1.5. PRINCIPAL PLANES AND PRINCIPAL STRESSES

The planes, which have no shear stress are known as principal planes. Hence principal planes are the planes of zero shear stress. These planes carry only normal stresses.

The normal stresses acting on a principal plane, are known as principal stresses.

1.5.1. METHODS FOR DETERMINING STRESSES ON OBLIQUE SECTION

The stresses on oblique section are determined by the following methods:

1. Analytical method and 2. Graphical method.

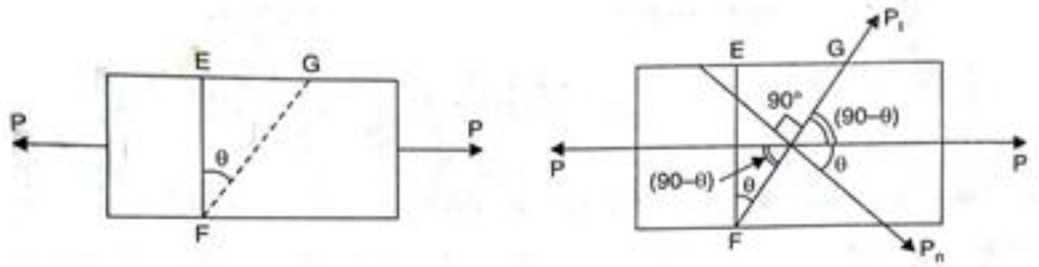
1.5.2 ANALYTICAL METHOD FOR DETERMINING STRESSES ON OBLIQUE SECTION

The following two cases will be considered for determining stresses on oblique plane.

1. A member subjected to a direct stress in one plane.
2. The member is subjected to like direct stresses in two mutually perpendicular directions.

1.5.2.1 A Member subjected to a Direct stress in one plane.

Figure shows a rectangular member of uniform cross sectional area A and of unit thickness



Let P = Axial force acting on the member

A = Area of cross section, which is perpendicular to the line of action of the force P.

The stress acting along X .axis, $\sigma = \frac{P}{A}$

Hence, the member is subjected to a stress along X axis.

Consider a cross section EF which is perpendicular to the line of action of the force P.

Then the area of section, EF = EF × 1 =A

The stress on the section EF is given by

$$\sigma = \frac{Force}{Area\ of\ EF} = \frac{P}{A}$$

The stress on the section EF is entirely normal stress. There is no shear stress (or tangential stress) on the section EF.

Now consider a section FG at an angle θ with the normal cross section EF as shown in Figure

Area of FG = FG × 1 =FG

$= \frac{EF}{cos\theta} \times 1$ (since In Δ EFG, $\theta = \frac{EF}{FG} \therefore FG = \frac{EF}{cos\theta}$)

$= \frac{A}{cos\theta}$

∴ Stress on the section, FG = $\frac{Force}{Area\ of\ sectiion\ FG} = \frac{P}{\frac{A}{cos\theta}} = \frac{P}{A}\cos\theta$

$= \sigma \cos \theta$ (since $\sigma = \frac{P}{A}$) …(i)

This stress on the section FG, is parallel to the axis of the member (i.e., this stress is along X axis). This stress may be resolved in two components. One component will be normal to the section FG whereas the second component will be along the section FG (i.e., tangential stress on the section FG) .The normal stress and tangential stress (i.e., shear stress) on the section FG are obtained as given below.(Refer above Fig.)

Let P_n = The component of the force P, normal to section FG = P cosθ

P_t = The component of force P, along the section FG (or tangential to the surface FG) = P sinθ

σ_n= Normal stress across the section FG

σ_t = Tangential stress (i.e., shear stress) across the section FG.

Normal stress $\sigma_n = \frac{Force\ normal\ to\ section\ FG}{Area\ of\ section\ FG}$

$= \frac{P_n}{\frac{A}{cos\theta}}$ $(since\ Pn = Pcos\theta)$

$= \frac{Pcos\theta}{\frac{A}{cos\theta}}$

$= \frac{P}{A}\cos\theta.\cos\theta = \frac{P}{A}\cos^2\theta$

$=\sigma\cos^2\theta$ …(ii) Tangential stress(or)

Shear stress,

$\sigma_t = \frac{Tangential\ force\ across\ section\ FG}{Area\ of\ section\ FG}$

$= \frac{P_t}{\frac{A}{cos\theta}}$ $(since\ Pt = Psin\theta)$

$= \frac{Psin\theta}{\frac{A}{cos\theta}}$

$= \frac{P}{A}\sin\theta.\cos\theta = \sigma\sin\theta.\cos\theta$

$= \boldsymbol{\frac{\sigma}{2}\sin 2\theta}$ …(iii)

$[\therefore since\ sin2\theta = 2sin\theta cos\theta\ sin\theta cos\theta = sin2\theta/2]$

From equation (ii) it is seen that the normal stress (σn) on the section FG will be maximum when $\cos^2\theta$ or $\cos\theta$ is maximum. And $\cos\theta$ will be maximum when $\theta=0^o$ as cos0=1. But when $\theta=0^o$, the section FG will coincide with section EF. But the section EF is normal to the line of action of the loading. This means the plane normal to the axis of loading. This means the plane normal to the axis of loading will carry the maximum normal stress.

$\therefore$Maximum normal stress, $= \sigma\cos^2\theta = \sigma\cos^2 0 = \sigma$ …(iv)

From equation (iii), it is observed that the tangential stress (i.e., shear stress) across the section FG will be maximum when $\sin2\theta=1$ or $2\theta=90^o$ or 270^o

or $\theta = 45^o$ or 135^o

This means the shear stress will be maximum on two planes inclined at 45^o and 135^o to the normal section EF as shown in fig.

$\therefore$ Max. value of shear stress $= \frac{\sigma}{2}\sin2\theta = \frac{\sigma}{2}\sin90^o = \frac{\sigma}{2}$ …(v)

From equations (iv) and (v) it is seen that maximum normal stress is equal to σ Whereas the maximum shear stress is equal to $\frac{\sigma}{2}$ or equal to half the value of greatest normal stress.

