

2.1. LINEAR TRANSFORMATION

Definition

Let V and W be vector spaces over F . A function $T: V \rightarrow W$ is called a linear transformation if for all $x, y \in V$ and $\alpha \in F$,

$$(a) T(x + y) = T(x) + T(y)$$

$$(b) T(\alpha x) = \alpha T(x)$$

Properties of linear transformation

1. If T is the linear, then $T(0) = 0$

Proof

$$T(0) = T(0 + 0)$$

$$T(0) = T(0) + T(0)$$

$\therefore T(0)$ is zero element of W .

Which implies,

$$T(0) = 0$$

2. T is linear if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$; for all $x, y \in V$ and $\alpha, \beta \in F$.

Proof

Assume T is linear.

$$\begin{aligned} T(\alpha x + \beta y) &= T(\alpha x) + T(\beta y) \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

Conversely.

$$\text{Assume } T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \dots (1)$$

Put $\alpha = 1, \beta = 1$ in (1). Then

$$T(x + y) = T(x) + T(y)$$

Put $y = 0$ in (1). Then

$$T(\alpha x + 0) = \alpha T(x) + T(0)$$

$$= \alpha T(x) + 0$$

$$= \alpha T(y)$$

$\therefore T$ is linear.

3. If T is linear, then $T(x - y) = T(x) - T(y)$; for all $x, y \in V$ Given T is linear

$$T(x - y) = T(x + (-y))$$

$$= T(x) + T(-y)$$

$$= T(x) - T(y)$$

Example 1. $T: R^2 \rightarrow R^2$ is defined by $T(a_1, a_2) = (2a_1 + a_2, a_1)$. Verif whether T is a linear transformation

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2)$ and $y = (b_1, b_2)$

$$x + y = (a_1 + b_1, a_2 + b_2)$$

Given

$$T(a_1, a_2) = (2a_1 + a_2, a_1)$$

To prove T is linear, we have to prove

$$(i) T(x + y) = T(x) + T(y)$$

$$(ii) T(\alpha x) = \alpha T(x)$$

Proof:

$$f(x) = T(a_1, a_2)$$

$$= (2a_1 + a_2, a_1)$$

$$T'(y) = T(b_1, b_2)$$

$$= (2b_1 + b_2, b_1)$$

$$\begin{aligned}
 \text{(i)} \quad r(x+y) &= T(a_1 + b_1, a_2 + b_2) \\
 &= (2(a_1 + b_1) + a_2 + b_2, a_1 + b_1) \\
 &= (2a_1 + 2b_1 + a_2 + b_2, a_1 + b_1) \\
 &= (2a_1 + a_2, a_1) + (2b_1 + b_2, b_1) \\
 &= T(a_1, a_2) + T(b_1, b_2) \\
 &= T(x) + T(y)
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad T(\alpha x) &= T(\alpha a_1, \alpha a_2) \\
 &= (2\alpha a_1 + \alpha a_2, \alpha a_1) \\
 &= \alpha(2a_1 + a_2, a_1) \\
 &= \alpha T(a_1, a_2) = \alpha T(x)
 \end{aligned}$$

$\therefore T: R^2 \rightarrow R^2$ is a linear transformation,

Example 2) $T: V_2(R) \rightarrow V_2(R)$ is defined by $T(a_1, a_2) = (3a_1 + 2a_2, 3a_1 - 4a_2)$. Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$$\therefore x = (a_1, a_2) \text{ and } y = (b_1, b_2)$$

$$x + y = (a_1 + b_1, a_2 + b_2)$$

Given

$$T(a_1, a_2) = (3a_1 + 2a_2, 3a_1 - 4a_2)$$

To prove T is linear, we have to prove

$$\text{(i)} \quad T(x + y) = T(x) + T(y)$$

$$\text{(ii)} \quad T(\alpha x) = \alpha T(y)$$

Proof:

$$T(x) = T(a_1, a_2)$$

$$= (3a_1 + 2a_2, 3a_1 - 4a_2)$$

$$T(y) = T(b_1, b_2)$$

$$= (3b_1 + 2b_2, 3b_1 - 4b_2)$$

$$\begin{aligned} \text{(i)} \quad T(x + y) &= T(a_1 + b_1, a_2 + b_2) \\ &= (3(a_1 + b_1) + 2(a_2 + b_2), 3(a_1 + b_1) - 4(a_2 + b_2)) \\ &= (3a_1 + 3b_1 + 2a_2 + 2b_2, 3a_1 + 3b_1 - 4a_2 - 4b_2) \\ &= (3a_1 + 2a_2 + 3b_1 + 2b_2, 3a_1 - 4a_2 + 3b_1 - 4b_2) \\ &= (3a_1 + 2a_2, 3a_1 - 4a_2) + (3b_1 + 2b_2, 3b_1 - 4b_2) \\ &= T(a_1, a_2) + T(b_1, b_2) \\ &= T(x) + T(y) \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad T(\alpha x) &= T(\alpha a_1, \alpha a_2) \\ &= (3(\alpha a_1) + 2(\alpha a_2), 3(\alpha a_1) - 4(\alpha a_2)) \\ &= (\alpha(3a_1 + 2a_2), \alpha(3a_1 - 4a_2)) \\ &= \alpha((3a_1 + 2a_2), (3a_1 - 4a_2)) \\ &= \alpha T(a_1, a_2) = \alpha T(x) \end{aligned}$$

$T: V_2(R) \rightarrow V_2(R)$ is a linear transformation.

Exmple 3. Define $T: R^3 \rightarrow R^3$ by $T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$.

Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$$\therefore x = (a_1, a_2, a_3) \text{ and } y = (b_1, b_2, b_3)$$

$$x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Given

$$T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$$

To prove T is linear, we have to prove

$$(i) T(x + y) = T(x) + T(y)$$

$$(ii) T(\alpha x) = \alpha T(x)$$

Proof

$$\begin{aligned} T(x) &= T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3) \\ T(y) &= T(b_1, b_2, b_3) \\ &= (2b_1 + b_2, b_2 - b_3, 2b_2 + 4b_3) \end{aligned}$$

$$\begin{aligned} (i) T(x + y) &= T(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (2(a_1 + b_1) + (a_2 + b_2), (a_2 + b_2) - (a_3 + b_3), 2(a_2 + b_2) + 4(a_3 + b_3)) \\ &= (2a_1 + 2b_1 + a_2 + b_2, a_2 + b_2 - a_3 - b_3, 2a_2 + 2b_2 + 4a_3 + 4b_3) \\ &= (2a_1 + a_2 + 2b_1 + b_2, a_2 - a_3 + b_2 - b_3, 2a_2 + 4a_3 + 2b_2 + 4b_3) \\ &= (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3) + (2b_1 + b_2, b_2 - b_3, 2b_2 + 4b_3) \\ &= T(a_1, a_2, a_3) + T(b_1, b_2, b_3) \\ &= T(x) + T(y) \end{aligned}$$

$$\begin{aligned} (ii) T(\alpha x) &= T(\alpha a_1, \alpha a_2, \alpha a_3) \\ &= (2\alpha a_1 + \alpha a_2, \alpha a_2 - \alpha a_3, 2\alpha a_2 + 4\alpha a_3) \\ &= (\alpha(2a_1 + a_2), \alpha(a_2 - a_3), \alpha(2a_2 + 4a_3)) \\ &= \alpha(2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3) \\ &= \alpha T(a_1, a_2, a_3) = \alpha T(x) \end{aligned}$$

$\therefore T$ is a linear transformation and hence a linear map on R^3 .

