SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

A solution or integral of a partial differential equation is a relation connecting the dependent and the independent variables which satisfies the given differential equation. A partial differential equation can result both from elimination of arbitrary constants and from elimination of arbitrary functions as explained in section 1.2. But, there is a basic difference in the two forms of solutions. A solution containing as many arbitrary constants as there are independent variables is called a complete integral. Here, the partial differential equations contain only two independent variables so that the complete integral will include two constants.A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Singular Integral

Let $f(x,y,z,p,q) = 0$ ---------- (1)

be the partial differential equation whose complete integral is

$\phi(x,y,z,a,b) = 0$----------- (2)

where a‟‟ and b‟‟ are arbitrary constants.

Differentiating (2) partially w.r.t. a and b, we obtain

$$\frac{\partial \phi}{\partial a} = 0 \qquad \text{-----------} \ (3)$$

and

$$\frac{\partial \phi}{\partial b} = 0 \qquad \text{-----------} \ (4)$$

The eliminant of „a‟‟ and „b‟‟ from the equations (2), (3) and (4), when it exists, is called the singular integral of (1).

General Integral

In the complete integral (2), put $b = F(a)$, we get

$\phi(x,y,z,a, F(a)) = 0$ ---------- (5)

Differentiating (2), partially w.r.t.a, we get

The eliminant of „a‟‟ between (5) and (6), if it exists, is called the general integral of (1).

SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS.

The first order partial differential equation can be written as

$$f(x,y,z, p,q) = 0,$$

where $p = \partial z/\partial x$ and $q = \partial z / \partial y$. In this section, we shall solve some standard forms of equations by special methods.

Standard I : f (p,q) = 0. i.e, equations containing p and q only.

Suppose that $z = ax + by +c$ is a solution of the equation $f(p,q) = 0$, where $f (a,b)= 0$.

Solving this for b, we get $b = F (a)$.

Hence the complete integral is $z = ax + F(a) y +c$------------ (1)

Now, the singular integral is obtained by eliminating a & c between

$$z = ax + y F(a) + c \quad 0 = x + y F'(a)$$

$$0 = 1.$$

The last equation being absurd, the singular integral does not exist in this case.

To obtain the general integral, let us take $c = \Phi (a)$.

Then, $z = ax + F(a) y + \Phi (a)$ -------------- (2)
Differentiating (2) partially w.r.t. a, we get

$$0 = x + F'(a). y + \Phi '(a) \quad \text{--------------- (3)}$$

Eliminating between (2) and (3), we get the general equations

Example 8

Solve $pq = 2$

The given equation is of the form f (p,q) = 0

The solution is $z = ax + by + c$, where $ab = 2$.
Solving, $b = 2/b$

The complete integral is

$Z = ax + 2/a\, y + c$ ---------- (1)

Differentiating (1) partially w.r.t „c‟, we

$0 = 1$,

which is absurd. Hence, there is no singular integral.

To find the general integral, put $c = \Phi(a)$ in (1), we get

$Z = ax + 2/a\, y + \Phi(a)$

Differentiating partially w.r.t „a‟, we get

$0 = x - 2/a^2\, y + \Phi'(a)$

Eliminating „a‟ between these equations gives the general integral.

Example 9

Solve $pq + p + q = 0$

The given equation is of the form f (p,q) = 0.

The solution is $z = ax + by + c$, where $ab + a + b = 0$.

Solving, we get

$$b = -\frac{a}{1+a}$$

Hence the complete Integral is $z = ax - \left(\frac{a}{1+a}\right) y + c$ ------ (1)

Differentiating (1) partially w.r.t. „c", we get

0 = 1.

The above equation being absurd, there is no singular integral for the given partial differential equation.

To find the general integral, put c = Φ (a) in (1), we have

$$z = ax - \left(\frac{a}{1+a}\right) y + \Phi(a) \qquad \text{------(2)}$$

Differentiating (2) partially w.r.t a, we get

$$0 = x - \frac{1}{(1+a)^2} y + \Phi'(a) \qquad 1 \qquad \text{----- (3)}$$

Example 10

Solve $p^2 + q^2 = npq$

The solution of this equation is z = ax + by + c, where $a^2 + b^2 = nab$.