1.5.2.1 A member subjected to like Direct stresses in two mutually perpendicular Directions.

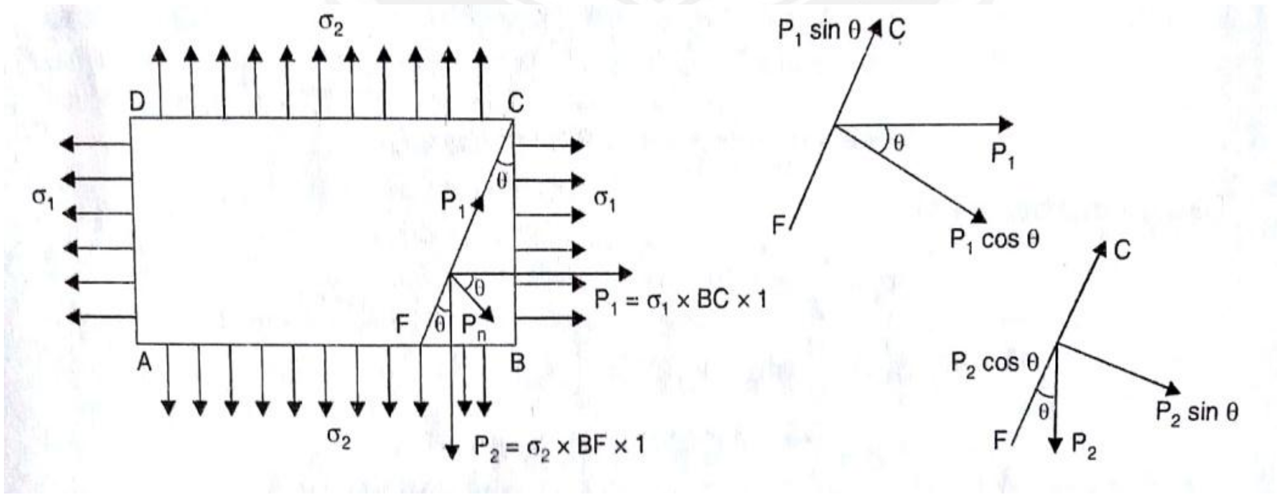


Fig. shows a rectangular bar ABCD of uniform cross sectional area A and of uniform thickness. The bar is subjected to two direct tensile stresses (or two principal tensile stresses) as shown in Fig

Let FC be the oblique section on which stresses are to be calculated. This can be done by converting the stresses σ_1 (acting along on face BC) and σ_2 (acting on face AB) into equivalent forces. Then these forces will be resolved along the incline plane FC and perpendicular to FC. Consider the forces acting on wedge FBC.

Let θ = Angle made by oblique section FC with normal cross section BC

σ_1 = Major tensile stress on face AD and BC

σ_2 = Minor tensile stress on face AB and CD

P_1 = Tensile force on face BC

P_2 = Tensile force on face FB.

The tensile force on face BC,

$P_1 = \sigma_1 \times$ Area of face BC $= \sigma_1 \times BC \times 1$

The tensile force on face FB,

$P_2 = \sigma_2 \times$ Area of face FB $= \sigma_2 \times FB \times 1$

The tensile forces P_1 and P_2 are also acting on the oblique section FC. The force P_1 is acting in the axial direction, whereas the force P_2 is acting downwards as shown in Fig. Two forces P_1 and P_2 each can be resolved into two components i.e., one normal to the plane FC and the other along the plane FC. The components of P_1 and $P_1\cos\theta$ normal to the plane FC and $P_1\sin\theta$ along the plane in the upward direction. The components of P_2 and $P_2\sin\theta$ normal to the plane FC and $P_2\cos\theta$ along the plane in the downward direction.

Let P_n = Total force normal to section FC

= Component of force P_1 normal to the section FC + Component of force P_2 normal to the section FC

$= P_1\cos\theta + P_2\sin\theta$

$= \sigma_1 \times BC \times \cos\theta + \sigma_2 \times FB \times \sin\theta$

(since $P_1 = \sigma_1 \times BC$, $P_2 = \sigma_2 \times FB$)

P_t = Total force along the section FC

= Component of force P1 along the section FC + Component of force P2 along the section FC

$= P_1\sin\theta - P_2\cos\theta$ (-ve sign is taken due to opposite direction)

$= \sigma_1 \times BC \times \sin\theta - \sigma_2 \times FB \times \cos\theta$ ($\therefore P_1 = \sigma_1 \times BC$, $P_2 = \sigma_2 \times FB$)

σ_n = Normal stress across the section FC

$$= \frac{Total\ force\ normal\ to\ the\ section\ FC}{Area\ of\ section\ FC}$$

$$= \frac{P_n}{FC \times 1} = \frac{\sigma_1 \times BC \times \cos\theta + \sigma_2 \times BF \times \sin\theta}{FC}$$

$= \sigma_1 \times \frac{BC}{FC} \times \cos\theta + \sigma_2 \times \frac{BF}{FC} \times \sin\theta$

$= \sigma_1 \times \cos\theta \times \cos\theta + \sigma_2 \times \sin\theta \sin\theta$

$= \sigma_1 \times \cos^2\theta + \sigma_2 \times \sin^2\theta$

$= \sigma_1\left[\frac{1+cos2\theta}{2}\right] + \sigma_2\left[\frac{1-cos2\theta}{2}\right]$

