UNITV

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

PROBLEMS BASED ON FINITE DIFFERENCE SOLUTION OF SECOND ORDER ORDINARY EQUATION

1. **Solve the equation $y'' = x + y$ with boundary conditions $y(0) = y(1) = 0$, numerically taking $\Delta x = 0.25$**

Solution :

Using the central difference approximation ,we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = x_i + y_i \qquad \left[h = \frac{1}{4}\right]$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{(\frac{1}{4})^2} = x_i + y_i$$

$$(16)(y_{i-1} - 2y_i + y_{i+1}) = x_i + y_i$$

$$(16y_{i-1} - 32y_i + 16y_{i+1}) = x_i + y_i$$

$$(16y_{i-1} - 33y_i + 16y_{i+1}) = x_i \qquad i = 1, 2, 3 \ldots\ldots$$

$$y_0 = y_4 = 0$$

$$i = 1 \Rightarrow (16y_0 - 33y_1 + 16y_2) = x_1$$

$$33y_1 + 16y_2 = \frac{1}{4} \ldots\ldots\ldots\ldots(i)$$

$$i = 2 \Rightarrow (16y_1 - 33y_2 + 16y_3) = x_2$$

$$(16y_1 - 33y_2 + 16y_3) = \frac{1}{2} \ldots\ldots\ldots\ldots\ldots..(2)$$

$$i = 3 \Rightarrow (16y_2 - 33y_3 + 16y_4) = x_3$$

$$16y_2 - 33y_3 = \frac{3}{4} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$Solving\ (1),(2) and\ (3) we\ get$$

$$y_1 = -0.0349$$

$$y_2 = -0.0563$$

$$y_3 = -0.050$$

2. Solve the equation $xy'' + y = 0$ with boundary conditions $y(1) = 1, y(2) = 2,$ numerically taking $h = 0.25$ by finite difference method

Solution :

Given : $xy'' + y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$ $here\ h = \frac{1}{4}$

Using the Central difference approximation , we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

(1) $\Rightarrow x_i \frac{y_{i-1}-2y_i+y_{i+1}}{(\frac{1}{4})^2} + y_i$=0

$$16x_i(y_{i-1} - 2y_i + y_{i+1}) + y_i = 0$$

$$16x_iy_{i-1} - 32x_iy_i + 16x_iy_{i+1} + y_i = 0$$

$16x_iy_{i-1} + (-32x_i+1)y_i + 16x_iy_{i+1} = 0$................(2)

The boundary condition

When $i = 1, h = \frac{1}{4}, x_i = 0.25, y_0 = 1$

x_0	x_1	x_2	x_3	x_4
1	1.25	1.5	1.75	2
1	-	-	-	2
y_0	y_1	y_2	y_3	y_4

In equation (2) put $i = 1,2,3$

$$i = 1 \rightarrow 16x_1y_0 + (-32x_1+1)y_1 + 16x_1y_2 = 0$$

$$16(1.25)(1) \quad + (-32(1.25)+1)y_1 + 16(1.25)y_2 = 0$$

$$20-39y_1 + 20y_2 = 0$$

$$-39y_1 + 20y_2 = -20$$

$$39y_1 - 20y_2 = 20 \ldots \ldots \ldots \ldots .(3)$$

$$i = 2 \rightarrow 16x_2y_1 + (-32x_2+1)y_2 + 16x_2y_3 = 0$$

$$16(1.5)y_1 + (-32(1.5)+1)y_2 + 16(1.5)y_3 = 0$$

$$24y_1 - 47y_2 + 24y_3 = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)$$

$$i = 3 \rightarrow 16x_3y_2 + (-32x_3+1)y_3 + 16x_3y_4 = 0$$

$$16(1.75)y_2 + (-32(1.75)+1)y_3 + 16(1.75)2 = 0$$

$$28y_2 + (55)y_3 + 56 = 0 \ldots \ldots \ldots \ldots \ldots (5)$$

Solving $(3),(4) and (5) we\ get$

$$\begin{bmatrix} 39 & -20 & 0 \\ 24 & -47 & 24 \\ 0 & 28 & -55 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ -56 \end{bmatrix}$$

By Using elimination method

$$\sim\begin{bmatrix} 39 & -20 & 0 & | & 20 \\ 24 & -47 & 24 & | & 0 \\ 0 & 28 & -55 & | & -56 \end{bmatrix}$$

$$\sim\begin{bmatrix} 39 & -20 & 0 & | & 20 \\ 0 & -1353 & 936 & | & -480 \\ 0 & 28 & -55 & | & -56 \end{bmatrix} \quad R_2 \leftrightarrow 39R_2 - 24R_1$$

$$\sim\begin{bmatrix} 39 & -20 & 0 & | & 20 \\ 0 & -1353 & 936 & | & -480 \\ 0 & 0 & -48207 & | & -89208 \end{bmatrix} \quad R_3 \leftrightarrow 1353R_3 + 28R_2$$

