

Limit of a function

Definition:

Suppose $f(x)$ is defined when x is near the number a , Then we write $\lim_{x \rightarrow a} f(x) = L$ and say the limit of $f(x)$, as x approaches a , equals L .

The above definition says that the value of $f(x)$ approach as x approaches a . In other words, the value of $f(x)$ tend to get closer and closer to L as x gets closer and closer to a from either side of a but $x \neq a$. The alternate notation for $\lim_{x \rightarrow a} f(x) = L$ is $f(x) \rightarrow L$ as $x \rightarrow a$.

One-sided Limits:

Left-hand limit of $f(x)$:

Suppose $f(x)$ is defined when x is near the number from left hand side of a , Then we write $\lim_{x \rightarrow a^-} f(x) = L$ and say the left-hand limit of $f(x)$, as x approaches a .

Right-hand limit of $f(x)$:

Suppose $f(x)$ is defined when x is near the number from right hand side of a , Then we write $\lim_{x \rightarrow a^+} f(x) = L$ and say the right-hand limit of $f(x)$, as x approaches a .

Definition:

Suppose $f(x)$ is defined when x is near the number a . Then we write $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$.

Infinite Limits:

Suppose $f(x)$ is defined on both sides of ' a ' except possibly at ' a ' itself. Then

(i) $\lim_{x \rightarrow a} f(x) = \infty$ means that the value of $f(x)$, can be made arbitrarily large by taking x to be sufficiently close to ' a ' but not equal to a .

(ii) $\lim_{x \rightarrow a} f(x) = -\infty$ means that the value of $f(x)$, can be made arbitrarily large negative by taking x to be sufficiently close to ' a ' but not equal to a .

Example:

 Evaluate $\lim_{x \rightarrow 2} x^2 - x + 2$
Solution:

Let $f(x) = x^2 - x + 2$

X	f(x)	x	f(x)
1.9	3.71	2.1	4.31
1.99	3.9701	2.01	4.0301
1.999	3.997001	2.001	4.003001
1.9999	3.99970001	2.0001	4.00030001
$x < 2$		$x > 2$	

From the table, $\lim_{x \rightarrow 2} x^2 - x + 2 = 4$

Example:

 Find the value of $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$
Solution:

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{(x-1)}$$

$$= x + 1, x \neq 1$$

X	f(x)	x	f(x)
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001
$x < 1$		$x > 1$	

 We can say $f(x)$ approaches the limit 2 as x approaches 1.

$$\therefore \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

Example:

 Investigate $\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$
Solution:

$$\text{Let } f(x) = \sin \frac{\pi}{x}$$

x	1	1/3	0.1	1/2	1.4	0.01
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f(x)	0	0	0	0	0	0
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Our guess $\lim_{x \rightarrow 0} \sin \frac{\pi}{x} = 0$ is wrong.

$$\therefore f\left(\frac{1}{n}\right) = \sin n\pi = 0 \text{ for any integer } n.$$

$$\therefore f\left(\frac{1}{n}\right) = 0 \text{ which is not possible.}$$

$\therefore$ The limit does not exist.

Example:

Use a table of values to estimate the value of the limit $\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$

Solution:

$$\text{Let } f(x) = \lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$$

x	f(x)	x	f(x)
-1	0.2679	1	0.2361
-0.5	0.2583	0.5	0.2426
-0.1	0.2516	0.1	0.2485
-0.05	0.2508	0.05	0.2492
-0.01	0.2502	0.01	0.2498
-0.001	0.25	0.001	0.25

$$\therefore \lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x} = 0.25 = \frac{1}{4}$$

Example:

Evaluate the limit and justify each step for the following:

(i) $\lim_{x \rightarrow -1} (x^4 - 3x)(x^2 + 5x + 3)$

(ii) $\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

(iii) $\lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6}$

Solution:

$$\begin{aligned}
 \text{(i)} \lim_{x \rightarrow -1} (x^4 - 3x)(x^2 + 5x + 3) &= \lim_{x \rightarrow -1} (x^4 - 3x) \lim_{x \rightarrow -1} (x^2 + 5x + 3) \\
 &= \left[\lim_{x \rightarrow -1} x^4 - 3 \lim_{x \rightarrow -1} x \right] \left[\lim_{x \rightarrow -1} x^2 + 5 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 3 \right] \\
 &= [(-1)^4 - 3(-1)][(-1)^2 + 5(-1) + 3] \\
 &= 4(-1) = -4
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x \rightarrow -2} (x^3 + 2x^2 - 1)}{\lim_{x \rightarrow -2} (5 - 3x)} \\
 &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \\
 &= \frac{-8 + 8 - 1}{5 + 6} = \frac{-1}{11}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6} &= \sqrt{\lim_{u \rightarrow -2} (u^4 + 3u + 6)} \\
 &= \sqrt{(-2)^4 + 3(-2) + 6} \\
 &= \sqrt{16 - 6 + 6} = \sqrt{16} = 4
 \end{aligned}$$