$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2} \cos2\theta$

σ_t = Tangential stress (or shear stress) along section FC

$= \frac{Total\, force\, along\, the\, section\, FC}{Area\, of\, section\, FC}$

$= \frac{P_t}{FC\times1} = \frac{\sigma_1 \times BC \times sin\theta - \sigma_2 \times BF \times cos\theta}{FC}$

$$= \sigma_1 \times \frac{BC}{FC} \times sin\theta - \sigma_2 \times \frac{BF}{FC} \times cos\theta$$

$$= \sigma1 \times cos\theta \times sin\theta - \sigma2 \times sin\theta \times cos\theta$$

$$= (\sigma1 - \sigma2)\ cos\theta \times sin\theta$$

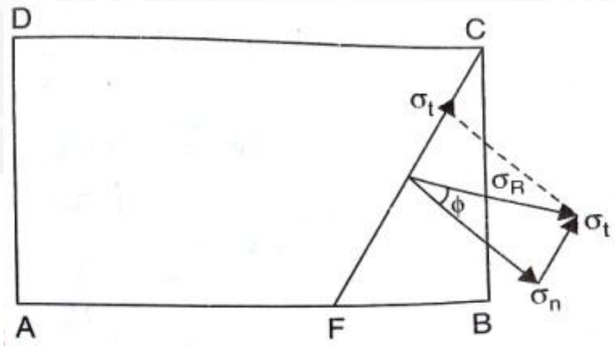
$= \frac{(\sigma_1-\sigma_2)}{2} \times 2\cos\theta \times \sin\theta$

$= \frac{(\sigma_1-\sigma_2)}{2} \sin2\theta$

The resultant stress on the section FC will be given as

$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$

Obliquity. The angle made by the resultant stress with the normal of the oblique plane, is known as obliquity as shown in fig. Mathematically, it is denoted by, $\tan\emptyset = \frac{\sigma_t}{\sigma_n}$



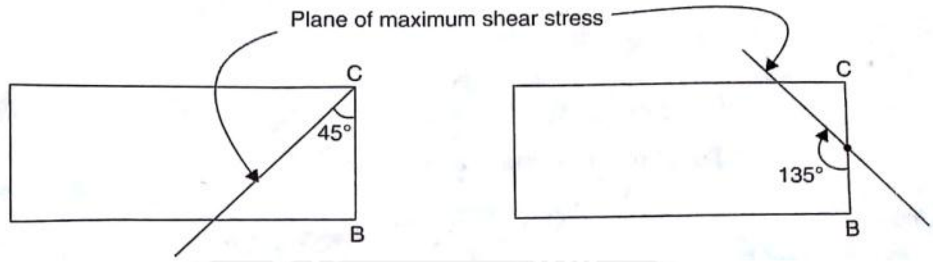
Maximum Shear stress. The shear stress is given by $\sigma_t = \frac{(\sigma_1-\sigma_2)}{2} \sin2\theta$. The shear stress will be maximum when

$\sin2\theta = 1$ or $2\theta = 90^o$ or 270^o

$\therefore \theta = 45^o$ or 135^o

And maximum shear stress $(\sigma_t)_{max} = \frac{(\sigma_1 - \sigma_2)}{2}$

The planes of maximum shear stress are obtained by making an angle of 45° and 135° with the plane BC (at any point on the plane BC) in such a way that the planes of maximum shear stress lie within the material as shown in Fig.



Principal Planes

Principal planes are the planes on which shear stress is zero. To locate the position of principal planes, the shear stress should be equated to zero.

∴ For principal planes, $\frac{(\sigma_1 - \sigma_2)}{2} \sin 2\theta = 0$

Or $\sin 2\theta = 0$ (since $(\sigma_1 - \sigma_2)$ cannot be=0)

Or $2\theta = 0$ or 180^o

Or $\theta = 0$ or 90^o

When $\theta = 0$,

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta$$

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 0$$

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}$$

$$= \sigma_1$$

When $\theta = 90^o$,

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos(2\times 90)^o$$

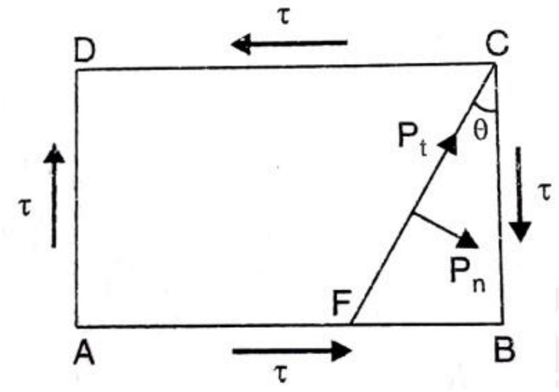
$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 180^o$$

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\times -1$$

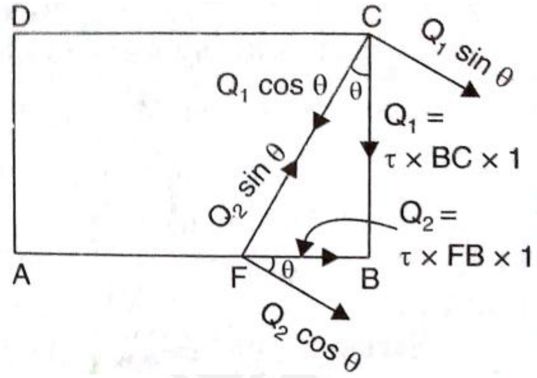
$$= \sigma_1$$

1.5.2.2 A member subjected to simple shear stress.

Fig. shows a rectangular bar ABCD of uniform cross sectional area A and of unit thickness. The bar is subjected to a simple shear stress (τ) across the faces BC and AD. Let FC be the oblique section on which normal and tangential stress are to be calculataed.



θ



= Let

Angle made by oblique section FC with normal cross section BC,

τ = Shear stress across faces BC and AD.

It has already been proved that a shear stress is always accompanied by an equal shear stress at right angles to it. Hence the faces AB and CD will also be subjected to a shear stress τ as shown in above Fig. Now these stress will be converted in to equal forces. Then these forces will be resolved along the inclined surface and normal to the inclined surface. Consider the forces acting on the wedge FBC of fig.

Let Q_1 = shear force on face BC

=Shear stress × Area of face BC

$= \tau \times BC \times 1$ (Since Area of BC = BC ×1)

$= \tau \times BC$

Q_2 = shear force on face FB

=Shear stress × Area of face FB

$= \tau \times FB \times 1$ (Since Area of FB = FB ×1)

$= \tau \times FB$

P_n= Total normal force on section FC

P_t = Total tangential force on section FC

The force Q_1 is acting along face CB as shown in Fig. This force is resolved into two components. i.e., $Q_1\cos\theta$ and $Q_1\sin\theta$ along the plane CF and normal to the plane CF respectively.

