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DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT : STATISTICS & NUMERICAL METHODS

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UNIT – V : NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

UNIT V - NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS.

01. **Obtain y by Taylor series method, given that y ' =xy + 1 , y (0) = 1 , for x = 0.1 and 0.2 correct to 4 decimal places.**

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + ...$$

When $n=0$, $y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + ...$

When $n=1$, $y_1 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + ...$

$y'' = xy' + y$

$y''' = xy'' + 2y'$

$y^{iv} = xy''' + 3y''$

$y_1 = 1.1053$ and $y_2 = 1.2228$

02.**Solve** $(1+x)\frac{dy}{dx} = -y^2, \quad y(0)=1$ **by Modified Euler's method by choosing h = 0.1, find y(0.1) and y(0.2).**

Solution: Given $x_0 = 0 \; y_0 = 1, \; h = 0.1$ and $f(x,y) = -\frac{y^2}{1+x}.$

$$y_{n+1} = y_n + h\left[f(x_n + \frac{h}{2}, y_n + \frac{h}{2}[f(x_n, y_n)]\right]$$

Put n = 0 we get $y_1 = y_0 + 0.1\left[f(x_0 + \frac{0.1}{2}, y_0 + \frac{0.1}{2}[f(x_0, y_0)]\right]$

$$y_1 = y(0.1) = 1 + 0.1\left[f(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}[f(0,1)]\right] = 0.91278$$

Put n = 1 we get $y_2 = y(0.2) = y_1 + 0.1\left[f(x_1 + \frac{0.1}{2}, y_1 + \frac{0.1}{2}[f(x_1, y_1)]\right] = 0.84550$

3) If $\frac{dy}{dx} = \log_{10}(x+y), \quad y(0)=2$ by Euler's method by choosing h = 0.2, find y(0.2) and y(0.4).

Given Data is : $x_0 = 0, y_0 = 2, h = 0.2$ and $f(x,y) = \log_{10}(x+y)$

Euler's Formula is $y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0,1,2,3,\cdots\cdots$

Put n = 0 we get, $y_1 = 2.0 + 0.2\log_{10}(0+2) = 2.0602$

Put n = 1 we get, $y_2 = 2.0656 + 0.2\log 10(0.2 + 2.0656) = 2.1366$

4)If $\frac{dy}{dx} = -\frac{y^2}{1+x}, \quad y(0)=1$ by Euler's method by choosing h = 0.1, find y(0.1) and y(0.2).

Given $x_0 = 0,\ y_0 = 1,\ h = 0.1$ and $f(x,y) = -\frac{y^2}{1+x}$.

Euler's Formula is $y_{n+1} = y_n + hf(x_n, y_n), \qquad n = 0,1,2,3,\cdots\cdots$

Put n = 0 we get, $y_1 = y(0.1) = 1 - 0.1\left(\frac{(1)^2}{1+0}\right) = 0.9$

Put n = 1 we get, $y_2 = y(0.2) = 0.9 - 0.1\left(\frac{(0.9)^2}{1+0.1}\right) = 0.82636$

5) Obtain the approximate value of y at x = 0.1 & 0.2 for the differential equation $\frac{dy}{dx} = 2y + 3e^x,\ y(0) = 0$, by Taylor's Series method. Compare the numerical solution obtained with the exact solution.

Given $x_0 = 0,\ y_0 = 0, h = 0.1$ and $y' = 2y + 3e^x$

Taylor's series expansion is $y_{n+1} = y_n + \frac{h}{1!}y_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \ldots\ldots.$

For $n = 0$, $\qquad y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots\ldots. \qquad (1)$

$$\frac{dy}{dx} = y'(x) = 2y + 3e^x \qquad y_0' = 3$$

Differentiating $y'(x) = 2y + 3e^x$ successively three times and putting x = y = 0, we get

$$y''(x) = 2y' + 3e^x \qquad y_0'' = 9$$

$$y'''(x) = 2y'' + 3e^x \qquad y_0''' = 21$$

$$y^{iv}(x) = 2y''' + 3e^x \qquad y_0^{iv} = 45$$

Putting the values in (1)

$$y(x) = 0 + 3h + \frac{9}{2}h^2 + \frac{21}{6}h^3 + \frac{45}{24}h^4 = 3(0.1)+4.5(0.1)^2+3.5(0.1)^3+1.875(0.1)^4 = 0.3486875$$

<u>Exact Solution:</u>

The given differential equation can be written as $\frac{dy}{dx} - 2y = 3e^x$ which is Leibnitz's linear differential equation.

