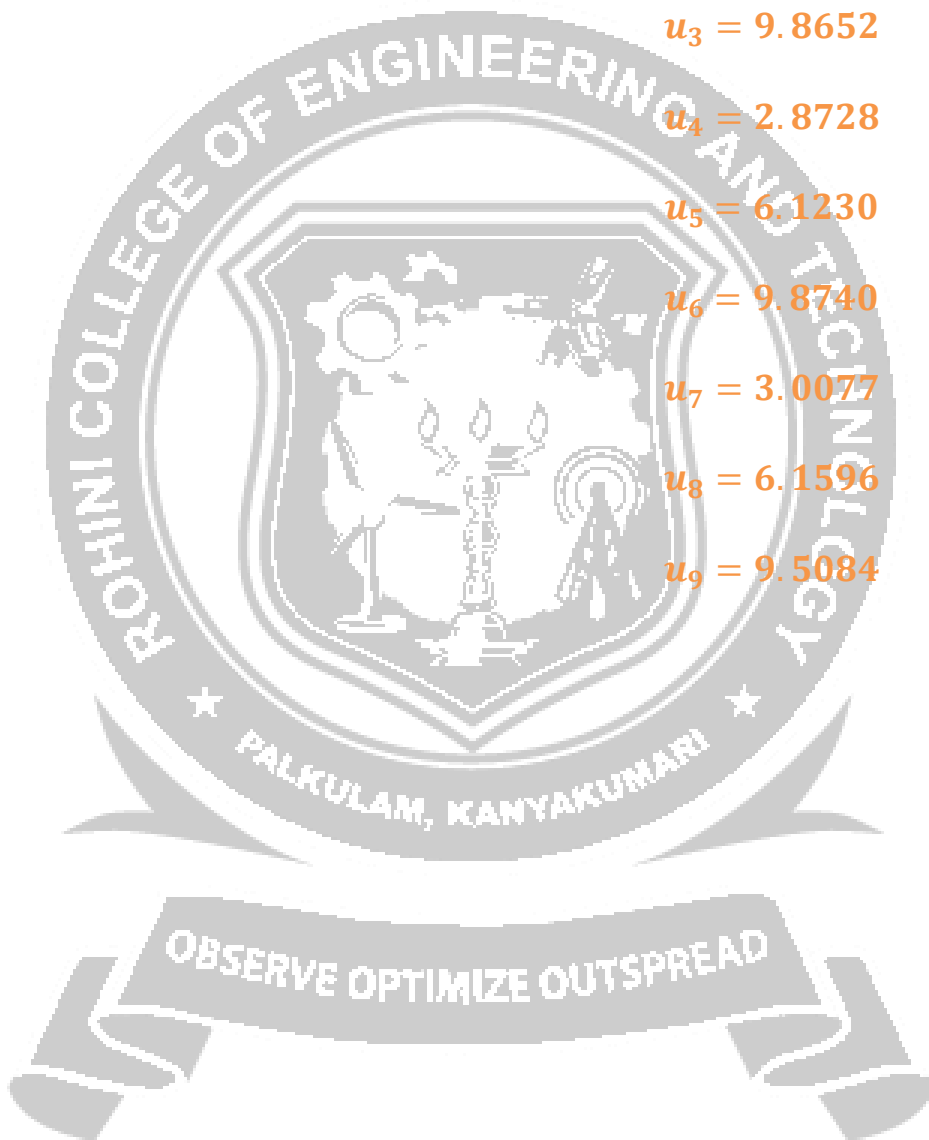
$$u_5 = 6.1230$$

$$u_6 = 9.8740$$

$$u_7 = 3.0077$$

$$u_8 = 6.1596$$

$$u_9 = 9.5084$$



Solve the Laplace Equation $u_{xx} + u_{yy} = 0$ given that

	11.1 17 19.7			18.6
0	U1	U2	U3	21.9
0	U4	U5	U6	21
0	U7	U8	U9	17
0	X=1,u=3	X=2,u=6	x=3,u=9	9
	8.7	12.1	12.5	