Solving, we get

$$b = a\left(\frac{n \pm \sqrt{(n^2-4)}}{2}\right)$$

Hence the complete integral is

$$z = ax + a\left(\frac{n \pm \sqrt{n^2-4}}{2}\right) y + c \qquad \text{-----------(1)}$$

Differentiating (1) partially w.r.t c, we get 0 = 1, which is absurd. Therefore, there is no singular integral for the given equation.
To find the general Integral, put C = Φ (a), we get

$$z = ax + a\left(\frac{n + \sqrt{n^2 - 4}}{2}\right) y + \Phi(a)$$

Differentiating partially w.r.t 'a', we have

$$0 = x + \left(\frac{n \pm \sqrt{n^2 - 4}}{2}\right) y + \Phi'(a)$$

The eliminant of a" between these equations gives the general integral

Standard II : Equations of the form f (x,p,q) = 0, f (y,p,q) = 0 and f (z,p,q) = 0. i.e, one of the variables x,y,z occurs explicitly.

(i) Let us consider the equation f (x,p,q) = 0.
Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or dz = pdx + qdy

Assume that q = a.

Then the given equation takes the form f (x, p,a) = 0

Solving, we get p = Φ(x,a).

Therefore, dz = Φ(x,a) dx + a dy.

(ii) Let us consider the equation f(y,p,q) = 0. Assume that p = a.

Then the equation becomes f (y,a, q) = 0 Solving, we get q = Φ (y,a).

Therefore, dz = adx + Φ(y,a) dy.

Integrating, z = ax + ∫Φ(y,a) dy + b, which is a complete Integral.

(iii) Let us consider the equation f(z, p, q) = 0.

Assume that q = ap.

Then the equation becomes f (z, p, ap) = 0

Solving, we get p = Φ(z,a). Hence dz = Φ(z,a) dx + a Φ(z, a) dy.

$$\text{ie, } \frac{dz}{\Phi(z,a)} = dx + ady.$$

$$\text{Integrating, } \int \frac{dz}{\Phi(z,a)} = x + ay + b, \text{ which is a complete Integral.}$$

Example 11

Solve $q = xp + p^2$

Given $q = xp + p^2$ -------------(1)

This is of the form f (x,p,q) = 0.

Put q = a in (1), we get

$a = xp + p^2$

i.e, $p^2 + xp - a = 0$.

Therefore,

$$p = \frac{-x + \sqrt{(x^2 + 4a)}}{2}$$

Integrating , $$z = \int \left(\frac{-x \pm \sqrt{x^2 + 4a}}{2} \right) dx + ay + b$$

Thus, $$z = -\frac{x^2}{4} \pm \left\{ \frac{x}{4} \sqrt{(4a + x^2)} + a \sinh^{-1}\left(\frac{x}{2\sqrt{a}} \right) \right\} + ay + b$$

Example 12

Solve $q = yp^2$

This is of the form f (y,p,q) = 0

Then, put p = a.

Therfore, the given equation becomes $q = a^2y$.

Since dz = pdx + qdy, we have

$dz = adx + a^2y\, dy$

Integrating, we get $z = ax + (a^2y^2/2) + b$

Example 13

Solve $9 (p^2z + q^2) = 4$

This is of the form f (z,p,q) = 0

Then, putting q = ap, the given equation becomes

$9 (p^2z + a^2p^2) = 4$

Then, putting q = ap, the given equation becomes

$9 (p^2z + a^2p^2) = 4$

Therefore, $p = \pm \dfrac{2}{3(\sqrt{z + a^2})}$

and $q = \pm \dfrac{2a}{3(\sqrt{z + a^2})}$

Since dz = pdx + qdy,

$$dz = \pm \frac{2}{3} \frac{1}{\sqrt{z + a^2}} dx \pm \frac{2}{3} a \frac{1}{\sqrt{z + a^2}} dy$$

Multiplying both sides by $\sqrt{z + a^2}$, we get

$\sqrt{z + a^2}\, dz = \dfrac{2}{3} dx + \dfrac{2}{3} a\, dy$, which on integration gives,

$$\frac{(z+a^2)^{3/2}}{3/2} = \frac{2}{3} x + \frac{2}{3} ay + b.$$

or $(z + a^2)^{3/2} = x + ay + b$.

Standard III : $f_1(x,p) = f_2(y,q)$. ie, equations in which 'z' is absent and the variables are separable.

Let us assume as a trivial solution that

f(x,p) = g(y,q) = a (say).

Solving for p and q, we get p = F(x,a) and q = G(y,a).

But $dz = \dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dy$

Hence dz = pdx + qdy = F(x,a) dx + G(y,a) dy

Therefore, z = ∫F(x,a) dx + ∫G(y,a) dy + b , which is the complete integral of the given equation containing two constants a and b. The singular and general integrals are found in the usual way.

Example 14

Solve pq = xy

The given equation can be written as

The given equation can be written as

$$\frac{p}{x} = \frac{y}{q} = a \text{ (say)}$$

Therefore, $\frac{p}{x} = a$ implies $p = ax$

and $\frac{y}{q} = a$ implies $q = \frac{y}{a}$

Since dz = pdx + qdy, we have

$$dz = ax\,dx + \frac{y}{a}\,dy,$$ which on integration gives.

