5.2 LINEAR SYSTEMS WITH RANDOM INPUTS

When the input to a continuous time linear system $y(t) = f[\,X(t)]$ is a random process {X(t)} , then the output will also a random process {Y(t)}.

(i.e)., Y(t) = f[X(t)]

For a linear time invariant system, we can express Y(t) as

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du.$$

Thus the system f accepts the random processes {x(t)} as input, yields the output random process {Y(t)}.

AUTO CORRELATION AND CROSS CORRELATION FUNCTIONS OF INPUT AND OUTPUT

1. **If the input of a time invariant stable linear system is a WSS process, then prove that the output is also a WSS process**.

Sol:

Given input [X(t)] is WSS

∴ E[X(t)] is a constant = μ_X and

$R_{XX}(t_1, t_2)$ = function of τ

Let [Y(t)] be the output . To prove [Y(t)] to be a WSS process, we have to prove

1) E[Y(t)] = constant

2) $R_{YY}(t_1, t_2)$ = function of τ

Since the system is stable , $\int_{-\infty}^{\infty} h(u) < \infty$

WKT., Y(t) = $\int_{-\infty}^{\infty} h(u)X(t-u)du$

1) E[Y(t)] = $E[\int_{-\infty}^{\infty} h(u)X(t-u)du]$ = $\int_{-\infty}^{\infty} h(u)E[X(t-u)]du$

$= \int_{-\infty}^{\infty} h(u)\mu_X du = \mu_X \int_{-\infty}^{\infty} h(u) < \infty$ ∵ the system is stable

∴ E[Y(t)] = constant

2) $R_{YY}(t_1, t_2)$ = E[Y(t_1)Y(t_2)]

$= E[\int_{-\infty}^{\infty} h(u)X(t_1-u)du \int_{-\infty}^{\infty} h(u)X(t_2-u)du]$

$= E[\int_{-\infty}^{\infty} h(u_1)X(t_1-u_1)du_1 \int_{-\infty}^{\infty} h(u_2)X(t_2-u_2)du_2]$

$= E[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)[X(t_1-u_1)\, X(t_2-u_2)]du_1du_2]$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)E[X(t_1-u_1)\, X(t_2-u_2)]du_1du_2$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(t_1-u_1, t_2-u_2)]du_1du_2$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(t_1-u_1-t_2+u_2)]du_1du_2$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(t_1-t_2-u_1+u_2)]du_1du_2$

$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(\tau-u_1+u_2)]du_1du_2$

Which is a function of τ.

Since the conditions (1) and (2) of WSS are satisfied, [Y(t)] is a WSS process.

Note:

The convolution of two functions f(t) and g(t) is defined as

$$f(t)*g(t) = \int_{-\infty}^{\infty} f(u)g(t-u)du = \int_{-\infty}^{\infty} f(t-u)g(u)du$$

$$f(t)*g(t) = \int_{-\infty}^{\infty} f(t+u)g(u)du = \int_{-\infty}^{\infty} f(u)g(t+u)du$$

2. State and prove the fundamental theorem of power spectrum.

(or)

State and prove the relation between PSD of input and output

(or)

Prove that $S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$

(or)

If [X(t)] is WSS process and if Y(t) = $\int_{-\infty}^{\infty} h(u)X(t-u)du$ then prove that (i) $R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)$

(ii) $R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$

(iii) $S_{XY}(\omega) = S_{XX}(\omega)H^*(\omega)$

(iv) $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

Sol:

We know that

Y(t) $=\int_{-\infty}^{\infty} h(u)X(t-u)du$ … … … (1)

(i) Multiplying X(t+τ) on both sides in (1) , we get

X(t+τ)Y(t) $=\int_{-\infty}^{\infty} h(u)X(t+\tau)X(t-u)du$

E[X(t+τ)Y(t)] $=\int_{-\infty}^{\infty} h(u)E[X(t+\tau)X(t-u)]du$

$= \int_{-\infty}^{\infty} h(u)R_{XX}(t+\tau, t-u\,)]du$

$= \int_{-\infty}^{\infty} h(u)R_{XX}(t+\tau-t+u\,)]du$

$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)R_{XX}(\tau+u\,)]du$

$R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)$ … … … (2)

ii) Multiplying Y(t$-\tau$) on both sides in (1), we get

Y(t)Y(t$-\tau$) $=\int_{-\infty}^{\infty} h(u)X(t-u)Y(t-\tau)du$

E[Y(t)Y(t$-\tau$)] $=\int_{-\infty}^{\infty} h(u)E[X(t-u)Y(t-\tau)]du$

$= \int_{-\infty}^{\infty} h(u)R_{XY}(t-u, t-\tau)]du$

$= \int_{-\infty}^{\infty} h(u)R_{XY}(t-u-t+\tau)]du$

$R_{YY}(\tau) = \int_{-\infty}^{\infty} h(u)R_{XY}(\tau-u\,)]du$

$R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$ … … … (3)

iii) Take Fourier transform on (2) , we get

$\therefore$ F[$R_{XY}(\tau)$] = F[$R_{XX}(\tau)$* h(-τ)]

$S_{XY}(\omega)$ = F[$R_{XX}(\tau)$]F[h(-τ)]

$S_{XY}(\omega)$ = $S_{XX}(\omega)$H*(ω) … … … (4)

iv) Take Fourier transform on (3) , we get

F[$R_{YY}(\tau)$]= F[$R_{XY}(\tau)$* h(τ)]

$S_{YY}(\omega)$ = F[$R_{XY}(\tau)$]F[h(τ)]

= $S_{XY}(\omega)$H(ω)

= $S_{XX}(\omega)$H*(ω)H(ω) From (4)

= $S_{XX}(\omega)|H(\omega)|^2$

$\therefore S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

3. If X(t) is a WSS process and if Y(t) = $\int_{-\infty}^{\infty} h(u)X(t-u)du$. then P.T $R_{YY}(\tau)$ = $R_{XX}(\tau)$* k(τ), where k(t) = h(t)*h(-t) = $\int_{-\infty}^{\infty} h(u)X(t+u)du$

Sol:

Given Y(t) = $\int_{-\infty}^{\infty} h(u)X(t-u)du$ … … … (1)

$R_{YY}(\tau)$ = E[Y(t+τ)Y(t)]

$$= E[\int_{-\infty}^{\infty} h(u_1)X(t + \tau - u_1)du_1 \int_{-\infty}^{\infty} h(u_2)X(t - u_2)du_2]$$

$$= E[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)X(t + \tau - u_1)X(t - u_2)du_1du_2]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)E[X(t + \tau - u_1)X(t - u_2)]du_1du_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}[(t + \tau - u_1),(t - u_2)]du_1du_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}[(t + \tau - u_1) - t + u_2)]du_1du_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}[(\tau - (u_1 - u_2))]du_1du_2$$

$\because$put v = $u_1 - u_2$;u_1=$u_2 + v$;du_1=dv

$$= \int_{-\infty}^{\infty} h(u_2) \int_{-\infty}^{\infty} h(u_1)R_{XX}[(\tau - (u_1 - u_2))]du_1du_2$$

$$= \int_{-\infty}^{\infty} h(u_2) \int_{-\infty}^{\infty} h(u_2 + v)R_{XX}[(\tau - (v))]dvdu_2$$

$$= [\int_{-\infty}^{\infty} h(u_2)\, h(u_2 + v)du_2]R_{XX}(\tau - v)dv$$

$$= \int_{-\infty}^{\infty} k(v)R_{XX}(\tau - v)dv$$

$\because$ k(t) = h(t)*h(-t) = $\int_{-\infty}^{\infty} h(u)X(t + u)du$

$R_{YY}(\tau)$ = $R_{XX}(\tau)$* k(τ)

Note

The following example is another method of proving fundamental theorem on the power spectrum.

4. State and prove Fundamental theorem on the power spectrum of the input of a linear system.

Statement:

The relation between the PSDs for the input and output process is

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

Proof:

Let X(t) be the input and Y(t) be the output.