Solution:

Let $u_1, u_2, u_3, \dots$ be the values of u at the internal grid points Here u_5 is the centre point

Rough values :

$$u_5^{(0)} = \frac{1}{4} [0 + 17 + 21 + 12.1] = 12.5 \quad (\text{SFPP})$$

$$u_1^{(0)} = \frac{1}{4}[0 + 12.5 + 17 + 0] = 7.4 \quad (\text{DFPF})$$

$$u_3^{(0)} = \frac{1}{4}[12.5 + 18.6 + 17 + 21] = 17.28 \quad (\text{DFPF})$$

$$u_7^{(0)} = \frac{1}{4}[12.5 + 0 + 0 + 12.1] = 6.15 \quad (\text{DFPF})$$

$$u_9^{(0)} = \frac{1}{4}[12.5 + 9 + 21 + 12.1] = 13.65 \quad (\text{DFPF})$$

$$u_2^{(0)} = \frac{1}{4}[17 + 12.5 + 7.4 + 17.3] = 13.55 \quad (\text{SFPP})$$

$$u_4^{(0)} = \frac{1}{4}[7.4 + 6.2 + 0 + 12.5] = 6.52 \quad (\text{SFPP})$$

$$u_6^{(0)} = \frac{1}{4}[17.3 + 13.7 + 12.5 + 21] = 16.12 \quad (\text{SFPP})$$

$$u_8^{(0)} = \frac{1}{4}[12.5 + 12.1 + 6.2 + 13.7] = 11.12 \quad (\text{SFPP})$$