The force Q_2 is acting along face FB as shown in Fig. This force is resolved into two components. i.e., $Q_2\sin\theta$ and $Q_2\cos\theta$ along the plane FC and normal to the plane FC respectively.

∴Total normal force of section FC,

$P_n = Q1\sin\theta + Q2\cos\theta$

$= \tau \times BC \sin\theta + \tau \times FB \cos\theta$ (since $Q_1 = \tau \times BC$ and $Q_2 = \tau \times FB$)

And total tangential force on section FC.

$P_t = Q_2\sin\theta - Q_1\cos\theta$ (-ve sign is taken due to opposite direction)

$= \tau \times FB \sin\theta - \tau \times BC \cos\theta$

Let σ_n = Normal stress on section FC

σ_t = Tangential stress on section FC

$$\sigma_n = \frac{Total\ normal\ force\ on\ section\ FC}{Area\ of\ section\ FC}$$

$$= \frac{P_n}{FC \times 1}$$

$$= \frac{\tau \times BC \sin\theta + \tau \times FB \cos\theta}{FC\times 1}$$

$$= \tau\ \frac{BC}{FC}.\sin\theta + \tau\ \frac{FB}{FC}.\cos\theta$$

$$= \tau \cos\theta \sin\theta + \tau \sin\theta \cos\theta$$

$$=2\ \tau \cos\theta \sin\theta$$

$$= \tau \sin 2\theta$$

$$\sigma_t = \frac{Total\ tangential\ force\ on\ section\ FC}{Area\ of\ section\ FC}$$

$$= \frac{P_t}{FC \times 1}$$

$$= \frac{\tau \times FB \sin\theta - \tau \times BC \cos\theta}{FC\times 1}$$

$$= \tau\ \frac{FB}{FC}.\sin\theta - \tau\ \frac{BC}{FC}.\cos\theta$$

$$= \tau \sin\theta \sin\theta - \tau \cos\theta \cos\theta$$

$$= \tau \sin^2\theta - \tau \cos^2\theta$$

$$=-\tau (\cos^2\theta - \sin^2\theta)$$

$$=-\ \tau \cos 2\theta$$

-ve sign shows that σt will be acting downwards on the plane CF.

1.5.2.3 A member subjected to Direct stresses in two mutually perpendicular Directions Accompanied by a simple shear stress.

Above fig. shows a rectangular bar ABCD of uniform cross sectional area A and of unit thickness. The bar is subjected to :

(i) tensile stress σ1 on the face BC and AD

(ii) tensile stress σ2 on the face AB and CD

(iii) a simple shear stress τ on face BC and AD.

The forces acting on the wedge FBC are:

P_1 = Tensile force on face BC due to tensile stress σ1

$= \sigma_1 \times$ Area of BC

$= \sigma_1 \times BC \times 1$

$= \sigma_1 \times BC$

P_2 = Tensile force on face BC due to tensile stress σ2

$= \sigma_2 \times$ Area of FB

$= \sigma_2 \times FB \times 1$

$= \sigma_2 \times FB$

Q_1 = Shear force on the face BC due to shear stress τ

$= \tau \times$ Area of BC

$= \tau \times BC \times 1$

$= \tau \times BC$

Q_2 = Shear force on the face FB due to shear stress τ

$= \tau \times$ Area of FB

$= \tau \times FB \times 1$

$= \tau \times FB$

Resolving the above four forces (i.e., P_1,P_2, Q_1 ,and Q_2) normal to the oblique section FC, we get

Total normal force,

$$P_n = P_1\cos\theta + P_2\sin\theta + Q_1\sin\theta + Q_2\cos\theta$$

Substituting the values of P1,P2,Q1,and Q2, we get

$$P_n = \sigma_1.BC.\cos\theta + \sigma_2.FB.\sin\theta + \tau.BC.\sin\theta + \tau.FB.\cos\theta$$

Similarly, the total tangential force (P_t) is obtained by resolving P_1,P_2,Q_1 and Q_2 along the oblique section FC

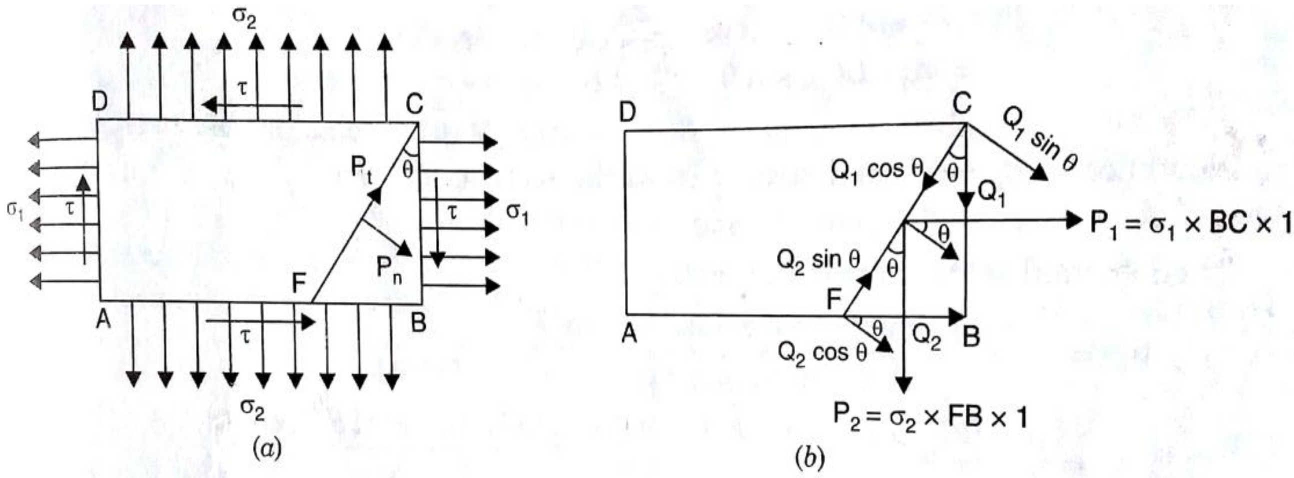
∴Total tangential force,

$$P_t = P_1\sin\theta - P_2\cos\theta - Q_1\cos\theta + Q_2\sin\theta$$

$= \sigma_1.BC \sin\theta - \sigma_2.FB \cos\theta - \tau.BC \cos\theta + \tau .FB \sin\theta$