Example 4. Define mapping $T: V_3(F) \rightarrow V_2(F)$ by $T(a_1, a_2, a_3) = (a_2, a_3)$. Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$$\therefore x = (a_1, a_2, a_3), y = (b_1, b_2, b_3)$$

$$x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Given

$$T(a_1, a_2, a_3) = (a_2; a_3)$$

To prove T is linear, we have to prove

$$(i) T(x + y) = T(x) + T(y)$$

$$(ii) T(\alpha x) = \alpha T(x)$$

Proof:

$$T(x) = T(a_1, a_2, a_3) = (a_2, a_3) \quad T(y) = T(b_1, b_2, b_3) = (b_2, b_3).$$

$$\begin{aligned} (i) \quad T(x + y) &= T(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (a_2 + b_2, a_3 + b_3) \\ &= (a_2, a_3) + (b_2, b_3) \\ &= T(a_1, a_2, a_3) + T(b_1, b_2, b_3) \\ &= T(x) + T(y) \end{aligned}$$

$$\begin{aligned} (i) T(\alpha x) &= T(\alpha a_1, \alpha a_2, \alpha a_3) \\ &= (\alpha a_2, \alpha a_3) \\ &= \alpha(a_2, a_3) \\ &= \alpha T(a_1, a_2, a_3) \\ &= \alpha T(x) \end{aligned}$$

$\therefore T$ is a linear transformation.

Example 5. Show that for $0 \leq \theta < 2\pi$, the transformation given by $T_\theta: R^2 \rightarrow R^2$, $T_\theta(a, b) = (a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$ is linear

Sol: $x, y \in R^2$ and $\alpha \in F$

$$\therefore x = (a_1, b_1), y = (a_2, b_2) \quad x + y = (a_1 + a_2, b_1 + b_2)$$

Given

$$T_\theta(a, b) = (a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$$

To prove T is linear, we have to prove

$$(i) T(x + y) = T(x) + T(y)$$

$$(ii) T(\alpha x) = \alpha T(x)$$

Proof

$$T(x) = T(a_1, b_1)$$

$$= (a_1 \cos \theta - b_1 \sin \theta, a_1 \sin \theta + b_1 \cos \theta)$$

$$T(y) = T(a_2, b_2)$$

$$= (a_2 \cos \theta - b_2 \sin \theta, a_2 \sin \theta + b_2 \cos \theta)$$

$$(i) T(x + y) = T(a_1 + a_2, b_1 + b_2)$$

$$= ((a_1 + a_2) \cos \theta - (b_1 + b_2) \sin \theta, (a_1 + a_2) \sin \theta + (b_1 + b_2) \cos \theta)$$

$$= (a_1 \cos \theta + a_2 \cos \theta - b_1 \sin \theta - b_2 \sin \theta, a_1 \sin \theta + a_2 \sin \theta - b_1 \cos \theta - b_2 \cos \theta)$$

$$= (a_1 \cos \theta - b_1 \sin \theta, a_1 \sin \theta + b_1 \cos \theta)$$

$$+ (a_2 \cos \theta - b_2 \sin \theta, a_2 \sin \theta - b_2 \cos \theta) = T(x) + T(y)$$

$$(ii) T(\alpha x) = T(\alpha a_1, \alpha b_1)$$

$$= (\alpha a_1 \cos \theta - \alpha b_1 \sin \theta, \alpha a_1 \sin \theta + \alpha b_1 \cos \theta)$$

Example 9. Let $M(R)$ be the vector space of all 2×2 matrices over R and B be a fixed non-zero element of $M(R)$. Show that the mapping $T: M(R) \rightarrow M(R)$ defined by $T(A) = AB + BA, \forall A \in M(R)$ is a linear transformation.

Sol:

Let $A, C \in M(R)$ and $\alpha \in R$

Given

$$T(A) = AB + BA, \forall A \in M(R) \text{ for a fixed non-zero element } B \text{ of } M(R)$$

To prove F is linear, we have to prove

$$(i) F(A + C) = F(A) + F(C)$$

$$(ii) F(\alpha A) = \alpha F(A)$$

Proof:

$$T(A) = AB + BAT(C) = CB + BC$$

$$(i) T(A+C) = (A+C)B + B(A+C)$$

$$= AB + CB + BA + BC$$

$$= (AB + BA) + (CB + BC)$$

$$= T(A) + T(C)$$

$$(ii) T(\alpha A) = (\alpha A)B + B(\alpha A)$$

$$= \alpha(AB + BA)$$

$$= \alpha T(A)$$

$\therefore T$ is a linear transformation.