Y(t) $=\int_{-\infty}^{\infty} h(u_1)X(t - u_1)du_1$

Y(t+τ) $=\int_{-\infty}^{\infty} h(u_2)X(t + \tau - u_2)du_2$

Y(t+τ)Y(t) $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)X(t + \tau - u_2)X(t - u_1)du_1du_2$

$R_{YY}(\tau)$ = E[Y(t+τ)Y(t)]

$= E[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)X(t + \tau - u_2)X(t - u_1)du_1du_2]$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)E[X(t + \tau - u_2)X(t - u_1)]du_1du_2$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}[(t + \tau - u_2 - t + u_1)]du_1du_2$

$R_{YY}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(\tau + u_1 - u_2)du_1du_2$ .... … … (1)

The PSD of output is given by

$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-i\omega\tau}d\tau$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(\tau + u_1 - u_2)e^{-i\omega\tau}du_1du_2d\tau$

From (1)

$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(v)e^{-i\omega(v-u_1+u_2)}du_1du_2dv$

$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)h(u_2)R_{XX}(v)e^{-i\omega v}e^{i\omega u_1}e^{-i\omega u_2}du_1du_2dv$

$=\int_{-\infty}^{\infty} h(u_1)e^{i\omega u_1}du_1\int_{-\infty}^{\infty} h(u_2)e^{-i\omega u_2}du_2\int_{-\infty}^{\infty} R_{XX}(v)e^{-i\omega v}dv$

$=\int_{-\infty}^{\infty} h(t)e^{i\omega t_1}dt\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$

= F[h(t)]*F[h(t)]$S_{XX}(\omega)$

= H*(ω)H(ω)$S_{XX}(\omega)$

= H(ω)H*(ω)$S_{XX}(\omega)$

$S_{YY}(\omega) = |H(\omega)|^2S_{XX}(\omega)$

5. **For a linear time- invariant system with a WSS process [X(t)] is the input ,show that the mean value of the output is given by $\mu_Y = \mu_X H(0)$**

Sol:

Given input [X(t)] is a WSS process

∴ E[X(t)] = constant = μ_X

By the definition of Y(t),

Y(t) $=\int_{-\infty}^{\infty} h(u)X(t-u)du$

μ_Y= E[Y(t)]

$=E[\int_{-\infty}^{\infty} h(u)X(t-u)du]$

$= \int_{-\infty}^{\infty} h(u)E[X(t-u)]du$

$= \int_{-\infty}^{\infty} h(u)\mu_X du$

$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(u)du$ … … … (1)

We know that

$H(\omega) = F[\, h(t)]$

$= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$

Taking $\omega = 0$, we get

$H(0) = \int_{-\infty}^{\infty} h(t)dt$ ∵Replace t by u

$= \int_{-\infty}^{\infty} h(u)du$ … …. … (2)

Substitute (2) in (1) we get

$\mu_Y = \mu_X H(0)$

6. **Consider a linear time invariant system with impulse response $h(t) = 4e^{-2t}u(t)$ and suppose that a WSS random process [X(t)] with mean $\mu_X = 2$ is used as input of the system. Find the mean value of the output of the system.**

Sol:

Given $\mu_X = 2$, $h(t) = 4e^{-2t}u(t)$

h(t) = { $4e^{-2t}$ $t \geq 0$

0 $t < 0$

H(ω) = F[h(t)] = $\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$

= $\int_{-\infty}^{\infty} 4e^{-2t}u(t)e^{-i\omega t}dt$ = $\int_{-\infty}^{0}$ 0 dt + $\int_{0}^{\infty} 4e^{-2t}e^{-i\omega t}dt$

= 4 $\int_{0}^{\infty} e^{-(2+i\omega)t}dt$ = 4 $[\frac{e^{-(2+i\omega)t}}{-(2+i\omega)}]_{0}^{\infty}$

= 4 $[0 - \frac{1}{-(2+i\omega)}]$

H(ω) = $\frac{4}{2+i\omega}$

It follows that H(0) = $\frac{4}{2+0} = 2$

The mean value of the output system is given by

$\mu_Y = \mu_X H(0)$= $2 * 2$

μ_Y= 4

7. **Consider a linear system with impulse $h(t) = 4e^{-t}cos2tu(t)$. Suppose that a WSS process [X(t)] with $\mu_X = 3$ is used to the input system. Find the mean value of the output system.**

Sol:

Given that $h(t) = 4e^{-t}\, cos2t\, u(t)$

H(ω) = F[h(t)]

$= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$

$= \int_{-\infty}^{\infty} 4e^{-t}\, cos2t\, u(t)e^{-i\omega t}dt$

H(ω) = $\int_{-\infty}^{0}$ 0 dt + $\int_{0}^{\infty} 4e^{-t}\, cos2t\, e^{-i\omega t}dt$

$= 4\int_{0}^{\infty} e^{-(1+i\omega)t}cos2t\, dt$

$= \frac{4(1+i\omega)}{(1+i\omega)^2+4}$

H(ω) = $\frac{4}{5}$

The mean value of the output is $\mu_Y = \mu_X H(0)$

$= 3 * \frac{4}{5} = \frac{12}{5}$

∴ The mean value of the output is $\mu_Y = \frac{12}{5}$

PROBLEMS UNDER RELATIONSHIP BETWEEN PSD'S OF INPUT AND OUTPUT

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

The following are some important formulae which are useful in this section

$$\int_0^\infty e^{-at}\cos bt\,dt = \frac{a}{a^2+b^2} \quad ; \quad \int_0^\infty e^{-at}\sin bt\,dt = \frac{a}{a^2+b^2}$$

$$\int_{-\infty}^\infty \frac{e^{i\omega\tau}}{\omega^2+a^2}d\omega = \frac{\pi}{a}e^{-a|\tau|} \quad ; \int_{-\infty}^\infty \frac{e^{i\omega\tau}}{(\omega^2+a^2)^2}d\omega = \frac{\pi}{2a^3}(1+a|\tau|)e^{-a|\tau|}$$

1. Consider a while Gaussian noise of zero mean with PSD = $\frac{N_0}{2}$ is applied to a low pass RC filter whose transfer function is H(f) = $\frac{1}{1+i2\pi fRC}$.Find the auto correlation function of output random process.

Sol.

Given: $S_{XX}(\omega)$= $\frac{N_0}{2}$; H(f) =$\frac{1}{1+i2\pi fRC}$.

∴H(ω) = $\frac{1}{1+i\omega RC}$ ∵ω= $2\pi f$

$|H(\omega)|$ = $\frac{1}{|1+i\omega RC|}$ = $\frac{1}{\sqrt{1+\omega^2R^2C^2}}$; $|H(\omega)|^2$= $\frac{1}{1+\omega^2R^2C^2}$

The relationship between the PSD's of the input and output is given by

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \frac{N_0}{2}\frac{1}{1+\omega^2R^2C^2}$

The ACF of the output is given by

$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2}\frac{1}{1+\omega^2R^2C^2}e^{i\omega\tau}d\omega$

$= \frac{N_0}{4\pi}\int_{-\infty}^{\infty} \frac{N_0}{2}\frac{e^{i\omega\tau}}{R^2C^2(\omega^2+\frac{1}{R^2C^2})}d\omega$

$= \frac{N_0}{4\pi R^2C^2}\int_{-\infty}^{\infty} \frac{N_0}{2}\frac{e^{i\omega\tau}}{(\omega^2+\frac{1}{R^2C^2})}d\omega$

$= \frac{N_0}{4\pi R^2C^2}\frac{\pi}{\frac{1}{RC}}\; e^{\frac{-1}{RC}|\tau|}$

$R_{YY}(\tau)= \frac{N_0}{4RC}\, e^{\frac{-1}{RC}|\tau|}$

2. **The input to the RC filter is a white noise process with autocorrelation function $R_{XX}(\tau)= \frac{N_0}{2}\delta(\tau)$, frequency response is H(ω) = $\frac{1}{1+i\omega RC}$. Find the auto correlation and mean square value of the output process.**

Sol.