Now,Let σ_n = Normal stress across the section FC, and

σ_t = Tangential stress across the section FC,



$$\sigma_n = \frac{\text{Total normal force across section FC}}{\text{Area of section FC}}$$

$$= \frac{P_n}{FC \times 1}$$

$$= \frac{\sigma_1 BC.\cos\theta + \sigma_2\, FB.\sin\theta + \tau BC.\sin\theta + \tau FB.\cos\theta}{FC \times 1}$$

$$= \sigma_1 .\frac{BC}{FC}.\cos\theta + \sigma_2 .\frac{FB}{FC}.\sin\theta + \tau .\frac{BC}{FC}.\sin\theta + .\frac{FB}{FC}.\cos\theta$$

$$= \sigma_1 \cos\theta.\cos\theta + \sigma_2 \sin\theta.\sin\theta + \tau\cos\theta.\sin\theta + \tau\sin\theta.\cos\theta$$

$$= \sigma 1 \cos^2\theta + \sigma 2 \sin\theta^2\theta + 2\tau\cos\theta.\sin\theta$$

$$= \sigma_1\left[\frac{1+\cos 2\theta}{2}\right] + \sigma_2\left[\frac{1-\cos 2\theta}{2}\right] + \tau\sin 2\theta$$

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau\sin 2\theta \qquad \ldots\text{(i)}$$

Tangential stress (i.e., shear stress) across the section FC,

$$\sigma_t = \frac{Total\ tangential\ force\ across\ section\ FC}{Area\ of\ section\ FC}$$

$$= \frac{P_t}{FC \times 1}$$

$$= \frac{\sigma_1 BC.sin\theta - \sigma_2\, FB.cos\theta - \tau BC.cos\theta + \tau FB.sin\theta}{FC \times 1}$$

$$= \sigma_1 .\frac{BC}{FC}.\sin\theta - \sigma_2 .\frac{FB}{FC}.\cos\theta - \tau .\frac{BC}{FC}.\cos\theta + .\frac{FB}{FC}.\sin\theta$$

$$= \sigma_1 \cos\theta.\sin\theta - \sigma_2 \sin\theta.\cos\theta - \tau\cos\theta.\cos\theta + \tau\sin\theta.\sin\theta$$

$= (\sigma_1 - \sigma_2).\cos\theta\sin\theta - \tau\cos^2\theta + \tau\sin^2\theta$

$= \frac{\sigma_1-\sigma_2}{2} 2\cos\theta\sin\theta - \tau(\cos^2\theta - \sin^2\theta)$

$= \frac{\sigma_1-\sigma_2}{2}\sin2\theta - \tau\cos2\theta$ …(ii)

Position of principal planes.

The planes on which shear stress (i.e., tangential stress) is zero are known as principal planes. And the stress acting on principal planes are known as principal stresses.

The position of principal planes are obtained by equating the tangential stress to zero

∴ For principal planes, $\sigma_t = 0$

Or $\frac{\sigma_1-\sigma_2}{2}\ sin2\theta - \tau\cos2\theta = 0$

Or $\frac{\sigma_1-\sigma_2}{2} sin2\theta = \tau\cos2\theta$

Or $\frac{sin2\theta}{\cos2\theta} = \frac{\tau}{\frac{\sigma_1-\sigma_2}{2}} = \frac{2\tau}{\sigma_1-\sigma_2}$

Or $tan2\theta = \frac{2\tau}{\sigma_1-\sigma_2}$

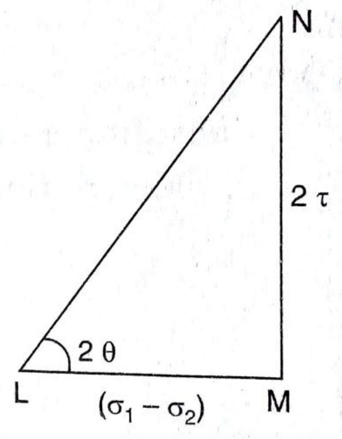
But the tangent of any angle in a right angled triangle

$= \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}} = \frac{2\tau}{\sigma_1-\sigma_2}$

Now diagonal of the right angled triangle

$= \sqrt{(\sigma_1-\sigma_2)^2 + (2\tau)^2}$

$= \pm\sqrt{(\sigma_1-\sigma_2)^2 + 4\tau^2}$



1st case

Diagonal = $\sqrt{(\sigma_1-\sigma_2)^2 + 4\tau^2}$

Then $\sin2\theta = \frac{Height}{Diagonal} = \frac{2\tau}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau^2}}$

And $\cos2\theta = \frac{Base}{Diagonal} = \frac{(\sigma_1-\sigma_2)}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau^2}}$

The value of major principal stress is obtained by substituting the values of sin2θ and cos2θ in equation (i)

∴ Major Principal stress

$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos2\theta + \tau\sin2\theta$

$$= \frac{\sigma_{1+\sigma_2}}{2} + \frac{\sigma_{1-\sigma_2}}{2}\frac{(\sigma_{1-\sigma_2})}{\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}} + \tau\ \frac{2\tau}{\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$$

$$= \frac{\sigma_{1+\sigma_2}}{2} + \frac{1}{2}\frac{(\sigma_{1-\sigma_2})^2}{\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}} + \frac{2\tau^2}{\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$$

$$= \frac{\sigma_{1+\sigma_2}}{2} + \frac{(\sigma_{1-\sigma_2})^2+4\tau^2}{2\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$$

$$= \frac{\sigma_{1+\sigma_2}}{2} + \frac{1}{2}\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}$$

$$= \frac{\sigma_{1+\sigma_2}}{2} + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2} \quad \text{...(iii)}$$