Example 14. Prove that there exists linear transformation $T: R^2 \rightarrow R^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

Sol: Let us express $(1,1)$ and $(2,3)$ as a linear combination of the standard basis vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ of R^2

$$(1,1) = 1(1,0) + 1(0,1) = 1e_1 + 1e_2$$

$$= e_1 + e_2 \dots (1)$$

$$(2,3) = 2(1,0) + 3(0,1) = 2e_1 + 3e_2$$

$$= 2e_1 + 3e_2 \dots (2)$$

Given

$$T(1,1) = (1,0,2)$$

$$\Rightarrow T(e_1 + e_2) = (1,0,2) \text{ [from (1)]}$$

$$\Rightarrow T(e_1) + T(e_2) = (1,0,2) \dots (3)$$

Also given

$$T(2,3) = (1,-1,4)$$

$$\Rightarrow T(2e_1 + 3e_2) = (1, -1, 4) \quad [\text{from (2)}]$$

$$\Rightarrow 2T(e_1) + 3T(e_2) = (1, -1, 4) \dots$$

Solve (3) and (4)

$$(3) \times 2 \Rightarrow 2T(e_1) + 2T(e_2) = (2, 0, 4)$$

$$(4) \Rightarrow 2T(e_1) + 3T(e_2) = (1, -1, 4)$$

$$\text{Subtracting} \quad -T(e_2) = (1, 1, 0)$$

$$T(e_2) = (-1, -1, 0)$$

$$(3) \Rightarrow T(e_1) + (-1, -1, 0) = (1, -1, 4)$$

$$\Rightarrow T(e_1) = (1, -1, 4) - (-1, -1, 0)$$

$$T(e_1) = (2, 0, 4)$$

To find the linear transformation:

Let $(a_1, a_2) \in R^2$. Then

$$(a_1, a_2) = a_1(1, 0) + a_2(0, 1)$$

$$= a_1e_1 + a_2e_2$$

$$T(x, y) = T(a_1e_1 + a_2e_2)$$

$$= a_1T(e_1) + a_2T(e_2)$$

$$= a_1(2, 0, 4) + a_2(-1, -1, 0)$$

$$T(a_1, a_2) = (2a_1 - a_2, -a_2, 4a_1)$$

$$T(8, 11) = (5, -11, 32)$$

Example 15. Is there a linear transformation $T: R^3 \rightarrow R^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$?

Sol: Let us express $(1, 0, 3)$ and $(-2, 0, -6)$ as a linear combination of the R^3

standard basis vectors $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ of R^3 .

$$(1,0,3) = 1(1,0,0) + 0(0,1,0) + 3(0,0,1)$$

$$= e_1 + 0e_2 + 3e_3$$

$$= e_1 + 3e_3 \dots (1)$$

$$(-2,0,-6) = -2(1,0,0) + 0(0,1,0) - 6(0,0,1)$$

$$= -2e_1 + 0e_2 - 6e_3$$

$$= -2e_1 - 6e_3 \dots (2)$$

$$T(1,0,3) = (1,1)$$

$$\Rightarrow T(e_1 + 3e_3) = (1,1) \text{ [from (1)]}$$

$$\Rightarrow T(e_1) + 3T(e_3) = (1,1) \dots (3)$$

Also given

$$T(-2,0,-6) = (2,1)$$

$$T(-2e_1 - 6e_3) = (2,1) \text{ [from (2)]}$$

$$\Rightarrow -2T(e_1) - 6T(e_3) = (2,1) \dots (4)$$

solve (3) and (4)

$$(3) \times 2 \Rightarrow 2T(e_1) + 6T(e_3) = (2,2)$$

$$(4) \Rightarrow -2T(e_1) - 6T(e_3) = (2,1)$$

$$\text{Adding} \quad 0 = (4,3)$$

It is not possible.

Therefore there is no linear transformation with the given data's.