Its I.F. is I.F = $e^{-\int 2dx} = e^{-2x}$

Therefore the general solution is, $ye^{-2x} = \int 3e^x(e^{-2x})dx + c = -3e^{-x} + c \Rightarrow y = -3e^x + ce^{2x}$ -- (2)

Using the given initial condition y = 0 when x = 0 in (3) we get c = 3.

Thus the exact solution is $y = 3\left(e^{2x} - e^x\right)$

When x = 0.1, the exact solution is y(0.1) = 0.348695

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6)If $\frac{dy}{dx} = \sin x + \cos y, \quad y(2.5) = 0$ **by Modified Euler's method by choosing h = 0.5, find y(3.5)**

Given Data is : $x_0 = 2.5, y_0 = 0, h = 0.5$ and $f(x, y) = \sin x + \cos y$

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

Put n = 0 we get $y_1 = y_0 + 0.5\left[f(x_0 + \frac{0.5}{2}, y_0 + \frac{0.5}{2}[f(x_0, y_0)]\right]$

$y_1 = 0 + 0.5[f(2.5 + 0.25, 0 + 0.25[f(2.5, 0)]] = 0.6354$

Put n = 1 we get $y_2 = y_1 + h\left[f(x_1 + \frac{0.5}{2}, y_1 + \frac{0.5}{2}[f(x_1, y_1)]\right]$

$y1 = 0.6354 + 0.5[f(3 + 0.25, 0.6354 + 0.25[f(3, 0.6354)]] = 0.93155$

7)Apply Runge – Kutta method, to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x + y, \quad y(0) = 1$

Given: $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = x + y$

Finding $y_1 = y(0.2)$:

R – K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ -------- (1)

$k_1 = hf(x_0, y_0) = 0.2 \times [0 + 1] = 0.2$; $k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.2}{2}\right)\right] = 0.2400$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.24}{2}\right)\right] = 0.2440$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times [(0 + 0.2) + (1 + 0.2440)] = 0.2888$

Using the values of k_1, k_2, k_3 and k_4 in (1), we get

$y_1 = y(0.2) = 1 + \frac{1}{6}(0.2 + (0.24 + 0.244)^2 + 0.2888) = 1.2468$

Hence the required approximate value of y is 1.2468.

8). Apply Runge – Kutta method, to find an approximate value of y when x = 1.2 in steps of 0.1 given that y ' = x^2 + y^2 and y (1) = 1.5

$k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

$$y_1 = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

9) Solve $\frac{dy}{dx} = y - x^2 + 1,\ y(0) = 0.5$ **by Modified Euler's method by choosing h = 0.1, find y(0.1) and y(0.2).**

$$y_{n+1} = y_n + h\,f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h\,f(x_n, y_n)\right)$$

$x_0 = 0$, $y_0 = 0.5$, $f(x , y) = y - x^2 + 1$

$y_1 = 0.655$ and $y_2 = 0.826475$

10)Using Runge – Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2},\quad y(0) = 1$ **at x = 0.2 & 0.4.**

Given: $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

Finding $y_1 = y(0.2)$

R – K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$ -------- (2)

$k_1 = hf(x_0, y_0) = 0.2 \times f(0,1) = 0.2$; $k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times f(0.1, 1.1) = 0.19672$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times f(0.1, 1.0936) = 0.1967;$

$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.2 \times f(0.2, 1.1967) = 0.1891$

Using the values of k_1, k_2, k_3 and k_4 in (2), we get

$y_1 = y(0.2) = 1 + \frac{1}{6}\left(0.2 + 2(0.19672) + 2(0.1967) + 0.1891\right)$ = 1+0.19599 = 1.19599

Hence the required approximate value of y is 1.19599.

