Cauchy's integral formula

Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected region R

and if 'a' is any point interior to C, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

(OR)

$$\int_c \frac{f(x)}{z-a}dz = 2\pi i\, f(a),$$

the integration around C being taken in the positive direction.

Cauchy's Integral formula for derivatives

Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected Region R

and if 'a' is any point interior to C, then

$$\int_c \frac{f(z)}{(z-a)^2}dz = 2\pi i f'(a)$$

$$\int_c \frac{f(z)}{(z-a)^3}dz = 2\pi i f''(a)$$

In general, $\int_c \frac{f(z)}{(z-a)^n}dz = 2\pi i f^{(n-1)}(a)$

Example: Evaluate $\int_c \frac{e^{2z}}{z^2+1}dz, where\ C\ is\ |z| = \frac{1}{2}$

Solution:

Given $\int_c \frac{e^{2z}}{z^2+1}dz$

$Dr = 0 \Rightarrow z^2 + 1 = 0 \Rightarrow z = \pm i$

Given $C\ is\ |z| = \frac{1}{2}$

$\Rightarrow |z| = |\pm i| = 1 > \frac{1}{2}$

∴Clearly both the points $z = \pm i$ lies outside C.

∴ By Cauchy's Integral Theorem, $\int_c \frac{e^{2z}}{z^2+1}dz = 0$

Example: Using Cauchy's integral formula Evaluate $\int_c \frac{z+1}{(z-3)(z-1)}dz, where\ C\ is\ |z| =$

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Solution:

Given $\int_c \frac{z+1}{(z-3)(z-1)}dz$

$Dr = 0 \Rightarrow z = 3, 1$

Given $C\ is\ |z| = 2$

∴Clearly $z = 1$ lies inside C and $z = 3$ lies outside C

$$\int_c \frac{z+1}{(z-3)(z-1)}dz = \int_c \frac{(z+1)/(z-3}{(z-1)}dz$$

∴ By Cauchy's Integral Theorem

$$\int_c \frac{(z+1)/(z-3)}{(z-1)}dz = 2\pi i f(1)$$ Where $f(z) = \frac{z+1}{z-3} \Rightarrow f(1) = \frac{2}{-2}$

$$= 2\pi i(-1) = -2\pi i$$

Example: Using Cauchy's integral formula, evaluate $\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}dz$ where C is the circle

$|z| = 4$.

Solution:

Given $\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}dz$

$Dr = 0 \Rightarrow z = 2, 3$

Given $C\ is\ |z| = 4$

∴Clearly $z = 2\ and\ 3$ lies inside C.

Consider, $\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$$\Rightarrow 1 = A(z-3) + B(Z = 2)$$

Put $z = -3 \Rightarrow 1 = B$

Put $z = 2 \Rightarrow -1 = A$

∴ $\frac{1}{(z-2)(z-3)} = -\frac{1}{z-2} + \frac{1}{z-3}$

$$\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}dz = -\int \frac{\sin\pi z^2+\cos\pi z^2}{z-2}dz + \int \frac{\sin\pi z^2+\cos\pi z^2}{z-3}dz$$

$$= -2\pi i f(2) + 2\pi i\, f(3)$$ Where $f(z) = \sin(\pi z^2) + \cos\pi z^2$

$$= -2\pi i(1) + 2\pi i(-1)$$ $f(2) = \sin 4\pi + \cos 4\pi = 1$

$$= -4\pi i$$ $f(3) = \sin 9\pi + \cos 9\pi - 1 =$
-1

Example: Evaluate $\int_c \frac{z+4}{z^2+2z+5}$ Where C is the circle $(i)|z+1+i| = 2$ $(ii)|z+1-i| = 2$

$(\boldsymbol{iii})\ |\boldsymbol{z}| = \mathbf{1}$

Solution:

Given $\int_c \frac{z+4}{z^2+2z+5} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 5 = 0$

$\Rightarrow z = \frac{-2\pm\sqrt{4-20}}{2}$

$\Rightarrow z = -1 \pm 2i$

$\therefore \int_c \frac{z+4}{z^2+2z+5} dz = \int_c \frac{(z+4)\ dz}{[z-(-1+2i)[z-(-1-2i)]}$

(i) $|z + 1 + i| = 2\ is\ the\ circle$

When $z = -1 + 2i, |-1 + 2i + 1 + i| = |3i| > 2$ lies outside C.