Given $R_{XX}(\tau)= \frac{N_0}{2}\delta(\tau)$

The PSD of the input is given by

$$S_{XX}(\omega)=\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\tau)e^{-i\omega\tau}d\tau$$

$$=\frac{N_0}{2}\int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}d\tau$$

$$=\frac{N_0}{2}(1)$$

$$S_{XX}(\omega)=\frac{N_0}{2}$$

Also given that H(ω) = $\frac{1}{1+i\omega RC}$

$$|H(\omega)| =\frac{1}{|1+i\omega RC|} =\frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

$$|H(\omega)|^2= \frac{1}{1+\omega^2R^2C^2}$$

The relationship between the PSD's of the input and output is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \frac{1}{1+\omega^2R^2C^2}\frac{N_0}{2}$$

The ACF output is given by

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2R^2C^2}\frac{N_0}{2}e^{i\omega\tau}d\omega$$

$$=\frac{N_0}{4\pi}\int_{-\infty}^{\infty} \frac{N_0}{2}\frac{e^{i\omega\tau}}{R^2C^2(\omega^2+\frac{1}{R^2C^2})}d\omega$$

$$=\frac{N_0}{4\pi R^2C^2}\int_{-\infty}^{\infty} \frac{N_0}{2}\frac{e^{i\omega\tau}}{(\omega^2+\frac{1}{R^2C^2})}d\omega$$

$= \frac{N_0}{4\pi R^2 C^2} \frac{\pi}{\frac{1}{RC}} \; e^{\frac{-1}{RC}|\tau|}$

$R_{YY}(\tau) = \frac{N_0}{4RC} e^{\frac{-1}{RC}|\tau|}$

Mean square value of output = E[$Y^2(t)$] = $\frac{N_0}{4RC}$

3. Consider a linear system as shown in the figure , where [X(t)] is the output of the system. The ACF of the input is $R_{XX}(\tau)$=2$\delta(\tau)$ and H(ω) = $\frac{1}{3+i\omega}$. Find the following i)The power spectral density of [Y(t)] ii)The auto correlation of [Y(t)] iii) The average power of the output [Y(t)].

Sol:

i) To find $S_{yy}(\omega)$:

Given $R_{XX}(\tau)$=2$\delta(\tau)$

The PSD of the output is given by

$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} 2\delta(\tau) e^{-i\omega\tau} d\tau$

$= 2\int_{-\infty}^{\infty} \delta(\tau) e^{-i\omega\tau} d\tau$

= 2×1

$S_{XX}(\omega)= 2$

We have $H(\omega) = \frac{1}{3+i\omega}$

$|H(\omega)|= \frac{1}{|3+i\omega|}$

$= \frac{1}{\sqrt{3^2+\omega^2}}$

$|H(\omega)|^2= \frac{1}{9+\omega^2}$

The relation between the PSD's of input and output is given by

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) = \frac{1}{9+\omega^2} \times 2$

$S_{YY}(\omega) = \frac{2}{9+\omega^2}$

ii) To find $R_{YY}(\tau)$

The ACF of the output is given by

$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{9+\omega^2}e^{i\omega\tau}d\omega$

$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{3^2+\omega^2}d\omega$

$= \frac{1}{\pi} \times \frac{\pi}{3} e^{-3|\tau|}$

$R_{YY}(\tau)=\frac{1}{3}e^{-3|\tau|}$

iii) Average power of the output = $R_{YY}(0)=\frac{1}{3}$

∴ Average power of the output = $\frac{1}{3}$

4. **Consider a circuit with input voltage {X(t)} and output voltage Y(t). If {X(t)} is a stationary process with mean zero and auto correlation function $R_{XX}(\tau)=3e^{-2|\tau|}$ and if the system transfer function is H(ω) = $\frac{1}{1+i\omega}$; find the following . (a) The mean of Y(t). (b) The input power spectral density function $S_{YY}(\omega)$ (c) The output power spectral density function $S_{YY}(\omega)$ (d) The ACF of the output.**

Sol:

a)To find μ_Y:

Given μ_X= 0.

The mean of {Y(t)} is given by

$\mu_Y = \mu_X H(0) = 0$

$\therefore \mu_Y = 0$

b)To find $S_{XX}(\omega)$:

Given that $R_{XX}(\tau)$=$3e^{-2|\tau|}$

The PSD of the input is given by

$S_{XX}(\omega)$= $\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$

=$\int_{-\infty}^{\infty} 3e^{-2|\tau|}e^{-i\omega\tau}d\tau$

= $3\int_{-\infty}^{\infty} e^{-2|\tau|}[cos\ \omega\tau\ - i\ sin\ \omega\tau]d\tau$

= $3[\int_{-\infty}^{\infty} e^{-2|\tau|}cos\ \omega\tau\ d\tau - i\int_{-\infty}^{\infty} e^{-2|\tau|}\ sin\ \omega\tau d\tau]$

=$3[2\int_{-\infty}^{\infty} e^{-2|\tau|}cos\ \omega\tau d\tau\ - i\ 0]$

= $6\int_{-\infty}^{\infty} e^{-2\tau}cos\ \omega\tau\ d\tau$

= $6 \times \frac{12}{4+\omega^2}$

c) To find $S_{YY}(\omega)$:

The relationship between the PSD's of the input and output is given by

$S_{YY}(\omega)$= $|H(\omega)|^2 S_{XX}(\omega)$

H(ω) = $\frac{1}{1+i\omega}$

$|H(\omega)|$= $\frac{1}{\sqrt{1+w^2}}$

$|H(\omega)|^2$= $\frac{1}{1+w^2}$

$\therefore S_{YY}(\omega)$ = $\frac{1}{1+w^2}\frac{12}{4+w^2}$= $\frac{12}{(1+w^2)(4+w^2)}$

d) To find $R_{YY}(\tau)$:

The ACF of the output is given by

$R_{YY}(\tau)= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{12}{(1+w^2)(4+w^2)}e^{i\omega\tau}d\omega$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{12}{4-1}(\frac{1}{1+w^2}-\frac{1}{4+w^2})e^{i\omega\tau}d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4(\frac{1}{1+w^2}-\frac{1}{4+w^2})e^{i\omega\tau}d\omega$

$= \frac{2}{\pi}[\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{1+w^2}d\omega \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{4+w^2}d\omega]$

$= \frac{2}{\pi}[\pi e^{-|\tau|} - \frac{\pi}{2}e^{-2|\tau|}]$

$R_{YY}(\tau)= 2e^{-|\tau|}$- $e^{-2|\tau|}$

5. **If {X(t)} is the input voltage to a circuit and {Y(t)} is the output voltage, {X(t)} is a stationary random process with μ_X= 0 and $R_{XX}(\tau)= e^{-\alpha|\tau|}$. Find μ_Y, $S_{YY}(\omega)$, $R_{YY}(\tau)$,if the power transfer function is H(ω) = $\frac{R}{R+iL\omega}$**

Sol.

i) To find μ_X:

Given μ_X= 0

The mean of {Y(t)} is given by

$\mu_Y= \mu_X$H(0) = 0 ×H(0)

$\therefore\mu_Y$= 0

ii) Given $R_{XX}(\tau)= e^{-\alpha|\tau|}$

The PSD of the input is given by

$$S_{XX}(\omega)=\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} e^{-\alpha|\tau|}e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} e^{-\alpha|\tau|}(\cos\omega\tau - i\sin\omega\tau)d\tau$$

$$=\int_{-\infty}^{\infty} e^{-\alpha|\tau|}\cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} e^{-\alpha|\tau|}\sin\omega\tau d\tau$$

$$= 2\int_{0}^{\infty} e^{-\alpha\tau}\cos\omega\tau\, d\tau$$

$$S_{XX}(\omega)= \frac{2\alpha}{\alpha^2+\omega^2}$$

iii) To find $S_{YY}(\omega)$:

Given $H(\omega) = \frac{R}{R+iL\omega}$

$$|H(\omega)|= \frac{R}{\sqrt{R^2+L^2\omega^2}}$$

$$|H(\omega)|^2= \frac{R^2}{R^2+L^2\omega^2}$$

The relation between PSD's of input and output is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$S_{YY}(\omega) = \frac{R^2}{R^2+L^2\omega^2} \times \frac{2\alpha}{\alpha^2+\omega^2}$$

iv) To find $R_{YY}(\tau)$

The ACF of the Output is given by

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\alpha R^2}{(\alpha^2+\omega^2)(R^2+L^2\omega^2)}e^{i\omega\tau}d\omega$$

$$=\frac{\alpha R^2}{\pi}\int_{-\infty}^{\infty} \frac{1}{(\alpha^2+\omega^2)L^2(\frac{R^2}{L^2}+\omega^2)}e^{i\omega\tau}d\omega$$

$$=\frac{\alpha R^2}{\pi L^2}\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}d\omega}{(\alpha^2+\omega^2)(\frac{R^2}{L^2}+\omega^2)}$$

$$=\frac{\alpha R^2}{\pi L^2}\int_{-\infty}^{\infty} \frac{1}{(\alpha^2+\omega^2)(\frac{R^2}{L^2}+\omega^2)}e^{i\omega\tau}d\omega$$

$$=\frac{\alpha R^2}{\pi L^2(\frac{R^2}{L^2}-\alpha^2)}\int_{-\infty}^{\infty} \left(\frac{1}{\alpha^2+\omega^2}-\frac{1}{\frac{R^2}{L^2}+\omega^2}\right)e^{i\omega\tau}d\omega$$

$$=\frac{\alpha R^2}{\pi L^2(\frac{R^2}{L^2}-\alpha^2)}\left[\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}d\omega}{\alpha^2+\omega^2}-\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}d\omega}{\frac{R^2}{L^2}+\omega^2}\right]$$

$$=\frac{\alpha R^2}{\pi L^2(\frac{R^2}{L^2}-\alpha^2)}\left(\frac{\pi}{\alpha}e^{-\alpha|\tau|}-\frac{\pi}{\frac{R}{L}}e^{\frac{-R}{L}|\tau|}\right)$$

$$=\frac{\alpha R^2}{\pi L^2(\frac{R^2}{L^2}-\alpha^2)}\left(\frac{\pi}{\alpha}e^{-\alpha|\tau|}-\frac{L\pi}{R}e^{\frac{-R}{L}|\tau|}\right)$$

$$R_{YY}(\tau)=\frac{\alpha R^2}{L^2(\frac{R^2}{L^2}-\alpha^2)}\left(\frac{e^{-\alpha|\tau|}}{\alpha}-\frac{L}{R}e^{\frac{-R}{L}|\tau|}\right)$$