2nd case

Diagonal =- $\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}$

Then $\sin 2\theta = \frac{Height}{Diagonal} = \frac{2\tau}{-\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$

And $\cos 2\theta = \frac{Base}{Diagonal} = \frac{(\sigma_{1-\sigma_2})}{-\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$

Substituting these values in equation (i), we get minor principal stress

∴ Minor principal stress

$$= \frac{\sigma_{1+\sigma_2}}{2} + \frac{\sigma_{1-\sigma_2}}{2}\cos 2\theta + \tau\sin 2\theta$$

$$= \frac{\sigma_{1+\sigma_2}}{2} + \frac{\sigma_{1-\sigma_2}}{2}\frac{(\sigma_{1-\sigma_2})}{-\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}} + \tau\ \frac{2\tau}{-\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$$

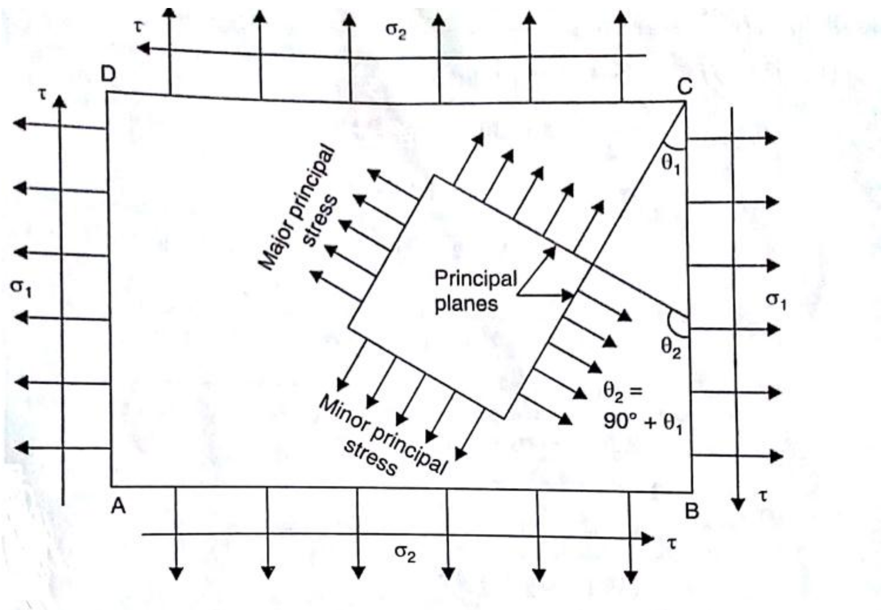
$$= \frac{\sigma_{1+\sigma_2}}{2} - \frac{1}{2}\frac{(\sigma_{1-\sigma_2})^2}{\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}} - \frac{2\tau^2}{\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$$

$$= \frac{\sigma_{1+\sigma_2}}{2} - \frac{(\sigma_{1-\sigma_2})^2+4\tau^2}{2\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}}$$

$$= \frac{\sigma_{1+\sigma_2}}{2} - \frac{1}{2}\sqrt{(\sigma_{1-}\sigma_2)^2+4\tau^2}$$

$$= \frac{\sigma_{1+\sigma_2}}{2} - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2} \quad \text{...(iv)}$$

Equation (iii) gives the maximum principal stress whereas equation (iv) gives the minimum principal stress. The two principal planes are at right angles.



Maximum shear stress.

The shear stress will be maximum or minimum when $\frac{d}{dt}(\sigma_t) = 0$

$$\frac{d}{dt}\left[\frac{\sigma_1-\sigma_2}{2} sin2\theta \;-\; \tau\cos2\theta\right] = 0$$

$$\frac{\sigma_1-\sigma_2}{2}(cos2\theta) \times 2 - \tau(-\sin2\theta)\times 2 = 0$$

$$(\sigma_1-\sigma_2).\cos2\theta + 2\tau(\sin2\theta) = 0$$

$$2\tau(\sin2\theta) = -(\sigma_1-\sigma_2).\cos2\theta$$

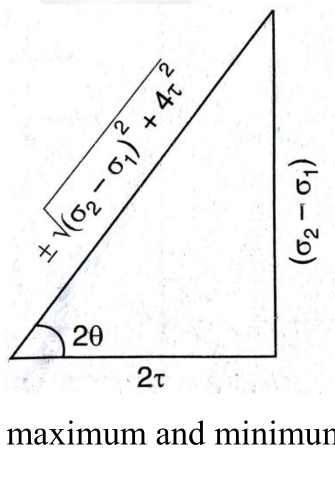
$$= (\sigma_2-\sigma_1).\cos2\theta$$

Or $$\frac{\sin2\theta}{\cos2\theta} = \frac{\sigma_2-\sigma_1}{2\tau}$$

Or $$\tan2\theta = \frac{\sigma_2-\sigma_1}{2\tau} \quad \ldots(v)$$

Equation (v) gives condition for maximum or minimum shear stress.

If $\tan2\theta = \frac{\sigma_2-\sigma_1}{2\tau}$



Then, $\sin2\theta = \frac{Height}{Diagonal} = \pm \frac{\sigma_2-\sigma_1}{\sqrt{(\sigma_2-\sigma_1)^2+4\tau^2}}$

And $\cos2\theta = \frac{Base}{Diagonal} = \pm \frac{2\tau}{\sqrt{(\sigma_2-\sigma_1)^2+4\tau^2}}$

Substituting the values of sin2θ and cos2θ in equation (ii), the maximum and minimum shear stresses are obtained.