When $z = -1 - 2i, |-1 - 2i + 1 + i| = |-i| < 2$ lies inside C.

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{[(z+1)/(z-(-1+2i)]}{[z-(-1-2i)]} dz = 2\pi i f(-1 - 2i)$ Where $f(z) = \frac{z+4}{[z-(-1+2i)]}$

$= 2\pi i \left[\frac{3-2i}{-4i}\right]$ $f(-1 - 2i) = \frac{-1-2i+4}{-1-2i+1-2i} = \frac{3-2i}{-4i}$

$= \frac{\pi}{2}(2i - 3)$

(ii) $|z + 1 - i| = 2$ is the circle

When $z = -1 + 2i, |-1 + 2i + 1 - i| = |i| < 2$ lies inside C

When $z = -1 - 2i, |-1 - 2i + 1 - i| = |-3i| > 2$ lies outside C

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{(z+1)/[z-(-1-2i)]}{[z-(-1+2i)]} dz = 2\pi i\ f(-1 + 2i)$ Where $f(z) = \frac{z+4}{z-(-1-2i)}$

$= 2\pi i \frac{[3+2i]}{4i}$ $f(-1 + 2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$

$= \frac{\pi}{2}(3 + 2i)$

(iii) $|z| = 1$ is the circle

When $z = -1 + 2i, 1 - 1 + 2i| = \sqrt{5} > 1$ lies outside C

When $z = -1 - 2i, 1 - 1 - 2i| = \sqrt{5} > 1$ lies outside C

$\therefore$ By Cauchy's Integral theorem

$$\int_c \frac{z+4}{z^2+2z+5} dz = 0$$

Example: Using Cauchy's integral formula, evaluate $\int_c \frac{z+1}{z^2+2z+4} dz$ where C is the circle

$|z+1+i| = 2$

Solution:

Given $\int_c \frac{z+1}{z^2+2z+4} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 4 = 0$

$\Rightarrow z = \frac{-2 \pm \sqrt{4-16}}{2}$

$\Rightarrow z = -1 \pm i\sqrt{3}$

$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \int_c \frac{(z+1\ )dz}{[z-(-1+i\sqrt{3})][z-(-1-i\sqrt{3}]}$

Given $C\ is\ |z+1+i| = 2$

When $z = -1 - i\sqrt{3}, \left|-1 - i\sqrt{3} + 1 + i\right| = |(1-\sqrt{3}i)| < 2$ lies inside C.

When $z = -1 + i\sqrt{3}, \left|-1 + i\sqrt{3} + 1 + i\right| = |i + \sqrt{3}i| > 2$ lies outside C.

$\therefore$ By Cauchy's Integral Formula

$\int_c \frac{(z+1)/[z-(-1+i\sqrt{3})]}{[z-(-1-i\sqrt{3})]} dz = 2\pi i\, f(-1-i\sqrt{3})$ Where $f(z) = \frac{z+1}{z-(-1+i\sqrt{3})}$

$= 2\pi i \left(\frac{1}{2}\right) = \pi i$ $f(-1-i\sqrt{3}) = \frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}} = \frac{\sqrt{3}i}{-2i\sqrt{3}} = \frac{1}{2}$

$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \pi i$

Example: Evaluate $\int_c \frac{z^2+1}{z^2-1} dz$ where C is the circle $(i)|z-1| = 1\ (ii)|z+1| = 1 (iii)|z-i| = 1$

Solution:

Given $\int_c \frac{z^2+1}{z^2-1} dz = \int_c \frac{z^2+1}{(z+1)(z-1)} dz$

$Dr = 0 \Longrightarrow z = 1, -1$

(i) $(z-1) = 1$ is the circle

When $z = 1$, $|1-1| = 0 < 1$ lies inside C

When $z = -1, |-1-1| = 2 > 1$ lies outside C

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{z^2+1}{(z+1)(z-1)} dz = \int_c \frac{(z^2+1)/z+1}{(z-1)} dz$