Example:

Evaluate $\lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$

Solution:

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(6+h)}{h} \\
 &= \lim_{h \rightarrow 0} 6 + h = 6
 \end{aligned}$$

Example:

Evaluate $\lim_{x \rightarrow -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4}$

Solution:

$$\begin{aligned}
 \lim_{x \rightarrow -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} &= \lim_{x \rightarrow -4} \frac{(x+1)(x+4)}{(x-1)(x+4)} \\
 &= \lim_{x \rightarrow -4} \frac{x+1}{x-1} \\
 &= \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}
 \end{aligned}$$

Example:

Evaluate the limit if it exists $\lim_{x \rightarrow 1} \frac{x^4 - 1}{x^3 - 1}$

Solution:

$$\lim_{x \rightarrow 1} \frac{x^4 - 1}{x^3 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^3 + x^2 + x + 1)}{(x-1)(x^2 + x + 1)}$$

$$= \lim_{x \rightarrow 1} \frac{(x^3 + x^2 + x + 1)}{(x^2 + x + 1)}$$

$$= \frac{1+1+1+1}{1+1+1} = \frac{4}{3}$$

Example:

Evaluate $\lim_{t \rightarrow 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$

Solution:

$$\lim_{t \rightarrow 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} = \lim_{t \rightarrow 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t} \times \frac{\sqrt{1+t} + \sqrt{1-t}}{\sqrt{1+t} + \sqrt{1-t}}$$

$$= \lim_{t \rightarrow 0} \frac{(\sqrt{1+t})^2 - (\sqrt{1-t})^2}{t(\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \rightarrow 0} \frac{1+t - (1-t)}{t(\sqrt{1+t} + \sqrt{1-t})}$$

$$= \lim_{t \rightarrow 0} \frac{2t}{t(\sqrt{1+t} + \sqrt{1-t})}$$

$$= \frac{2}{\sqrt{1+0} + \sqrt{1-0}}$$

$$= \frac{2}{2} = 1$$

Example:

Evaluate $\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x}$

Solution:

$$\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x} = \lim_{x \rightarrow -4} \frac{\frac{x+4}{4x}}{4+x}$$

$$= \lim_{x \rightarrow -4} \frac{1}{4x} = \frac{1}{-16}$$

Example:

Evaluate the limit if it exists $\lim_{x \rightarrow 2} \frac{x^2 - x + 6}{x - 2}$

Solution:

$$\lim_{x \rightarrow 2} \frac{x^2 - x + 6}{x - 2} = \frac{8}{0} = \infty$$

So the limit does not exist.

Example:

Evaluate the limit if it exists $\lim_{x \rightarrow -1} \frac{x^2 - 4x}{(x-4)(x+1)}$

Solution:

$$\lim_{x \rightarrow -1} \frac{x^2 - 4x}{(x-4)(x+1)} = \lim_{x \rightarrow -1} \frac{x(x-4)}{(x-4)(x+1)}$$

$$= \lim_{x \rightarrow -1} \frac{x}{(x+1)}$$

$$= \frac{-1}{0} = \infty$$

∴ The limit does not exist.

Example:

Prove that $\lim_{x \rightarrow 0} |x| = 0$

Solution:

$$|x| = f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0 \text{ for } |x| = x, x > 0$$

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0 \text{ for } |x| = -x, x < 0$$

$$\therefore \lim_{x \rightarrow 0^+} |x| = 0 = \lim_{x \rightarrow 0^-} |x|$$

$$\lim_{x \rightarrow 0} |x| = 0$$

Example:

Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

Solution:

$$\text{Let } f(x) = \frac{|x|}{x}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \left(\frac{x}{x} \right) = \lim_{x \rightarrow 0^+} (1)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \left(\frac{-x}{x} \right) = \lim_{x \rightarrow 0^-} (-1) = -1$$

$$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

∴ $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

Example:

Let $g(x) = \frac{x^2+x-6}{|x-2|}$ does $\lim_{x \rightarrow 2} g(x)$ exist?

Solution:

$$\begin{aligned} \lim_{x \rightarrow 2^-} g(x) &= \lim_{x \rightarrow 2^-} \frac{x^2+x-6}{-(x-2)} \\ &= \lim_{x \rightarrow 2^-} \frac{(x-2)(x+3)}{-(x-2)} \\ &= \lim_{x \rightarrow 2^-} -(x+3) \\ &= -(2+3) = -5 \end{aligned}$$

$$\lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} \frac{x^2+x-6}{(x-2)}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 2^+} \frac{(x-2)(x+3)}{(x-2)} \\
 &= \lim_{x \rightarrow 2^+} (2+3) \\
 &= 5
 \end{aligned}$$

$$\lim_{x \rightarrow 2^-} g(x) \neq \lim_{x \rightarrow 2^+} g(x)$$

$\therefore \lim_{x \rightarrow 2} g(x)$ does not exist.