6. **A linear system is described by the impulse response h(t) = $\boldsymbol{\frac{1}{RC} e^{\frac{1}{RC}} u(t)}$. Assume an input process whose ACF is A$\boldsymbol{\delta(\tau)}$. Find the mean and ACF of the output process.**

Sol.

i) To find H(ω)

Given h(t) = $\frac{1}{RC} e^{\frac{-t}{RC}} u(t)$;

H(ω) = F[h(t)]

$$=\int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$=\int_{-\infty}^{\infty} \frac{1}{RC} e^{\frac{-t}{RC}} u(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{0} 0\, dt + \int_{0}^{\infty} \frac{1}{RC} e^{\frac{-t}{RC}} (1) e^{-i\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{\frac{-t}{RC} - i\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-(\frac{1}{RC} + i\omega)t} dt$$

$$= \frac{1}{RC} \left[\frac{e^{-(\frac{1}{RC} + i\omega)t}}{-(\frac{1}{RC} + i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{RC} \frac{1}{(\frac{1}{RC} + i\omega)}$$

$$= \frac{1}{RC} \left(\frac{RC}{1 + i\omega RC}\right)$$

$$H(\omega) = \frac{1}{1 + i\omega RC}$$

$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$

$|H(\omega)|^2 = \frac{1}{1+\omega^2R^2C^2}$

ii) To find $S_{XX}(\omega)$

Given $R(\tau) = A\delta(\tau)$

The PSD of the input is given by

$$\boldsymbol{S_{XX}(\omega)} = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} A\delta(\tau)e^{-i\omega\tau}d\tau$$

$$= A\int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}d\tau$$

$$= A(1) = A$$

iii) To find $R_{YY}(\tau)$

The relation between PSD's of input and output is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{1\times A}{1+\omega^2R^2C^2} \quad = \frac{A}{R^2C^2(\omega^2+\frac{1}{R^2C^2})}$$

The ACF of the output is given by

$$\boldsymbol{R_{YY}(\tau)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A}{R^2C^2(\omega^2+\frac{1}{R^2C^2})}e^{i\omega\tau}d\omega$$

$$= \frac{A}{2\pi R^2C^2}\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\omega^2+\frac{1}{R^2C^2})}d\omega$$

$=\frac{A}{2\pi R^2C^2}\frac{\pi}{\frac{1}{RC}}e^{\frac{-1}{RC}|\tau|}$

$\boldsymbol{R_{YY}(\tau)}=\frac{A}{2RC}e^{\frac{-|\tau|}{RC}}$

iv) To find μ_Y

The mean value of the output is

$$\mu_Y=\sqrt{\lim_{r\to\infty}R_{YY}(\tau)}=\sqrt{\lim_{r\to\infty}\frac{A}{2RC}e^{\frac{-|\tau|}{RC}}} \quad \Rightarrow \quad \mu_Y = 0$$

7. **Consider a system with transfer function $\frac{1}{1+j\omega}$. An input signal with auto correlation function m$\delta(\tau)+m^2$ is fed as input to the system. Find the mean-square value of the input.**

Sol:

Given R(τ) = $m\delta(\tau)+m^2$

A. To find S_{XX} (ω) :

S(ω) = $\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}$ dτ

$=\int_{-\infty}^{\infty}(m\delta(\tau)+m^2)\ e^{-i\omega\tau}\ d\tau$

$=m\int_{-\infty}^{\infty}\delta(\tau)e^{-i\omega\tau}\ d\tau+m^2\int_{-\infty}^{\infty}e^{-i\omega\tau}\ d\tau$

$= \mathrm{m}(1)+ m^2 2\pi\delta(\omega)$

$S(\omega)= \mathrm{m} + 2\pi m^2\delta(\omega)$

Given $H(\omega) = \frac{1}{1+j\omega} \Rightarrow \; ||H(\omega)| = \frac{1}{\sqrt{1^2+\omega^2}}$

ii) To find $S_{XX}(\omega)$:

The relationship between the PSD's of the input and output is given by

$S_{YY}(\omega) = |H(\omega)|^2 S_{YY}(\omega)$

$= \frac{1}{1+\omega^2}[\mathrm{m} + 2\pi m^2\delta(\omega)]$

$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY(\omega)}\, e^{i\omega\tau} d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\mathrm{m} + 2\pi m^2\delta(\omega))}{1+\omega^2}\, e^{i\omega\tau} d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\frac{m}{1+\omega^2} + \frac{2\pi m^2 \delta(\omega)}{1+\omega^2}] \, e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi} m \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{1+\omega^2} \, d\omega + \frac{1}{2\pi} 2\pi m^2 \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}\delta(\omega)}{1+\omega^2} d\omega$$

$$= \frac{m}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{1+\omega^2} \, d\omega + m^2 \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}\delta(\omega)}{1+\omega^2} \, d\omega$$

$$= \frac{m}{2\pi} \pi \, e^{-|\tau|} + m^2(1) \qquad \therefore \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}\delta(\omega)}{1+\omega^2} \, d\omega = 1$$

$$R_{YY}(\tau) = \frac{m}{2} \; e^{-|\tau|} + m^2$$

The mean value of the output { Y(t)} is given by

$$R_{YY}(0) = \frac{m}{2} \; e^{0} + m^2$$

$$= \frac{m}{2} + m^2$$

8. **A random process {X(t)} having the autocorrelation function $R_{XX}(\tau) = Pe^{-\alpha|\tau|}$ Where P and α are constants is applied to the input of the system with impulse response h(t) = $e^{-bt}u(t)$ where 'b' is a constant, Find the autocorrelation of the output {Y(t)}**

Sol.

i)To find $S_{XX}(\omega)$

Given $R_{XX}(\tau) = Pe^{-\alpha|\tau|}$

The PSD of the input is given by

$$S_{XX}(\omega)= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} Pe^{-\alpha|\tau|}e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} Pe^{-\alpha|\tau|}[\cos\omega\tau - i\sin\omega\tau]d\tau$$

$$= \int_{-\infty}^{\infty} Pe^{-\alpha|\tau|}\cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} Pe^{-\alpha|\tau|}\sin\omega\tau\, d\tau$$

$$= P[2\int_{0}^{\infty} e^{-\alpha\tau}\cos\omega\tau\, d\tau - i(0)] \qquad = 2P\int_{0}^{\infty} e^{-\alpha\tau}\cos\omega\tau\, d\tau$$

$$S_{XX}(\omega)= \frac{2P\alpha}{\alpha^2+\omega^2}$$

ii) To find H(ω)

Given h(t) = $e^{-bt}u(t)$

$H(\omega) = F[h(t)]$

$$= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}d\tau$$

$=\int_{-\infty}^{\infty} e^{-bt}u(t)e^{-i\omega t}d\tau$

$= \int_{-\infty}^{0} 0 + \int_{0}^{\infty} e^{-bt}e^{-i\omega t}(1)d\tau$

$=\int_{0}^{\infty} e^{-bt}e^{-i\omega t}d\tau$

$=\int_{0}^{\infty} e^{-(b+i\omega)t}d\tau$

$= [\frac{e^{-(b+i\omega)t}}{-(b+i\omega)}]_0^{\infty}$

$= \frac{-1}{b+i\omega}[e^{-\infty} - e^{o}]$

$H(\omega) = \frac{1}{b+i\omega}$

$|H(\omega)| = \frac{1}{\sqrt{b^2+\omega^2}}$

$|H(\omega)|^2 = \frac{1}{b^2+\omega^2}$

iii) To find $S_{YY}(\omega)$

The relation between the PSD's of input and output is given by

$S_{YY}(\omega)= |H(\omega)|^2 S_{XX}(\omega)= \frac{2P\alpha}{(\omega^2+\alpha^2)(b^2+\omega^2)}$