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + b$$

Example 15

Solve $p^2 + q^2 = x^2 + y^2$

The given equation can be written as $p^2 - x^2 = y^2 - q^2 = a^2$ (say)

$p^2 - x^2 = a^2$ Implies $p = \sqrt{(a^2 + x^2)}$

and $y^2 - q^2 = a^2$ Implies $q = \sqrt{(y^2 - a^2)}$

But dz = pdx + qy

ie, $dz = \sqrt{a^2 + x^2}\, dx + \sqrt{y^2 - a^2}\, dy$

Integrating, we get

$$z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{y}{a}\right) + b$$

Standard IV (Clairaut's) form

Equation of the type $z = px + qy + f(p,q)$ ------(1) is known as Clairaut"s
Differentiating (1) partially w.r.t x and y, we get p = a and q = b.

Therefore, the complete integral is given by
$z = ax + by + f(a,b)$.

Example 16

Solve $z = px + qy + pq$

The given equation is in Clairaut"s. form

Putting p = a and q = b, we have

$z = ax + by + ab$ -------- (1)

which is the complete integral.

To find the singular integral, differentiating (1) partially w.r.t a and b, we get

$0 = x + b$

$0 = y + a$

Therefore we have, a = -y and b= -x.

Substituting the values of a & b in (1), we get

$$z = -xy - xy + xy$$

or $z + xy = 0$, which is the singular integral.

To get the general integral, put $b = \Phi(a)$ in (1).

Then $z = ax + \Phi(a)y + a\,\Phi(a)$ ---------- (2)

Differentiating (2) partially w.r.t a, we have

$$0 = x + \Phi'(a)\,y + a\Phi'(a) + \Phi(a) \qquad \text{---------- (3)}$$

Eliminating a‟ between (2) and (3), we get the general integral.

Example 17

Find the complete and singular solutions of

$$z = px + qy + \sqrt{1 + p^2 + q^2}$$

The complete integral is given by

$$z = ax + by + \sqrt{1 + a^2 + b^2} \qquad \text{-------- (1)}$$

To obtain the singular integral, differentiating (1) partially w.r.t a & b. Then,

$$0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}}$$

$$0 = y + \frac{b}{\sqrt{1 + a^2 + b^2}}$$

Therefore,

$$x = \frac{-a}{\sqrt{(1 + a^2 + b^2)}} \qquad \text{-----------(2)}$$

and

$$y = \frac{-b}{\sqrt{(1 + a^2 + b^2)}} \qquad \text{---------- (3)}$$

Squaring (2) & (3) and adding, we get

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

Now, $$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2}$$

i.e, $$1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

Therefore,

$$\sqrt{(1 + a^2 + b^2)} = \frac{1}{\sqrt{1 - x^2 - y^2}} \qquad \text{-----------}(4)$$

Using (4) in (2) & (3), we get

$$x = -a\sqrt{1 - x^2 - y^2}$$

and $$y = -b\sqrt{1 - x^2 - y^2}$$

Hence, $$a = \frac{-x}{\sqrt{1-x^2-y^2}} \quad \text{and} \quad b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Substituting the values of a & b in (1) , we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

which on simplification gives

$$z = \sqrt{1 - x^2 - y^2}$$

or $x^2 + y^2 + z^2 = 1$, which is the singular integral.

EQUATIONS REDUCIBLE TO THE STANDARD FORMS

Sometimes, it is possible to have non –linear partial differential equations of the first order which do not belong to any of the four standard forms discussed earlier. By changing the variables suitably, we will reduce them into any one of the four standard forms.

Type (i) : Equations of the form F(x^m p, y^nq) = 0 (or) F (z, x^mp, y^nq) = 0.
Case(i) : If m ≠1 and n ≠1, then put x^{1-m} = X and y^{1-n} = Y.

$$\text{Now, } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1-m)\,x^{-m}$$

$$\text{Therefore, } x^m p = \frac{\partial z}{\partial X}(1-m) = (1-m)P, \text{ where } P = \frac{\partial z}{\partial X}$$

$$\text{Similarly, } y^n q = (1-n)Q, \text{ where } Q = \frac{\partial z}{\partial Y}$$

Hence, the given equation takes the form F(P,Q) = 0 (or) F(z,P,Q) = 0.

Case(ii) : If m = 1 and n = 1, then put log x = X and log y = Y.

$$\text{Now, } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\,\frac{1}{x}$$

$$\text{Therefore, } xp = \frac{\partial z}{\partial X} = P.$$

Similarly, yq =Q.