Example:

Find the limit if it exist $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

Solution:

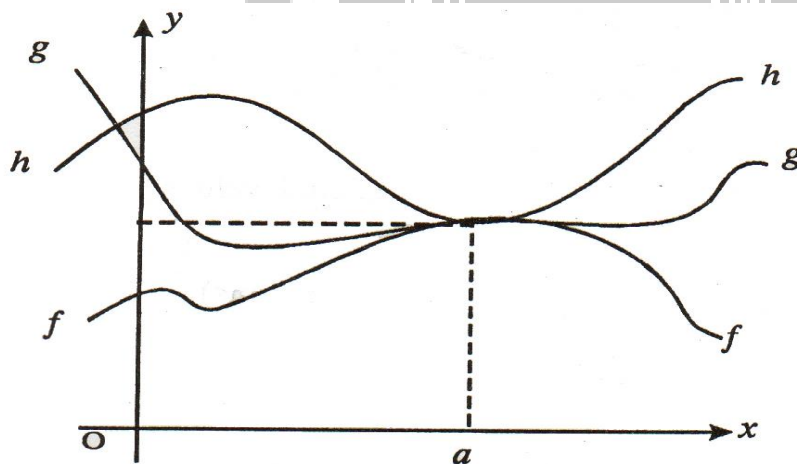
$$\begin{aligned}
 \lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) &= \lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{-x} \right) \\
 &= \lim_{x \rightarrow 0^-} \left(\frac{1}{x} + \frac{1}{x} \right) \\
 &= \lim_{x \rightarrow 0^-} \left(\frac{2}{x} \right) \\
 &= \frac{2}{0} = \infty
 \end{aligned}$$

$\therefore$ Limit does not exist.

Squeeze theorem (or) Sandwich theorem (or) Pinching theorem:

Statement:

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$ then $\lim_{x \rightarrow a} g(x) = L$



ie , If $g(x)$ is squeezed in between $h(x)$ and $f(x)$ which have the same limit L then $g(x)$ also forced to have the same limit L .

Example:

Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$

Solution:

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = \lim_{x \rightarrow 0} x^2 \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

Here $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist.

$\therefore$ By applying $x \rightarrow 0$ Squeeze theorem,

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$-x^2 \leq \sin \frac{1}{x} \leq x^2$$

$$\lim_{x \rightarrow 0} (-x^2) = 0 \text{ and } \lim_{x \rightarrow 0} (x^2) = 0$$

By Squeeze theorem, $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$

Example:

Find $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta}$

Solution:

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta \cos \theta} \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta} \right) \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) \cdot \lim_{\theta \rightarrow 0} \left(\frac{1}{\cos \theta} \right) \\ &= 1 \cdot 1 = 1 \end{aligned}$$

Example:

Find $\lim_{\theta \rightarrow 0} \frac{1 - \cos x}{x}$

Solution:

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{\theta \rightarrow 0} \frac{2 \sin^2 \left(\frac{x}{2} \right)}{x} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin^2 \left(\frac{x}{2} \right)}{\left(\frac{x}{2} \right)} \times \frac{\left(\frac{x}{2} \right)}{\left(\frac{x}{2} \right)} \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \left(\frac{x}{2} \right)}{\left(\frac{x}{2} \right)} \right)^2 \times \lim_{\theta \rightarrow 0} \left(\frac{x}{2} \right) \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \left(\frac{x}{2} \right)}{\left(\frac{x}{2} \right)} \right)^2 \times \lim_{\theta \rightarrow 0} \left(\frac{x}{2} \right) \\ &= 1 \times 0 = 0 \end{aligned}$$

Example:

Find $\lim_{x \rightarrow \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2\cos^2 x}{(\pi-2x)^2} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 \sin\left(\frac{\pi}{2}-x\right)}{2^2\left(\frac{\pi}{2}-x\right)^2} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{2} \left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} \right]^2 \\ &= \frac{1}{2} \lim_{\left(x-\frac{\pi}{2}\right) \rightarrow 0} \left[\frac{\sin(-)\left(x-\frac{\pi}{2}\right)}{-\left(x-\frac{\pi}{2}\right)} \right]^2 = \frac{1}{2} (1) = \frac{1}{2}\end{aligned}$$

Example:

Find $\lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2}$

Solution:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2} &= \lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2\left(\frac{1}{3^2}\right)} \times \left(\frac{1}{3^2}\right) \\ &= \lim_{x \rightarrow 0} \frac{\sin^2\left(\frac{x}{3}\right)}{\left(\frac{x}{3}\right)^2} \times \left(\frac{1}{3^2}\right) \\ &= \lim_{x \rightarrow 0} \left[\frac{\sin\left(\frac{x}{3}\right)}{\left(\frac{x}{3}\right)} \right]^2 \times \lim_{x \rightarrow 0} \left(\frac{1}{9}\right) \\ &= 1 \times \frac{1}{9} = \frac{1}{9}\end{aligned}$$