$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2P\alpha}{(\omega^2+\alpha^2)(b^2+\omega^2)}e^{i\omega\tau}d\omega$

$=\frac{P\alpha}{\pi}\int_{-\infty}^{\infty} \frac{1}{(\omega^2+\alpha^2)(b^2+\omega^2)}e^{i\omega\tau}d\omega$

$$= \frac{P\alpha}{\pi(b^2-\alpha^2)} \int_{-\infty}^{\infty} \left[\frac{1}{\omega^2+\alpha^2} - \frac{1}{\omega^2+b^2}\right] e^{i\omega\tau} d\omega$$

$$= \frac{P\alpha}{\pi(b^2-\alpha^2)} \left[\int_{-\infty}^{\infty} \frac{e^{i\omega\tau} d\omega}{\omega^2+\alpha^2} - \int_{-\infty}^{\infty} \frac{e^{i\omega\tau} d\omega}{\omega^2+b^2}\right]$$

$$= \frac{P\alpha}{\pi(b^2-\alpha^2)} \left[\frac{\pi}{\alpha} e^{-\alpha|\tau|} - \frac{\pi}{b} e^{-b|\tau|}\right]$$

$$R_{YY}(\tau) = \frac{P\alpha}{b^2-\alpha^2} \left[\frac{e^{-\alpha|\tau|}}{\alpha} - \frac{e^{-b|\tau|}}{b}\right]$$

9 . **A wide sense stationary random process $\{X(t)\}$ with auto correlation function $R_{XX}(\tau) = e^{-|\tau|}$, where A and α are real positive constants, is applied to the input system with impulse response $h(t) = e^{-bt}u(t)$ where b is a real positive constant. Find the autocorrelation of the output $\{Y(t)\}$ of the** system.

i) To find $S_{XX}(\omega)$:

Sol: Given: $R_{XX}(\tau) = e^{-|\tau|}$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-|\tau|} [\cos \omega\tau - i\sin \omega\tau] d\tau$$

$$= \int_{-\infty}^{\infty} e^{-|\tau|} \cos \omega\tau d\tau - i\int_{-\infty}^{\infty} e^{-|\tau|} \sin \omega\tau d\tau$$

$= \left[2\int_0^\infty e^{-\tau}\cos\omega\tau d\tau - i(0)\right] = 2\int_0^\infty e^{-\tau}\cos\omega\tau d\tau$

$$S_{XX}(\omega) = \frac{2}{1+\omega^2}\Bigg] \because \int_0^\infty e^{-at}\cos bt\, dt = \frac{a}{a^2+b^2}$$

ii) To find (ω) :

Given: $h(t) = e^{-bt}u(t)$

$$H(\omega) = F[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-bt}u(t)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{\infty} e^{-bt}e^{-i\omega t}(1)dt = \int_{0}^{\infty} e^{-bt}e^{-i\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-(b+i\omega)t}dt \quad u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$= \left[\frac{e^{-(b+i\omega)t}}{-(b+i\omega)}\right]_0^\infty$$

$$= \frac{-1}{b+i\omega}\left[e^{-\infty} - e^{0}\right]$$

$$H(\omega) = \frac{1}{b+i\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{b^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{b^2+\omega^2}$$

iii) To find $S_{YY}(\omega)$:

The relation between the PSD's of input and output is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$
$$= \frac{2}{(\omega^2+1)(\omega^2+b^2)}$$
$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{(\omega^2+1)(\omega^2+b^2)}e^{i\omega\tau}d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{(\omega^2+1)(\omega^2+b^2)}e^{i\omega\tau}d\omega$$
$$= \frac{1}{\pi(b^2-1)}\int_{-\infty}^{\infty}\left[\frac{1}{\omega^2+1}-\frac{1}{\omega^2+b^2}\right]e^{i\omega\tau}d\omega$$
$$= \frac{1}{\pi(b^2-1)}\left[\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}d\omega}{(\omega^2+1)}-\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}d\omega}{(\omega^2+b^2)}\right]$$
$$= \frac{1}{\pi(b^2-1)}\left[\pi e^{-|\tau|}-\frac{\pi}{b}e^{-b|\tau|}\right]$$
$$R_{YY}(\tau) = \frac{1}{b^2-1}\left[e^{-|\tau|}-\frac{e^{-b|\tau|}}{b}\right]$$

10. **Let $X(t)$ be the input voltage to a circuit, $Y(t)$ be the output voltage and $\{X(t)\}$ be a stationary random process with $\mu_x = 0$ and $R_{XX}(\tau) = e^{-2|\tau|}$. Find the Mean, Auto correlation and power spectral density of the output $Y(t)$ if the system function is given by $H(\omega) = \frac{1}{\omega+2i}$.**

Solution:

Given $\mu_X = 0, R_{XX}(\tau) = e^{-2|\tau|}$

$$H(\omega) = \frac{1}{\omega+2i}$$

(i) To find μ_Y :

Mean of $Y(t)$ is given by $\mu_Y = \mu_X H(0) = 0 \times H(0)$

$$\therefore \mu_Y = 0$$

(ii) To find the power spectral density of $Y(t)$

The relation between PSD of input and output is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \ldots\ldots (1)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} e^{-2|\tau|} [\cos\omega\tau - i\sin\omega\tau] d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} \cos\omega\tau dt - i \int_{-\infty}^{\infty} e^{-2|\tau|} \sin\omega\tau dt = 2\int_0^{\infty} e^{-2|\tau|} \cos\omega d\tau - i(0)$$

$$= 2\int_0^{\infty} e^{-2\tau} \cos\omega\tau d\tau = 2\left[\frac{2}{4+\omega^2}\right] = \frac{4}{4+\omega^2}$$

$$S_{XX}(\omega) = \frac{4}{\omega^2 + 4}$$

Given $H(\omega) = \dfrac{1}{\omega + 2i}$

$$|H(\omega)| = \frac{1}{|\omega + 2i|} = \frac{1}{\sqrt{\omega^2 + 4}}$$

$$|H(\omega)|^2 = \frac{1}{\omega^2 + 4}$$

$$(1) \Rightarrow S_{YY}(\omega) = \frac{1}{\omega^2 + 4} \times \frac{4}{\omega^2 + 4} = \frac{4}{(\omega^2 + 4)^2}$$

(iii) To find $R_{YY}(\tau)$:

The ACF of output is given by $R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega$

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{(\omega^2+4)^2} e^{i\omega\tau}d\omega$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\omega^2+2^2)^2} d\omega$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2\times 2^3}(1+2|\tau|)e^{-2|\tau|}\right] \quad \because \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\omega^2+a^2)^2} d\omega$$

$$= \frac{\pi}{2a^3}(1+a|\tau|)e^{-a|\tau|}$$

$R_{YY}(\tau) = \frac{1}{8}(1+2|\tau|)e^{-2|\tau|}$

11. **A random process $\{X(t)\}$ is applied to a network with response $h(t) = te^{-bt}u(t)$, where $b > 0$ is a constant. The cross function of $X(t)$ with the output $Y(t)$ is known to have the same $i.e$ ACF $\cdot$ $R_{XY}(\tau) = \tau e^{-b\tau}u(\tau)$. Find the ACF of the output $\{Y(t)\}$.**

Sol: i) To find (ω) :

$$h(t) = te^{-bt}u(t)H(\omega) = F[h(t)]$$

$$= \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \ = \int_{-\infty}^{\infty} t e^{-bt} u(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{m} t e^{-bt} (1) e^{-i\omega t} dt = \int_{0}^{\infty} t e^{-(b+\mathrm{i}\omega)\mathrm{t}} dt$$

$$H(\omega) \quad = \left[t \frac{e^{-(b+i\omega)t}}{-(b+i\omega)} - (1) \frac{e^{-(b+\mathrm{i}\omega)t}}{(b+i\omega)^2} \right]_0^{\infty} = \frac{1}{(b+i\omega)^2}$$

$$\text{Also } H^*(\omega) \quad = \frac{1}{(b-i\omega)^2}$$

ii) To find $S_{XY}(\omega)$:

The cross power spectrum of $\{X(t)\}$ and $[Y(t)]$ is given by

$$S_{XY}(\omega) \quad = F[R_{XY}(t)] \ = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \tau e^{-b\tau} u(\tau) e^{-i\omega\tau} d\tau = \left[\int_{-\infty}^{0} 0 + \int_{0}^{\infty} \tau e^{-b\tau} (1) e^{-i\omega\tau} d\tau \right]$$

$$= \int_{0}^{\infty} \tau e^{-(b+i\omega)\tau} d\tau = \left[\tau \frac{e^{-(b+i\omega)t}}{-(b+i\omega)} \right] - \left[\frac{e^{-(b+i\omega)\tau}}{(-(b+i\omega))^2} \right]_0^{\infty}$$

$$S_{XY}(\omega) \quad = \frac{1}{(b+i\omega)^2}$$

iii) To find $R_{YY}(\tau)$:

We know that,

$$S_{YY}(\omega) = H^*(\omega)S_{XY}(\omega)$$
$$= \frac{1}{(b-i\omega)^2}\frac{1}{(b+i\omega)^2} = \frac{1}{((b-i\omega)(b+i\omega))^2}$$
$$= \frac{1}{(b^2+\omega^2)^2}$$

The ACF of the output $Y(t)$ is given by

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{i\omega\tau}d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(b^2+\omega^2)^2}e^{i\omega\tau}d\tau$$

$$= \frac{1}{2\pi}\frac{\pi}{2b^3}[1+b|\tau|]e^{-b|\tau|} \quad |\because \int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{(\omega^2+a^2)^2}d\omega = \frac{\pi}{2a^3}(1+a|\tau|)e^{-a|\tau|}$$

$$R_{YY}(\tau) = \frac{1}{4b^3}[1+b|\tau|]e^{-b|\tau|}$$

12. **Assume a random process $\{X(t)\}$ is given to a system with for system transfer function $H(\omega) = \begin{cases} 1 & \text{for } |\omega| \le \omega_0 \\ 0 & \text{else} \end{cases}$**

If the ACF of input is $\frac{N_0}{2}\delta(\tau)$, find the ACF of output.

Sol: i) To find $S_{XX}(\omega)$:

Given the ACF of the input is

$$R_{XX}(\tau) = \frac{N_0}{2}\delta(\tau)$$

The PSD of the input is given by

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau \\ &= \int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\tau)e^{-i\omega\tau}d\tau = \frac{N_0}{2}\int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}d\tau \\ &= \frac{N_0}{2}(1) \end{aligned}$$

Given $H(\omega) = \begin{cases} 1 & ; |\omega| \le \omega_0 \\ 0 & ;\text{else} \end{cases}$

ii) To find $R_{YY}(\tau)$: The relation between the PSD's of input and output is given by

$$\begin{aligned} S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) \\ &= (1)^2\frac{N_0}{2};\ |\omega| \le \omega_0 \\ \therefore S_{YY}(\omega) &= \frac{N_0}{2}; |\omega| \le \omega_0 \end{aligned}$$

The ACF of the output is given by

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \frac{N_0}{2}e^{i\omega\tau}d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\omega_0}^{\omega_0} e^{i\omega\tau}d\omega \qquad = \frac{N_0}{4\pi}\left[\frac{e^{i\omega r}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{N_0}{4\pi i\tau}[e^{t\omega_0 t} - e^{-\omega\omega_0 \mathrm{r}}] \quad = \frac{N_0}{4\pi i\tau}2i\sin\omega_0\tau$$

$$R_{YY}(\tau) = \frac{N_0}{2\pi\tau}\sin\omega_0\tau$$

13. An LTI system has an impulse response $h(t) = e^{-\beta t}u(t)$. Find the output auto correlation $R_{YY}(\tau)$ corresponding to an input $X(t)$

Sol: i) To find (ω) :

Given $h(t) = e^{-\beta t}u(t)$

$$H(\omega) = F[h(t)]$$

$$= \int_{-\infty}^{\infty} e^{-\beta t}u(t)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{\infty} e^{-\beta t}(1)e^{-iat}dt = \int_{-\infty}^{\infty} e^{-(\beta+i\omega)t}dt$$

$$= \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)}\right]_0^{\infty}$$

$$|H(\omega)| = \frac{1}{\sqrt{\beta^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2+\omega^2}$$

ii) To find $R_{YY}(\tau)$:

The relation between the PSD’s of input and output is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$
$$= \frac{1}{\beta^2+\omega^2} S_{XX}(\omega)$$
$$R_{YY}(\tau) = F^{-1}[S_{YY}(\omega)] \text{ [}\therefore S_{XX}(\omega) \text{ not known]}$$
$$= F^{-1}\left[\frac{1}{\beta^2+\omega^2}\right] \cdot F^{-1}[S_{XX}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\beta^2+\omega^2} e^{-\omega\tau} dt * R_{XX}(\tau)$$
$$= \frac{1}{2\pi}\frac{\pi}{\beta} e^{-\beta|\tau|} * R_{XX}(\tau) = \frac{1}{2\beta} e^{-\beta|\tau|} * R_{XX}(\tau)$$

$R_{YY}(\tau) = \int_{-\infty}^{\infty} \frac{1}{2\beta} e^{-\beta|u|} R_{XX}(\tau - u) du$ (Since $f(t) * g(t) = \int_0^t f(u) g(t - u) du$)

14. **A random process $X(t)$ having ACF $R_{XX}(\tau) = Ce^{-a|\tau|}$, where C and α are real constants, applied to the input of the system with impulse response $h(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t < 0 \end{cases}$ where $\lambda > 0$. Find the ACF of the output response $\{Y(t)\}$ and cross correlation function $R_{XY}(\tau)$.**

Sol: i) To find $H(\omega)$:

Given

$$h(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t < 0 \end{cases}$$
$$H(\omega) = F[h(t)]$$
$$= \int_{-\infty}^{\infty} h(t) e^{-t\omega t} dt = \int_{-\infty}^{0} 0 + \int_{0}^{\infty} \lambda e^{-\lambda t} e^{-i\omega t} dt$$

$$= \lambda \int_0^{\infty} e^{-\lambda t} e^{-i\omega t} dt = \lambda \int_0^{\infty} e^{-(\lambda+i\omega)t} dt$$

$$= \lambda \left[\frac{e^{-(\lambda+i\omega)}}{-(\lambda+i\omega)}\right]_0^{\infty} = \lambda \left[0 - \frac{1}{-(\lambda+i\omega)}\right] = \frac{\lambda}{\lambda+i\omega}$$

ii) To find $S_{XX}(\omega)$:

Given $R_{XX}(\tau) = Ce^{-a|\tau|}$

$$\begin{aligned}
S_{XX}(\omega) &= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \\
&= C \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-i\omega\tau} d\tau = C \int_{-\infty}^{\infty} e^{-a|\tau|} [\cos \omega\tau - i\sin \omega\tau] d\tau \\
&= C \int_{-\infty}^{\infty} e^{-a|\tau|} \cos \omega\tau d\tau - i \int_{-\infty}^{\infty} e^{-a|r|} \sin \omega\tau d\tau \\
&= 2C \int_0^{\infty} e^{-a|r|} \cos \omega d\tau - i(0) = 2C \int_0^{\infty} e^{-ar} \cos \omega\tau d\tau \\
S_{yY}(\omega) &= \frac{2C\alpha}{\alpha^2 + \omega^2}
\end{aligned}$$

iii) To find $R_{YY}(\tau)$:

The relation between the PSD's of input and output is given by

$$\begin{aligned}
S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) = \frac{\lambda^2}{\lambda^2 + \omega^2} \frac{2C\alpha}{\alpha^2 + \omega^2} \\
&= 2C\alpha\lambda^2 \left[\frac{1}{(\lambda^2 + \omega^2)(\alpha^2 + \omega^2)}\right] = \frac{2C\alpha\lambda^2}{\alpha^2 - \lambda^2} \left[\frac{1}{\lambda^2 + \omega^2} - \frac{1}{\alpha^2 + \omega^2}\right]
\end{aligned}$$