Example 18

Solve $x^4p^2 + y^2zq = 2z^2$

The given equation can be expressed as

$(x^2p)^2 + (y^2q)z = 2z^2$

Here m = 2, n = 2

Put $X = x^{1-m} = x^{-1}$ and $Y = y^{1-n} = y^{-1}$.
We have $x^m p = (1-m) P$ and $y^n q = (1-n)Q$
i.e, $x^2 p = -P$ and $y^2 q = -Q$.
Hence the given equation becomes
$P^2 - Qz = 2z^2$ ----------(1)
This equation is of the form f (z,P,Q) = 0.

Let us take Q = aP.

Then equation (1) reduces to
$P^2 - aPz = 2z^2$

Hence, $$P = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z$$

and $$Q = a\left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z$$

Since $dz = PdX + QdY$, we have

$$dz = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z\,dX + a\left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z\,dY$$

i.e, $$\frac{dz}{z} = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right)(dX + a\,dY)$$

Integrating, we get

$$\log z = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)(X + aY) + b$$

Therefore, $\log z = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right)\left(\frac{1}{x} + \frac{a}{y}\right) + b$ which is the complete solution.

Example 19

Solve $x^2p^2 + y^2q^2 = z^2$

The given equation can be written as

$(xp)^2 + (yq)^2 = z^2$

Here m = 1, n = 1.

Put X = log x and Y = log y.
Then xp = P and yq = Q.

Hence the given equation becomes
$P^2 + Q^2 = z^2$ ------------- (1)
This equation is of the form F(z,P,Q) = 0.

Therefore, let us assume that Q = aP.

Now, equation (1) becomes,
$P^2 + a^2 P^2 = z^2$

$$P^2 + a^2 P^2 = z^2$$

Hence $$P = \frac{z}{\sqrt{(1+a^2)}}$$

and $$Q = \frac{az}{\sqrt{(1+a^2)}}$$

Since dz = PdX + QdY, we have

$$dz = \frac{z}{\sqrt{(1+a^2)}}dX + \frac{az}{\sqrt{(1+a^2)}}dY.$$

i.e, $$\sqrt{(1+a^2)}\frac{dz}{z} = dX + a\,dY.$$

Integrating, we get

$\sqrt{(1+a^2)} \log z = X + aY + b.$

Therefore, $\sqrt{(1+a^2)} \log z = \log x + a \log y + b$, which is the complete solution.

Type (ii) : Equations of the form $F(z^k p, z^k q) = 0$ (or) $F(x, z^k p) = G(y, z^k q)$.

Case (i) : If $k \neq -1$, put $Z = z^{k+1}$,

$$\text{Now } \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = (k+1)z^k.\frac{\partial z}{\partial x} = (k+1)z^k p.$$

$$\text{Therefore, } z^k p = \frac{1}{k+1}\frac{\partial Z}{\partial x}$$

$$\text{Similarly, } z^k q = \frac{1}{k+1}\frac{\partial Z}{\partial y}$$

Case (ii) : If $k = -1$, put $Z = \log z$.

$$\text{Now, } \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = \frac{1}{z}p$$

$$\text{Similarly, } \frac{\partial Z}{\partial y} = \frac{1}{z}q.$$

Example 20

Solve $z^4q^2 - z^2p = 1$

The given equation can also be written as

$(z^2q)^2 - (z^2p) = 1$
Here $k = 2$. Putting $Z = z^{k+1} = z^3$, we get

$$Z^k p = \frac{1}{k+1}\,\frac{\partial Z}{\partial x} \quad \text{and} \quad Z^k q = \frac{1}{k+1}\,\frac{\partial Z}{\partial y}$$

$$\text{i.e, } Z^2 p = \frac{1}{3}\,\frac{\partial Z}{\partial x} \quad \text{and} \quad Z^2 q = \frac{1}{3}\,\frac{\partial Z}{\partial y}$$

Hence the given equation reduces to

$$\left(\frac{Q}{3}\right)^2 - \left(\frac{P}{3}\right) = 1$$

i.e, $Q^2 - 3P - 9 = 0,$

i.e, $Q^2 - 3P - 9 = 0,$

which is of the form $F(P,Q) = 0$.

Hence its solution is $Z = ax + by + c$, where $b^2 - 3a - 9 = 0$.

Solving for b, $b = \pm \sqrt{(3a + 9)}$
Hence the complete solution is

$Z = ax \pm \sqrt{(3a+9)} \,.\, y + c$

or $z^3 = ax \pm \sqrt{(3a+9)}\, y + c$