Finding $y_2 = y(0.4)$:

We have $x_1 = 0.1, y_1 = 1.19599$ and $h = 0.2$

R – K method (for n = 1) is: $y_2 = y(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ -------- (3)

$k_1 = hf(x_1, y_1) = 0.2 \times f(0.2, 1.19599) = 0.1891$; $k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2 \times f(0.3, 1.2906)$ = 0.1795

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2 \times f(0.3, 1.2858)$= 0.1793

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 \times f(0.4, 1.3753)$ = 0.1688

Using the values of k_1, k_2, k_3 and k_4 in (3), we get

$y_2 = y(0.4) = 1.19599 + \frac{1}{6}\left(0.1891 + 2(0.1795) + 2(0.1793) + 0.1688\right)$= 1.19599 + 0.1792 = 1.37519

Hence the required approximate value of y is 1.37519

11) Use Milne's method to find y (0.) , given y' = $\frac{1}{x+y}$, y (0) = 2 , y (0.2) = 2.0933 , y (0.4) = 2.1755 , y (0.6) = 2.2493

$x_0 = 0$, $y_0 = 2$, $x_1 = 0.2$, $y_1 = 2.0933$, $x_2 = 0.4$, $y_2 = 2.1755$, $x_3 = 0.6$, $y_3 = 2.2493$

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$y_{4,p}$ = 2.3162

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

$y_{4,c}$ = 2.3163

12)Given $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, y(0.2) = 0.02, y(0.4) = 0.0795 and y(0.6) = 0.1762. Compute y(0.8) using Milne's Method.

Solution:

$\frac{dy}{dx} = x - y^2 = f(x, y)$,first find y(0.8) and then y(1).

<u>Finding y(0.8)</u>

Given:

$x_0 = 0, y_0 = 0$ and $f_0 = f(x_0, y_0) = 0$; $x_1 = 0.2, y_1 = 0.02$ and $f_1 = f(x_1, y_1) = 0.1996$

$x_2 = 0.4, y_2 = 0.0795$ and $f_2 = f(x_2, y_2) = 0.3937$

$x_3 = 0.6, y_3 = 0.1762$ and $f_3 = f(x_3, y_3) = 0.56895$

To Find : $y_4 = y(x_4) = y(0.8)$

Predictor Method

$y_4^{(P)} = y(0.8) = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3] = 0 + \frac{4(0.2)}{3}[(2\times 0.1996) - 0.3937_2 + 2\times 0.56895]$ **= 0.30491**

Now we compute $f_4 = f(0.8, 0.30491) = 0.7070$

Corrector Method

$y_4^{(C)} = y(0.8) = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4] = 0.0795 + \frac{0.2}{3}[0.3937 + 4\times 0.56895 + 0.7070]$ **= 0.3046**

13) Given $y'' + xy'$ +y = 0,y(0)=1,$y'(0)$= 0,find the value of y(0.1)by Runge-Kutta method of fourth order.

Solution:

Given $y'' + xy'$ +y = 0, y(0)=1,$y'(0)$= 0, h=0.1, y_0 = 1,x_0 = 0 , y_1 = y(0.1) ,put y' = z. The equation becomes ,y'' = z' = -xz - y, let y' = f(x,y,z),

z' = g(x,y,z),given y_0 = 1,z_0 = y'_0= 0.By the formula

$$k_1 = hf(x_{0,} y_0, z_0)$$

$$l_1 = hg(x_{0,} y_0, z_0)$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$$

$$l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$$

$$l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$$

$$k_4 = hf(x_{0,} + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = hg(x_{0,} + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

$$\Delta z = \frac{1}{6}(l_1 + 2(l_2 + l_3) + l_4)$$

k_1 = h f(0,1,0) = 0,l_1=g f(0,1,0) = -0.1,k_2 = -0.005, l_2 = -0.09975, k_3 = -0.00499,

l_3= -0.0995, k_4 = -0.00995,l_4 = -0.0985 ,∴ y_1 = y(0.1) = 0.995.