The ACF of output is given by

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2C\alpha\lambda^2}{a^2-\lambda^2}\left[\frac{1}{\lambda^2+\omega^2}-\frac{1}{a^2+\omega^2}\right]e^{i\omega\tau}d\omega$$

$$= \frac{C\alpha\lambda^2}{\pi(\alpha^2-\lambda^2)}\left[\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{\omega^2+\lambda^2}d\omega - \int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{\omega^2+\alpha^2}d\omega\right]$$

$$= \frac{C\alpha\lambda^2}{\pi(\alpha^2-\lambda^2)}\left[\frac{\pi}{\lambda}e^{-\lambda|\tau|} - \frac{\pi}{\alpha}e^{-\alpha|\tau|}\right]$$

$$R_{YY}(\tau) = \frac{C\alpha\lambda^2}{\alpha^2-\lambda^2}\left[\frac{e^{-\lambda|\tau|}}{\lambda} - \frac{e^{-\alpha|\tau|}}{\alpha}\right]$$

iv) To find $R_{XY}(\tau)$:

The cross correlation function of $X(t)$ and $Y(t)$ is given by

$$R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau) = \int_{-\infty}^{\infty} R_{XX}(u)h(\tau+u)du$$

$$= \left[\int_{-\infty}^{0} 0 + \int_{0}^{\infty} Ce^{-\alpha|u|}\lambda e^{-\lambda(\tau+u)}du\right]$$

$$= C\lambda\int_{0}^{\infty} e^{-\alpha u}e^{-\lambda\tau-\lambda u}du = C\lambda\int_{0}^{\infty} e^{-\alpha u}e^{-\lambda\tau}e^{-\lambda u}du$$

$$= C\lambda e^{-\lambda\tau}\int_{0}^{\infty} e^{-au-\lambda u}du$$

$$= C\lambda e^{-\lambda\tau}\int_{0}^{\infty} e^{-(\lambda+a)u}du = C\lambda e^{-\alpha\lambda}\left[\frac{e^{-(\lambda+a)u}}{-(\lambda+a)}\right]_0^{\infty}$$

$$= C\lambda e^{-\lambda\tau}\int_{0}^{\infty}\left[0 - \frac{1}{-(\lambda+a)}\right]$$

$$R_{X\gamma}(\tau) = \frac{C\lambda e^{-\lambda\tau}}{\lambda+a}$$

15. **A circuit has unit impulse response gives by** $(t) = \begin{cases} \frac{1}{T} & 0 \le t \le T \\ 0 & else \end{cases}$ **,**

Evaluate $S_{YY}(\omega)$ in terms of $S_{XX}(\omega)$.

Sol: (i) To find H(ω):

Given $h(t) = \begin{cases} \frac{1}{T} & 0 \le t \le T \\ 0 & else \end{cases}$

H(ω) = F[h(t)]

$$= \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{T} e^{-t\omega t} dt \quad = \frac{1}{T} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_0^T$$

$$= \frac{-1}{i\omega T} \left[e^{-i\omega t} - e^0 \right] \quad = \frac{1}{i\omega T} \left[1 - e^{-i\omega T} \right]$$

$$= \frac{1}{i\omega T} \left[1 - e^{-i\omega T} \right] \quad = \frac{1}{i\omega T} [1 - (\cos \omega T - i \sin \omega T)]$$

$$= \frac{1}{i\omega T} [1 - \cos \omega T + i \sin \omega T]$$

$$= \frac{1}{i\omega T} \left[2\, sin^2 \frac{\omega T}{2} + i\, 2\, sin \frac{\omega T}{2} cos \frac{\omega T}{2} \right]$$

$$H(\omega) = \frac{2 \sin \frac{\omega T}{2}}{i\omega T} \left[sin \frac{\omega T}{2} + i\, cos \frac{\omega T}{2} \right]$$

$$H(\omega) = \frac{2 \sin \frac{\omega T}{2}}{\omega T} \left[\sqrt{sin^2 \frac{\omega T}{2} + cos^2 \frac{\omega T}{2}} \right]$$

$$|H(\omega)| = \frac{2 \sin \frac{\omega T}{2}}{\omega T}$$

(ii) To find $S_{YY}(\omega)$:

The relation between the PSD's of input and out put

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$S_{YY}(\omega) = \frac{4\sin^2\frac{\omega T}{2}}{T^2\omega^2} S_{XX}(\omega)$

16. A circuit has unit impulse response given by $h(t) = \begin{cases} \frac{1}{2c} & |t| \le \varepsilon \\ 0 & else \end{cases}$. Evaluate

$S_{YY}(\omega)$ in terms of $S_{XX}(\omega)$

Sol: (i) To find H(ω):

Given $h(t) = \begin{cases} \frac{1}{2c} & |t| \le \varepsilon \\ 0 & else \end{cases}$

H(ω) = F[h(t)]

$= \int_{-\varepsilon}^{\varepsilon} h(t) e^{-i\omega t} dt$

$= \int_{-\varepsilon}^{\varepsilon} \frac{1}{2C} e^{-i\omega t} dt \quad = \frac{1}{2C}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-\varepsilon}^{\varepsilon}$

$= \frac{-1}{2ci\omega}\left[e^{-i\omega t} + e^{-i\omega t}\right] \quad = \frac{1}{2ci\omega}[2i\sin\omega\varepsilon]$

$H(\omega) = \frac{\sin\omega\varepsilon}{c\omega}$

(ii) To find $S_{YY}(\omega)$:

The relation between the PSD's of input and out put

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$S_{YY}(\omega) = \frac{4\sin\omega}{T^2\omega^2} S_{XX}(\omega)$

17. If the input X(t)is defined as Y(t) = $\frac{1}{T}\int_{t-T}^{t} X(s)$ ds , prove that X(t) and Y(t) are related by means of convolution integral. Find the unit impulse response of the system?

Sol: Given Y(t) = $\frac{1}{T}\int_{t-T}^{t} X(s)$ ds

= $\frac{1}{T}\int_{T}^{0} X(t-u)(-du)$

= $\frac{1}{T}\int_{0}^{T} X(t-u)$ du = $\int_{0}^{T}\frac{1}{T}X(t-u)$ du

Y(t) = $\int_{-\infty}^{\infty} h(u)X(t-u)du$ = h(t) * X(t)

Where the unit impulse response of the system is $h(t) = \begin{cases} \frac{1}{T} & 0 < t < T \\ 0 & else \end{cases}$

18. Given that Y(t) = $\frac{1}{T}\int_{t-\varepsilon}^{t+\varepsilon} X(\alpha)\ d\alpha$, where {X(t)} is a WSS process. Prove that $S_{YY}(\omega) = \frac{\sin^2 \varepsilon\omega}{\varepsilon^2\omega^2} S_{XX}(\omega)$. Find also the output autocorrelation function.

Sol: Given that

Y(t) = $\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} X(\alpha)\ d\alpha$,

= $\frac{1}{2\varepsilon}\int_{\varepsilon}^{-\varepsilon} X(t-\varepsilon)\ (-du)$

= $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} X(t-\varepsilon)\ du$

Y(t)= $\int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}X(t-\varepsilon)\ du$ ……….. (1)

Define

H(t) = $\begin{cases} \frac{1}{2\varepsilon} & |t| \le \varepsilon \\ 0 & else \end{cases}$

Then (1) => Y(t))= $\int_{-\infty}^{\infty} h(u)X(t-u)\ du$

We know that

H(ω) = F[h(t)]

$=\int_{-\infty}^{\infty} \text{h(t)} e^{-i\omega t} dt$

$=\int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} e^{-t\omega t} dt \quad = \frac{1}{2\varepsilon}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-\varepsilon}^{\varepsilon}$

$=\frac{-1}{2\varepsilon i\omega}\left[e^{-i\omega t} - e^{i\omega t}\right] \qquad =\frac{1}{2\varepsilon i\omega}\left[e^{i\omega t} - e^{-i\omega t}\right]$

$= \quad \frac{1}{2\varepsilon i\omega}$ 2i sin $\varepsilon\omega = \quad \frac{\sin\varepsilon\omega}{\varepsilon\omega}$

The relationship between the PSD's of the input and output is given by

$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

$S_{YY}(\omega) = \frac{sin^2\omega}{\varepsilon^2\omega^2} S_{XX}(\omega)$

We know that $R_{YY}(\tau) = F^{-1}[S_{YY}(\omega)] \quad = F^{-1}[\ \frac{sin^2\omega}{\varepsilon^2\omega^2} S_{XX}(\omega)\ ]$

$= F^{-1}\left(\frac{sin^2\omega\varepsilon}{\varepsilon^2\omega^2}\right) * F^{-1}\left(S_{XX}(\omega)\right)$