$= 2\pi i f(1)$ where $f(z) = \frac{z^2+1}{z+1} \Rightarrow f(1) = 1$

$= 2\pi i(1)$

$= 2\pi i$

(ii)$|z+1| = 1$ is the circle

When $z = 1, |1+1| = 2 > 1$ lies outside C

When $z = -1, |-1+1| = 0 < 1$ lies inside C

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{(z^2+1)/(z-1)}{z+1} dz = 2\pi i\, f(-1)$ where $f(z) = \frac{z^2+1}{z-1} \Rightarrow$ $f(-1) = -1$

$= 2\pi i(-1) = -2\pi i$

(iii) $|z-i| = 1$ is the circle

When $z = 1, |1-i| = \sqrt{2} > 1$ lies outside C

When $z = -1, |-1-i| = \sqrt{2} > 1$ lies outside C

$\therefore$ By Cauchy's Integral Formula

$$\int_c \frac{(z^2+1)}{(z+1)(z-1)} dz = 0$$

Example: Using Cauchy's Integral formula evaluate $\int_c \frac{zdz}{(z-1)(z-2)^2}$ where C is the circle $|z-2| = \frac{1}{2}$

Solution:

Given $\int_c \frac{zdz}{(z-1)(z-2)^2}$

$Dr = 0 \Longrightarrow z = 1$ is a pole of order 1, $z = 2$ is a pole of order 2.

Given C is $|z-2| = \frac{1}{2}$

When $z = 1, |1-2| = 1 > \frac{1}{2}$ lies outside C.

When $z = 2, |2-2| = 0 < \frac{1}{2}$ lies inside C.

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{z/z-1}{(z-2)^2} dz = 2\pi i\, f'(2)$ Where $f(z) = \frac{z}{z-1}$

$= 2\pi i(-1)$ $f'(z) = \frac{(z-1)1-z(1)}{(z-1)^2} \Rightarrow f'(2) = -1$

$= -2\pi i$

Example: Evaluate $\int_c \frac{sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is the circle $|z| = 1$

Solution:

Given $\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3} dz$

$Dr = 0 \Rightarrow z = \frac{\pi}{6}$ is a pole of order 3.

Give C is $|z| = 1$.

Clearly $z = \frac{\pi}{6}$ lies inside the circle $|z| = 1$

∴ By Cauchy's Integral formula

$\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3} dz = \frac{2\pi i}{2!} f''(\pi/6)$ Where $f(z) = sin^2 z$

$= \frac{2\pi i}{2!}(1)$ $f'(z) = 2\sin z \cos z = \sin 2z$

$= \pi i$ $f''(z) = \cos 2z(2) \Rightarrow f''\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{2\pi}{6}\right)$

$= 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$

Example: Evaluate $\int_c \frac{z}{(z-1)^3} dz$ where C is the circle $|z| = 2$, using Cauchy's Integral formula

Solution:

Given $\int_c \frac{z}{(z-1)^3} dz$

$Dr = 0 \Rightarrow z = 1$ is a pole of order 3.

Given C is $|z| = 2$.

Clearly $z = 1$ lies inside the circle C

∴ By Cauchy's Integral formula

$\int_c \frac{sin^2 z}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$ Where $f(z) = z \Rightarrow f'(z) = 1$

$= \frac{2\pi i}{2!}(0)$ $\Rightarrow f''(z) = 0 \Rightarrow f''(1) = 0$

$= 0$

Example: Evaluate $\int_c \frac{z^2}{(2z-1)^2} dz$ where C is the circle $|z| = 1$

Solution:

Given $\int_c \frac{z^2}{(2z-1)^2} dz$

$Dr = 0 \Rightarrow 2z = 0 \Rightarrow z = \frac{1}{2}$ is a pole of order 2.

Given C is $|z| = 1$.

Clearly $z = \frac{1}{2}$ lies inside the circle C

∴ By Cauchy's Integral formula

$$\int_c \frac{z^2}{2^2(z-\frac{1}{2})^2}dz = \frac{1}{4}\int_c \frac{z^2}{\left(z-\frac{1}{2}\right)^2}dz \qquad \text{Where } f(z) = z^2 \Rightarrow f'(z) = 2z$$

$$= \frac{1}{4}\left(2\pi i f'\left(\frac{1}{2}\right)\right) \qquad \Rightarrow f'\left(\frac{1}{2}\right) = 1$$

$$= \frac{1}{2}\pi i(1)$$

$$= \frac{\pi i}{2}$$