$(\sigma_t)\text{max} = \frac{\sigma_1 - \sigma_2}{2} sin2\theta - \tau\cos 2\theta$

$= \pm \frac{\sigma_1 - \sigma_2}{2} \times \frac{\sigma_2 - \sigma_1}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \tau \frac{2\tau}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$

$= \pm \frac{(\sigma_2 - \sigma_1)^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$

$= \pm \frac{(\sigma_2 - \sigma_1)^2 + 4\tau^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$

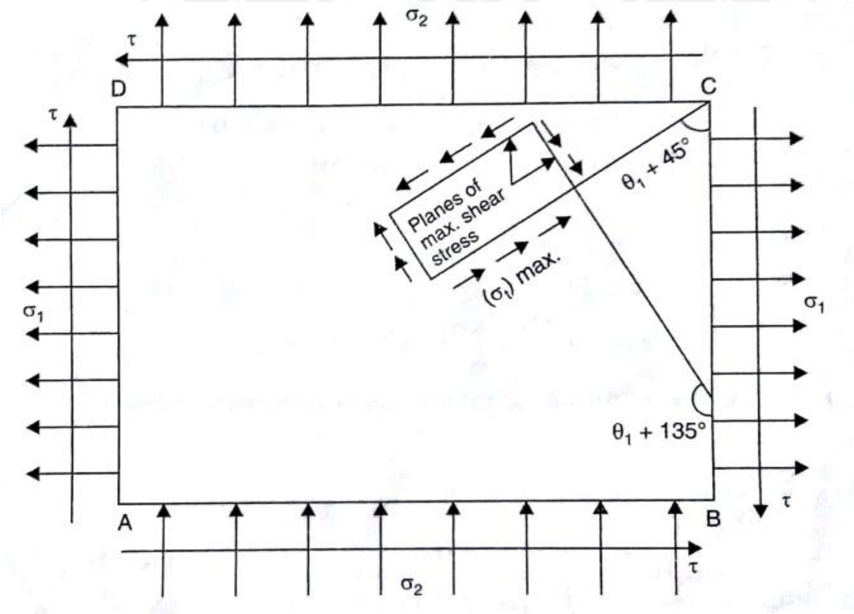
$= \pm \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + (2\tau)^2}$

$(\sigma_t)\text{max} = \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$

$= \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$ …(vi)

The planes on which maximum shear stress is acting, are obtained after finding the two values of θ from equation (v). These two values of θ will differ by 90^o.

The second method of finding the planes of maximum shear stress is to find first principal planes and principal stresses. Let θ_1 is the angle of principal plane with plane BC of Fig. Then the planes of maximum shear stress will be at θ_1+45^o and θ_1+135^o with plane BC as shown in below Fig.



Problem.1.21. The tensile stresses at a point across two mutually perpendicular planes are $120 N/mm^2$ and $60 N/mm^2$. Determine the normal, tangential and resultant stresses on a plane inclined at 30^o to the axis of minor stress

Given Data

Major principal stress, $\sigma_1 = 120 N/mm^2$

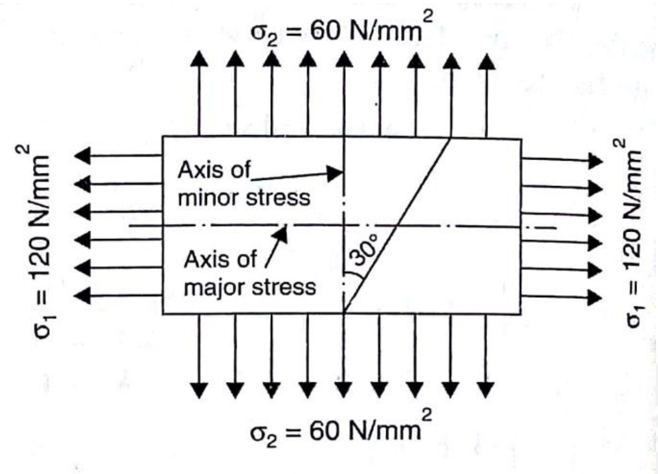
Minor principal stress, $\sigma_2 = 60 N/mm^2$

Angle of oblique plane with the axis of minor principal stress,

$$\theta = 30^o$$

To find

The normal, tangential and resultant stresses.



Normal stress(σ_n)

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta$$

$$= \frac{120+60}{2} + \frac{120-60}{2}\cos 2\times 30^o$$

$$= \mathbf{105 N/mm^2}$$

Tangential stress(σ_t)

$$\sigma_t = \frac{(\sigma_1-\sigma_2)}{2}\sin 2\theta$$

$$= \frac{120-60}{2}\sin 2\times 30 = \mathbf{25.98 N/mm^2}$$

Resultant stress(σ_R)

The resultant stress on the section FC will be given as

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$$

$$= \sqrt{105^2 + 25.98^2} = \mathbf{108.16 N/mm^2}$$

Problem.1.5.4.The stresses at a point in a bar are 200N/mm^2 (tensile) and 100N/mm^2(compressive). Determine the resultant stress in magnitude and direction on a plane inclined at 60^o to the axis of major stress. Also determine the maximum intensity of shear stress in the material at the point.

Given Data

Major principal stress, σ_1 = 200N/mm^2

Minor principal stress, σ_2 = -100N/mm^2 (-ve sign is due to compressive stress)

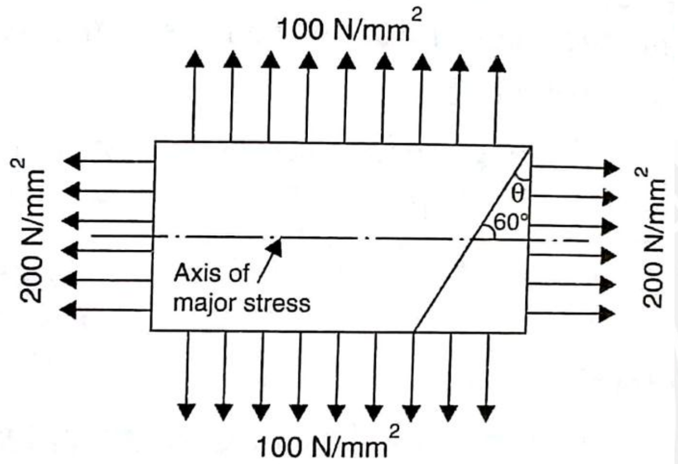
Angle of oblique plane with the axis of minor principal stress,