By back substitution method

$-48207y_3 = -89208$

$$y_3 = 1.8505$$

$$-1353y_2 + 936y_3 = -480$$

$$-1353y_2 + 936(1.8505) = -480$$

$$-1353y_2 + 1732.068 = -480$$

$$-1353y_2 = -480 - 1732.068$$

$$-1353y_2 = -2212.068$$

$$y_2 = 1.6349$$

$$-39y_1 - 20y_2 = 20$$

$$-39y_1 - 20(1.6349) = 20$$

$$-39y_1 - 32.698 = 20$$

$$39y_1 = 52.698$$

$$y_1 = 1.3512$$

x_0	x_1	x_2	x_3	x_4
1	1.25	1.5	1.75	2
1	1.3512	1.6349	1.8505	2
y_0	y_1	y_2	y_3	y_4

3. **Solve the equation $y'' + x^2 y = 0$with boundary conditions $y(0) = 0, y(1) = 1,$ numerically taking $h = 0.25$ by finite difference method**

Solution :

Given $y'' + x^2 y = 0$

Using the Central difference approximation , we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$(1) \rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + x_i{}^2 y_i = 0$$

$$y_{i-1} - 2y_i + y_{i+1} + h^2 x_i{}^2 y_i = 0$$

$$y_{i-1} + (-2 + h^2 x_i{}^2) y_i + y_{i+1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

$$\textit{The boundary conditions are } y(0) = 0, y(1) = 1,$$

$$x_0 = 0, y_0 = 0 \quad x_4 = 1, y_4 = 1$$

To find : $y_1 = y\left(\frac{1}{4}\right), y_2 = y\left(\frac{2}{4}\right) y_3 = y\left(\frac{3}{4}\right) y_4 = y\left(\frac{4}{4}\right) = y(1) = 1$

For $i = 1$ in $(2) we\ get$

$$y_0 + \left(-2 + (\frac{1}{4})^{2^{2}} (\frac{1}{4})^2\right) y_1 + y_2 = 0$$

$$0 + \left(-2 + \frac{1}{256}\right) y_1 + y_2 = 0$$

$$\left(\frac{-511}{256}\right)y_1 + y_2 = 0$$

$$-511y_1 + 256y_2 = 0$$

$$511y_1 - 256y_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

For $i = 2\ in\ (2) we\ get$

$$y_1 + \left(-2 + (\frac{1}{4})^{2^2} (\frac{2}{4})^2\right)y_2 + y_3 = 0$$

$$y_1 + \left(-2 + \frac{4}{256}\right)y_2 + y_3 = 0$$

$$y_1 + \left(\frac{-127}{64}\right)y_2 + y_3 = 0$$

$$64y_1 - 127y_2 + 64y_3 = 0 \ldots\ldots\ldots\ldots\ldots..(4)$$

For $i = 3\ in\ (2) we\ get$

$$y_2 + \left(-2 + (\frac{1}{4})^{2^2} (\frac{3}{4})^2\right)y_3 + y_4 = 0$$

$$y_2 + \left(-2 + \frac{9}{256}\right)y_3 + y_4 = 0$$

$$y_2 + \left(\frac{-503}{256}\right)y_3 + y_4 = 0$$

$$256y_2 - 503y_3 + 256y_4 = 0$$

$$256y_2 - 503y_3 = -256y_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Solving (3) ,(4) and (5) we get

$$y_1 \quad = 0.2617$$

$$y_2 = 0.5223$$

$$y_3 = 0.7748$$

4. **Using finite difference method ,compute $y(0.5)$, $given\ y'' - 64y + 16 = 0, y(0) = y(1) = 0$, Subdividing the interval into (i) 4 equal parts (ii) 2 equal parts**

Solution : $Given\ y'' - 64y + 16 = 0$

(i) we take $h = 0.025$

Using the Central difference approximation , we have

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - 64y_i + 10 = 0$$

$$y_{i-1} - (2 + 64h^2)y_i + y_{i+1} = -10h^2$$

$$y_{i-1} - 6y_i + y_{i+1} = -0.625 \ldots\ldots\ldots (i)$$

$$Put\ i = 1, 2, 3\ \ we\ get$$

$$-6y_1 + y_2 = -0.625 \ldots\ldots\ldots\ldots\ldots (2)$$

$$y_1 - 6y_2 + y_3 = -0.625 \ldots\ldots\ldots\ldots\ldots (3)$$

$$y_2 - 6y_3 = -0.625 \ldots\ldots\ldots\ldots\ldots (4)$$

$y_1 = y_3$ by (2) and (4)

$$-6y_1 + y_2 = -0.625$$

$$2y_1 - 6y_2 = -0.625$$

$$-17y_2 = -2.5$$

$$y_2 = 0.1470$$

$$y_1 = 0.1287$$

$$y_1 = y_3 = 0.1287$$

$$y(0.25) = y(0.75) = 0.1287$$

(ii) Let n=2 then $h = \frac{1}{2}$

$$y_{i-1} - (2 + 64h^2)y_i + y_{i+1} = -10h^2$$

$$y_{i+1} - (2 + 64h^2)y_i + y_{i-1} + 10h^2 = 0$$

$$y_{i+1} - y_{i-1} - 18y_i = -\frac{5}{2}$$

Put i=1, we get

$$y_2 - y_0 - 18y_1 = -\frac{5}{2}$$

$-18y_1 = -\frac{5}{2}$ $\quad [\therefore y_0 = y_2 = 0]$

$y_1 = 0.1389$

$$y(0.5) = 0.1389$$