$R_{YY}(\tau) \quad = F^{-1}\left(\frac{sin^2\omega\varepsilon}{\varepsilon^2\omega^2}\right) * R_{YY}(\tau)$ ……………(4)

Let us assume that f(τ) = $\begin{cases} 1 - \frac{|\tau|}{a} & |\tau| \le a \\ 0 & else \end{cases}$

$F[f(\tau)] \quad = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau$

$$= \int_{-a}^{a}\left(1-\frac{|\tau|}{a}\right)e^{-i\omega\tau}d\tau$$

$$= \int_{-a}^{a}\left(1-\frac{|\tau|}{a}\right)(\cos\omega\tau - i\sin\omega\tau)\,d\tau$$

$$= \int_{-a}^{a}\left(1-\frac{|\tau|}{a}\right)\cos\omega\tau - i\int_{-a}^{a}\left(1-\frac{|\tau|}{a}\right)\sin\omega\tau\,d\tau$$

$$= 2\left[\left(1-\frac{|\tau|}{a}\right)\left(\frac{\sin\omega\tau}{\omega}\right)-\left(\frac{-1}{a}\right)\left(\frac{\cos\omega\tau}{\omega^2}\right)\right]_0^a$$

$$=2\left[0-\left(\frac{\cos\omega\tau}{a\omega^2}\right)+\frac{1}{a\omega^2}\right]$$

$$=\frac{2sin^2\left[\frac{\omega\varepsilon}{2}\right]}{a\omega^2}$$

Put a =2 ε, we get

$$F[f(\tau)] = \frac{4sin^2\left[\frac{\omega\varepsilon}{2}\right]}{2\varepsilon\omega^2}$$

$$=\frac{2\,\varepsilon\,sin^2\left[\frac{\omega\varepsilon}{2}\right]}{\varepsilon^2\omega^2}$$

⇨ $f(\tau) = F^{-1}\left[\frac{2sin^2\left[\frac{\omega\varepsilon}{2}\right]}{\varepsilon^2\omega^2}\right]$

$$f(\tau) = 2\,\varepsilon\,F^{-1}\left[\frac{sin^2\left[\frac{\omega\varepsilon}{2}\right]}{\varepsilon^2\omega^2}\right]$$

$$F^{-1}\left[\frac{sin^2\left[\frac{\omega\varepsilon}{2}\right]}{\varepsilon^2\omega^2}\right] = \frac{1}{2\varepsilon}f(\tau)$$

(4) => $R_{YY}(\tau) = \frac{1}{2\varepsilon} f(\tau) * R_{XX}(\tau)$

$$=\int_{-\infty}^{\infty}\frac{1}{2\varepsilon}\,f(u)\;R_{XX}(\tau-u)\,du$$

$$=\int_{-\infty}^{\infty} \frac{1}{2\varepsilon}\left(1-\frac{|u|}{2\varepsilon}\right) R_{XX}(\tau-u)\, du$$

$$f(t)=\begin{cases} 1-\frac{|\tau|}{2\varepsilon} & |\tau| \le 2\varepsilon \\ 0 & else \end{cases}$$

19. The relation between input{X(t) }and output {Y(t)} of the diode is expressed as Y(t) = X^2(t). Let {X(t)} be a zero mean stationary Gaussian random process with ACF $R_{XX}(\tau) = e^{-\alpha|\tau|}$; $\alpha > 0$. Find the output auto correlation $R_{YY}(\tau)$ of the input.

Sol: Given Y(t) =X^2(t), where X(t) is the zero mean stationary Gaussian random process.

$$R_{YY}(\tau) = E[Y(t_1)\,Y(t_2)] = E[X^2(t_1)\,X^2(t_2)]$$

$$=E[(X(t_1)X(t_2))^2]$$

$$= E[X^2(t_1)]\,E[X^2(t_2)] + 2\,[E[(X(t_1)X(t_2)]]^2$$

$$=R_{XX}(0)\,R_{XX}(0)+ 2\,[R_{XX}(\tau)\,]^2 \qquad \ldots\ldots\ldots\ldots(1)$$

Given $R_{XX}(\tau) = e^{-\alpha|\tau|}$

$$R_{XX}(0) = e^0 = 1$$

$$(1) \Rightarrow R_{YY}(\tau) = 1+2\left[e^{-\alpha|\tau|}\right]^2$$

$$= 1+2\, e^{-\alpha|\tau|}$$

The PSD of output is given by

$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-i\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} 1 + 2\, e^{-2\alpha|\tau|}\, e^{-i\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} e^{-i\omega\tau} d\tau + 2 \int_{-\infty}^{\infty} 1 + 2\, e^{-2\alpha|\tau|}\, d\tau$

$=2\pi\delta(\omega) + 2\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\,[\cos\omega\tau - i\sin\omega\tau] d\tau$

$=2\pi\delta(\omega) + 2\left[\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\sin\omega\tau\, d\tau\right]$

$=2\pi\delta(\omega) + 2\left[2\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\cos\omega\, d\tau - i(0)\right]$

$=2\pi\delta(\omega) + 4\int_{-\infty}^{\infty} e^{-2\alpha|\tau|}\cos\omega\tau\, d\tau$

$S_{YY}(\omega) = 2\pi\delta(\omega) + \frac{8\alpha}{4\alpha^2+\omega^2}$ $\qquad \int_{0}^{\infty} e^{-at}\cos bt\, dt = \frac{a}{a^2+b^2}$

20. Find the auto correlation of the band limited white noise {N(t)} with PSD given by $S_{NN}(\omega) = \frac{N_0}{2}; |\omega - \omega_0| < \omega_B;$

Sol: $S_{NN}(\omega)=\frac{N_0}{2}; |\omega - \omega_0| < \omega_B;$

$|\omega - \omega_0| < \omega_B => -\omega_B < \omega - \omega_0 < \omega_B < \omega < \omega_0 + \omega_B$

The ACF of {N(t)} is given by

$R(\tau)= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\, e^{i\omega\tau} d\omega$

$= \frac{1}{2\pi}\int_{\omega_0-\omega_B}^{\omega_0+\omega_B} \frac{N_0}{2}\, e^{i\omega\tau} d\omega$

$= \frac{N_0}{4\pi}\int_{\omega_0-\omega_B}^{\omega_0+\omega_B} e^{i\omega\tau} d\omega$

$=\frac{N_0}{4\pi}\left(\frac{e^{i\omega\tau}}{i\tau}\right)_{\omega_0-\omega_B}^{\omega_0+\omega_B}$

$=\frac{N_0}{4\pi}\frac{\left(e^{i(\omega_0+\omega_B)\tau}-e^{i(\omega_0-\omega_B)\tau}\right)}{i\tau}$

$=\frac{N_0}{4\pi}\frac{e^{i\omega_B\tau+i\omega_B\tau}-e^{i\omega_B\tau-i\omega_B\tau}}{i\tau}$

$=\frac{N_0}{4\pi}\frac{e^{i\omega_0\tau}e^{i\omega_B\tau}-e^{i\omega_0\tau}e^{i\omega_B\tau}}{i\tau}$

$=\frac{N_0}{4\pi}e^{i\omega_0\tau}\frac{e^{i\omega_B\tau}-e^{-i\omega_B\tau}}{i\tau}$

$=\frac{N_0 e^{i\omega_0\tau}}{4\pi}\frac{2i\ sin\omega_{B\tau}}{i\tau}$

$=\frac{N_0}{4\tau\pi}e^{i\omega_0\tau}sin\omega_B\tau$

$=\frac{N_0}{4\tau\pi}(cos\omega_0\,\tau+i\ sin\omega_0\,\tau)sin\omega_B\,\tau$

$=\frac{N_0}{2\tau\pi}\ cos\omega_0\,\tau\ sin\omega_B\,\tau+i\,\frac{N_0}{2\tau\pi}\ sin\omega_0\,\tau\ sin\omega_B\,\tau$

Since ACF is real, equating the real part, we get

R(τ) = $\frac{N_0}{2\tau\pi}\ cos\omega_0\,\tau\ sin\omega_B\,\tau$

$=\frac{N_0}{2\pi}\ cos\omega_0\,\tau\ \frac{sin\omega_B\,\tau}{\omega_B\,\tau}\omega_B$

R(τ) = $\frac{N_0\omega_B}{2\tau\pi}\ cos\omega_0\,\tau\ \frac{sin\omega_B\,\tau}{\omega_B\,\tau}$