Example 16. Find a linear transformation $T: R^3 \rightarrow R^3$ such that $T(1,1,1) = (1,1,1)$, $T(1,2,3) = (-1,-2,3)$ and find $T(1,1,2) = (2,2)$

Sol: Let us express $(1,1,1)$, $(1,2,3)$ and $(1,1,2)$ as a linear combination basis vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ of R^3

$$\begin{aligned}(1,1,1) &= 1(1,0,0) + 1(0,1,0) + 1(0,0,1) = 1e_1 + 1e_2 + 1e_3 \\ &= e_1 + e_2 + e_3 \dots (1)\end{aligned}$$

$$\begin{aligned}(1,2,3) &= 1(1,0,0) + 2(0,1,0) + 3(0,0,1) = 1e_1 + 2e_2 + 3e_3 \\ &= e_1 + 2e_2 + 3e_3 \dots (2)\end{aligned}$$

$$\begin{aligned}(1,1,2) &= 1(1,0,0) + 1(0,1,0) + 2(0,0,1) = 1e_1 + 1e_2 + 2e_3 \\ &= e_1 + e_2 + e_3 \dots (3)\end{aligned}$$

Given

$$\begin{aligned}T(1,1,1) &= (1,1,1) \\ \Rightarrow T(e_1 + e_2 + e_3) &= (1,1,1) \quad [\text{from (1)}] \\ \Rightarrow T(e_1) + T(e_2) + T(e_3) &= (1,1,1) \dots (4)\end{aligned}$$

Also given

$$\begin{aligned}T(1,2,3) &= (-1, -2, 3) \\ \Rightarrow T(e_1 + 2e_2 + 3e_3) &= (-1, -2, 3) \quad [\text{from (1)}] \\ \Rightarrow T(e_1) + 2T(e_2) + 3T(e_3) &= (-1, -2, 3) \dots (5)\end{aligned}$$

Also given

$$\begin{aligned}T(1,1,2) &= (2, 2, 4) \\ \Rightarrow T(e_1 + e_2 + 2e_3) &= (2, 2, 4) \quad [\text{from (1)}] \\ \Rightarrow T(e_1) + T(e_2) + 2T(e_3) &= (2, 2, 4) \dots (6)\end{aligned}$$

Solve (4), (5) and (6)

$$\begin{aligned}(4) &\Rightarrow T(e_1) + T(e_2) + T(e_3) = (1, 1, 1) \\ (6) &\Rightarrow \frac{T(e_1) + T(e_2) + 2T(e_3) = (2, 2, 4)}{-T(e_3) = (-1, -1, -3)}\end{aligned}$$

Subtracting

$$T(e_3) = (1,1,3)$$

$$(4) \Rightarrow T(e_1) + T(e_2) + (1,1,3) = (1,1,1)$$

$$\Rightarrow T(e_1) + T(e_2) = (1,1,1) - (1,1,3)$$

$$= (0,0,-2) \dots (7)$$

$$(5) \Rightarrow T(e_1) + 2T(e_2) + 3(1,1,3) = (-1,-2,3)$$

$$\Rightarrow T(e_1) + 2T(e_2) + (3,3,9) = (-1,-2,3)$$

$$\Rightarrow T(e_1) + 2T(e_2) = (-1,-2,3) - (3,3,9)$$

$$= (-4,-5,-6) \dots (8)$$

Solve (7) and (8)

$$(7) \Rightarrow T(e_1) + T(e_2) = (0,0,-2)$$

$$\text{Subtracting (6)} \Rightarrow \frac{T(e_1) + 2T(e_2) = (-4,-5,-6)}{-T(e_2) = (4,5,4)}$$

$$T(e_2) = (-4,-5,-4)$$

$$(7) \Rightarrow T(e_1) + (-4,-5,-4) = (0,0,-2)$$

$$\Rightarrow T(e_1) = (0,0,-2) - (-4,-5,-4)$$

$$T(e_1) = (4,5,2)$$

To find the linear transformation:

Let $(x, y, z) \in R^3$. Then

$$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$= xe_1 + ye_2 + ze_3$$

$$r(x, y, z) = T(xe_1 + ye_2 + ze_3)$$

$$= xT(e_1) + yT(e_2) + zT(e_3)$$

$$= x(4, 5, 2) + y(-4, -5, -4) + z(1, 1, 3)$$

$$= (4x, 5x, 2x) + (-4y, -5y, -4y) + (z, z, 3z)$$

$$T(x, y) = (4x - 4y + z, 5x - 5y + z, 2x - 4y + 3z)$$