First iteration

$$u_1^{(1)} = \frac{1}{4}[11.1 + 6.52 + 13.55 + 0] = 7.79$$

$$u_2^{(1)} = \frac{1}{4}[7.79 + 17 + 12.5 + 17.28] = 13.64$$

$$u_3^{(1)} = \frac{1}{4}[13.64 + 19.7 + 16.12 + 21.9] = 12.84$$

$$u_4^{(1)} = \frac{1}{4}[0 + 7.79 + 6.15 + 12.5] = 6.61$$

$$u_5^{(1)} = \frac{1}{4}[6.61 + 13.64 + 11.12 + 16.12] = 11.88$$

$$u_6^{(1)} = \frac{1}{4}[11.88 + 17.84 + 13.65 + 21] = 16.09$$

$$u_7^{(1)} = \frac{1}{4}[0 + 6.61 + 8.7 + 11.12] = 6.61$$

$$u_8^{(1)} = \frac{1}{4}[6.61 + 11.88 + 12.1 + 13.65] = 11.06$$

$$u_9^{(1)} = \frac{1}{4}[11.06 + 16.09 + 12.5 + 17] = 12.238$$

Second iteration

$$u_1^{(2)} = \frac{1}{4}[11.1 + 0 + 6.61 + 13.64] = 7.84$$

$$u_2^{(2)} = \frac{1}{4}[7.84 + 17 + 11.88 + 17.84] = 16.64$$

$$u_3^{(2)} = \frac{1}{4}[13.64 + 19.7 + 16.09 + 21.9] = 17.83$$

$$u_4^{(2)} = \frac{1}{4}[0 + 7.84 + 6.61 + 11.88] = 6.58$$

$$u_5^{(2)} = \frac{1}{4}[6.58 + 13.64 + 11.06 + 16.09] = 11.84$$

$$u_6^{(2)} = \frac{1}{4}[11.84 + 17.83 + 14.24 + 21] = 16.23$$

$$u_7^{(2)} = \frac{1}{4}[0 + 6.58 + 8.7 + 11.06] = 6.58$$

$$u_8^{(2)} = \frac{1}{4}[6.58 + 11.84 + 12.1 + 14.24] = 11.19$$

$$u_9^{(2)} = \frac{1}{4}[11.9 + 16.23 + 12.5 + 17] = 14.30$$

Third iteration

$$u_1^{(3)} = \frac{1}{4}[0 + 11.1 + 6.58 + 13.64] = 7.83$$

$$u_2^{(3)} = \frac{1}{4}[7.83 + 17 + 11.84 + 17.83] = 13.637$$

$$u_3^{(3)} = \frac{1}{4}[13.63 + 19.7 + 16.23 + 21.9] = 17.86$$

$$u_4^{(3)} = \frac{1}{4}[0 + 7.83 + 6.58 + 11.84] = 6.56$$

$$u_5^{(3)} = \frac{1}{4}[6.56 + 13.63 + 11.19 + 16.23] = 11.90$$

$$u_6^{(3)} = \frac{1}{4}[11.90 + 17.86 + 14.30 + 21] = 16.27$$

$$u_7^{(3)} = \frac{1}{4}[0 + 6.56 + 8.7 + 11.19] = 6.61$$

$$u_8^{(3)} = \frac{1}{4}[6.61 + 11.90 + 12.1 + 14.30] = 11.23$$

$$u_9^{(3)} = \frac{1}{4}[11.23 + 16.27 + 12.5 + 17] = 14.32$$

Fourth iteration

$$u_1^{(4)} = \frac{1}{4}[0 + 11.1 + 6.56 + 13.637] = 7.82$$

$$u_2^{(4)} = \frac{1}{4}[7.82 + 17 + 11.90 + 17.86] = 13.65$$

$$u_3^{(4)} = \frac{1}{4}[13.637 + 19.7 + 16.27 + 21.9] = 17.88$$

$$u_4^{(4)} = \frac{1}{4}[0 + 7.82 + 6.56 + 11.90] = 6.58$$

$$u_5^{(4)} = \frac{1}{4}[6.56 + 13.637 + 11.23 + 16.27] = 11.94$$

$$u_6^{(4)} = \frac{1}{4}[11.94 + 17.88 + 14.32 + 21] = 16.28$$

$$u_7^{(4)} = \frac{1}{4}[0 + 6.58 + 8.7 + 11.23] = 6.63$$

$$u_8^{(4)} = \frac{1}{4}[6.63 + 11.94 + 12.1 + 14.38] = 11.25$$

$$u_9^{(4)} = \frac{1}{4}[11.25 + 16.28 + 12.5 + 17] = 14.38$$

Fifth iteration

$$u_1^{(5)} = \frac{1}{4}[0 + 11.1 + 6.58 + 13.65] = 7.83$$

$$u_2^{(5)} = \frac{1}{4}[7.83 + 17 + 11.94 + 17.88] = 13.66$$

$$u_3^{(5)} = \frac{1}{4}[13.637 + 19.7 + 16.27 + 21.9] = 17.89$$

$$u_4^{(5)} = \frac{1}{4}[0 + 7.83 + 6.58 + 11.95] = 6.50$$

$$u_5^{(5)} = \frac{1}{4}[6.58 + 13.65 + 11.25 + 16.29] = 11.95$$

$$u_6^{(5)} = \frac{1}{4}[11.95 + 17.89 + 14.34 + 21] = 16.29$$

$$u_7^{(5)} = \frac{1}{4}[0 + 6.50 + 8.7 + 11.25] = 6.64$$

$$u_8^{(5)} = \frac{1}{4}[6.64 + 11.95 + 12.1 + 14.34] = 11.25$$

$$u_9^{(5)} = \frac{1}{4}[11.25 + 16.29 + 12.5 + 17] = 14.34$$

$$u_1 = 7.83$$

$$u_2 = 13.66$$

$$u_3 = 17.89$$

$$u_4 = 6.6$$

$$u_5 = 11.95$$

$$u_6 = 16.29$$

$$u_7 = 6.64$$

$$u_8 = 11.25$$

$$u_9 = 14.34$$