$\theta = 90^o - 60^o = 30^o$

To find

The Magnitude and direction Resultant stress and maximum intensity of shear stress

Solution



Normal stress(σ_n)

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta$$

$$= \frac{200-100}{2} + \frac{200+100}{2}\cos 2\times 30^o = \mathbf{125 N/mm^2}$$

Tangential stress(σ_t)

$$\sigma_t = \frac{(\sigma_1-\sigma_2)}{2}\sin 2\theta$$

$$= \frac{200+100}{2}\sin 2\times 30 = \mathbf{129.9 N/mm^2}$$

Resultant stress(σ_R)

The resultant stress on the section FC will be given as

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$$

$$= \sqrt{125^2 + 129.9^2} = \mathbf{180.27 N/mm^2}$$

Direction of Resultant stress

$$\tan\emptyset = \frac{\sigma_t}{\sigma_n} = \frac{129.9}{125} = 1.04$$

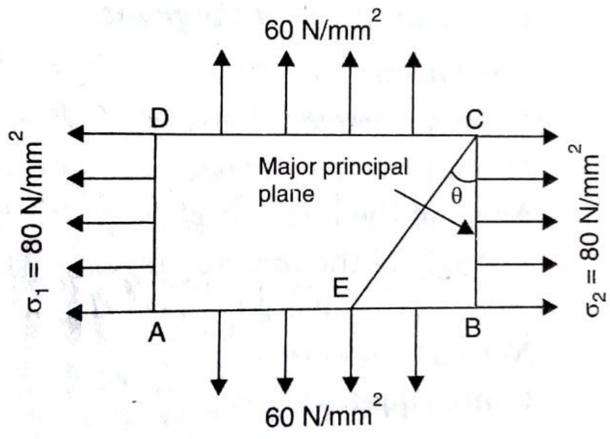
$$\emptyset = \tan^{-1}1.04 = \mathbf{46^o\ 6^1}$$

Maximum Shear stress

$(\sigma_t)_{max} = \frac{(200+100)}{2}$

$= \mathbf{150 N/mm^2}$

Problem.1.5.5. At a point within a body subjected to two mutually perpendicular directions, the stresses are 80 N/mm^2 tensile and 40N/mm^2 tensile. Each of the above stresses is accompanied by a shear stress of 60N/mm^2. Determine the normal stress, shear stress and resultant stress on an oblique plane inclined at an angle of 45^o with the axis of minor tensile stress.



Given Data

Major principal stress, $\sigma_1 = 80N/mm^2$

Minor principal stress, $\sigma_2 = 40N/mm^2$

Shear stress $\tau = 60N/mm^2$

Angle of oblique plane with the axis of minor principal stress, $\theta = 45^o$

To find

The normal, tangential and resultant stresses.

Solution.

Normal stress(σ_n)

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{80+40}{2} + \frac{80-40}{2}\cos(2\times45) + 60\times\sin 2\times45$$

$= \mathbf{120N/mm^2}$

Tangential stress(σ_t)

$$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta - \tau\cos 2\theta$$

$$= \frac{80-40}{2}\sin(2\times45) - 60\times\cos(2\times45)$$

$= \mathbf{20N/mm^2}$**.**

Resultant stress(σ_R)

The resultant stress on the section FC will be given as

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$$

$= \sqrt{120^2 + 20^2}$ = $\mathbf{121.655N/mm^2}$

Problem1.5.6.A rectangular block of material is subjected to a tensile stress of 110N/mm^2 on one plane and a tensile stress of 47N/mm^2 on the plane right angles on the former. Each of the above stresses is accompanied by a shear stress of 63N/mm^2 and that associated with the former tensile stress tends to rotate the block anticlockwise. Find (i) the direction and magnitude of each of the principal stress and (ii) magnitude of the greatest shear stress.

Given Data

Major principal stress, σ_1 = 110N/mm^2

Minor principal stress, σ_2 = 47N/mm^2

Shear stress τ = 63N/mm^2

To find

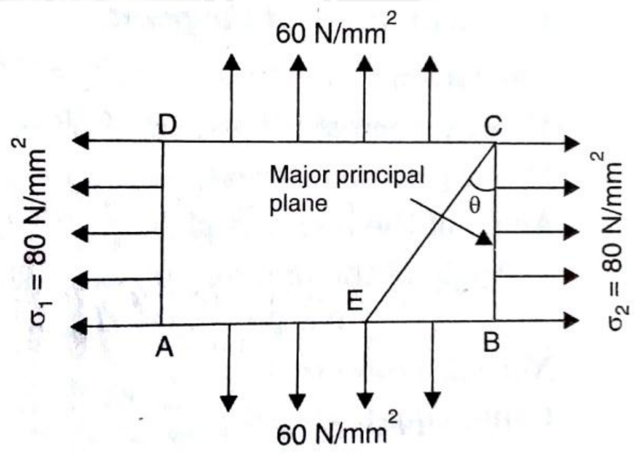
(i) The direction and magnitude of each of the principal stress and

(ii) The magnitude of the greatest shear stress.

Solution

(i)Direction and Magnitude of principal stresses.

Major Principal stress



$$= \frac{\sigma_1+\sigma_2}{2} + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{110+47}{2} + \sqrt{\left(\frac{110-47}{2}\right)^2 + 63^2}$$

= **148.936N/mm^2.**

Minor Principal stress

$$= \frac{\sigma_1+\sigma_2}{2} - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{110+47}{2} - \sqrt{\left(\frac{110-47}{2}\right)^2 + 63^2}$$ = **8.064N/mm^2**

Direction of Resultant stress

$$\tan 2\emptyset = \frac{2\tau}{\sigma_1-\sigma_2} = \frac{2\times 63}{(110-47)} = 2$$

$2\emptyset = \tan^{-1}2 = 63^o\, 26^1$ or $243^o\, 26^1$

Or $\emptyset$ = **$31^o\, 43^1$ or $121^o\, 26^1$**

(ii)Magnitude of greatest shear stress

$$(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(110-47)^2 + 4\times 63^2}$$ = **70.436N/mm^2**