1.3 MOMENTS

Definition:

The rth moment about origin is $\mu_r' = E[\, x_r']$

First moment about origin $\mu_1' = E[X]$

= $E(X^2) - [E(X)]^2$

Variance $\sigma^2 = \mu_2' - (\mu_1')^2$

The rth moment about mean is $\mu_r = E[(X-\mu)^r], where\ \mu\ is\ mean\ of\ X.$

$$\mu_1 = E[(X-\mu)^1]$$

$$= E[X] - E[\mu] = \mu - \mu = 0$$

$$\therefore \mu_1 = 0$$

$$\mu_2 = E[(X-\mu)^2]$$

$$= E[X^2 + \mu^2 - 2X\mu]$$

$$= E[X^2] + \mu^2 - 2E[X]\mu$$

$$= E(X^2) + [E(X)]^2 - 2E(X)E(X)$$

$$= E(X^2) + [E(X)]^2 - 2[E(X)]^2$$

$$= E(X^2) - [E(X)]^2 = \sigma^2$$

$$\therefore \mu_2 = \sigma^2$$

If the probability density of X is given $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases}$

Find its r[th] **moment about origin. Hence find evaluate** $E[(2X+1)^2]$

Solution:

The r[th] moment about origin is given by

$$\mu_r' = E[\, x_r'] = \int_0^1 x^r f(x) dx$$

$$= \int_0^1 x^r\, 2(1-x) dx$$

$$= 2\int_0^1 (x^r - x^{r+1}) dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+1+1}}{r+2}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+2)(r+1)}\right] = \frac{2}{r^2+3r+2}$$

$$E[(2X+1)^2] = E[4X^2 + 4X + 1]$$

$$= 4E[X^2] + 4E[X] + 1$$

$$= 4\mu_2' + 4\mu_1' + 1$$

$$= 4\frac{2}{2^2+3(2)+2} + 4\frac{2}{2^2+3(2)+2} + 1$$

$$= \frac{8}{12} + \frac{8}{6} + 1 = 3$$

$$\therefore E[(2X+1)^2] = 3$$

1.4 MOMENT GENERATIVG FUNCTION (MGF)

Let X be a random variable. Then the MGF of X is $M_X(t) = E[e^{tx}]$

If X is a discrete random variable, then the MGF is given by

$$M_X(t) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable, then the MGF is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Define MGF and why it is called so?

Sol: Let X be a random variable. Then the MGF of X is $M_x(t) = E[e^{tX}]$ Let X be a continuous random variable. Then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} \left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots \frac{t^r x^r}{r!} + \cdots\right] f(x)dx$$

$$= \int_{-\infty}^{\infty} \left[f(x) + \frac{tx}{1!} f(x) + \frac{t^2x^2}{2!} f(x) + \cdots \frac{t^r x^r}{r!} f(x) + \cdots\right] dx$$

$$= \int_{-\infty}^{\infty} f(x)dx + \frac{t}{1!}\int_{-\infty}^{\infty} x f(x)dx + \frac{t^2}{2!}\int_{-\infty}^{\infty} x^2 f(x)dx \ldots + \frac{t^r}{r!}\int_{-\infty}^{\infty} x^r f(x)dx + \cdots$$

$$M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots \frac{t^r}{r!}\mu_r' + \cdots \ldots \ldots$$

$\because M_X(t)$ generates moments therefore it is moment generation function

NOTE

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=0}$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

PROBLEMS UNDER MGF OF DISCRETE RANDOM VARIABLE

$$M_X(t) = \sum_x e^{tx} p(x)$$

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=}$

1. **Let X be the number occur when a die is thrown. Find the MGF and hence find Mean and Variance of X.**

   Solution:

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

(i) $M_X(t) = \sum_{x=1}^{6} e^{tx} p(x)$

$= e^t P(1) + e^{2t} P(2) + e^{3t} P(3) + e^{4t} P(4) + e^{5t} P(5) + e^{6t} P(6)$

$= e^t \frac{1}{6} + e^{2t} \frac{1}{6} + e^{3t} \frac{1}{6} + e^{4t} \frac{1}{6} + e^{5t} \frac{1}{6} + e^{6t} \frac{1}{6}$

$M_X(t) = \frac{1}{6}[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$

$= \frac{1}{6}[1 + 2 + 3 + 4 + 5 + 6] = \frac{21}{6}$

$E(X) = 3.5$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

=15.1

(iii) Variance of $X = E(X^2) - [E(X)]^2 = 15.1 - 12.25$

$\sigma_X = 2.85$

2. Find the moment generating function for the distribution

where $(X = x) = \begin{cases} \frac{2}{3}; \boldsymbol{x = 1} \\ \frac{1}{3}; \boldsymbol{x = 2} \\ \mathbf{0} \text{ ; otherwise} \end{cases}$ **. Also find its mean & variance,**

Sol: The probability distribution of X is given by

x	1	2
$p(x)$	$2/3$	$1/3$

$M_X(t) = E[e^{tx}] \qquad = \sum_{x=1}^{2} e^{tx}p(x)$

$$= e^t p(X = 1) + e^{2t} p(X = 2) = e^t \frac{2}{3} + e^{2t} \frac{1}{3}$$

$$M_X(t) = \frac{1}{3}(2e^t + e^{2t})$$

$$E(X) = M_X'(0)$$

$$= \left[\frac{d}{dt}\left[\frac{1}{3}(2e^t + e^{2t})\right]\right]_{t=0} \qquad = \frac{1}{3}(2e^t + 2e^{2t})$$

E(X)= $\frac{4}{3}$

$$E(X^2) = M_X''(0) = \left[\frac{d^2}{dt^2}\left[\frac{1}{3}(2e^t + e^{2t})\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[\frac{1}{3}(2e^t + 2e^{2t})\right]\right]_{t=0}$$

$$= \left[\frac{1}{3}(2e^t + 4e^{2t})_{t=0}\right] = \frac{6}{3} = 2$$

Variance of $X = E(X^2) - [E(X)]^2 = 2 - \left(\frac{4}{3}\right)^2$

$$\mathrm{Var}(X) = \frac{2}{9}$$

3. Let X be a RV with PMF $P(x) = \left(\frac{1}{2}\right)^x$; $x = 1, 2, 3, \ldots$. Find MGF and hence find mean and variance of X.

Sol:

(i) $M_X(t) = E[e^{tX}]$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx}\left(\frac{1}{2}\right)^x \qquad = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$$

$$= \left[\frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right] \qquad = \frac{e^t}{2}\left(1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right)$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} \qquad = \frac{e^t}{2}\frac{e^t}{2-e^t}$$

$$M_X(t) = \frac{e^t}{2-e^t}$$

$$(ii)\ E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)e^t - e^t(0-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2}\right]_{t=0} \quad \because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \left[\left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}\right] = \frac{2}{1}$$

$$E(X) = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{2e^t}{(2-e^t)^2}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 2e^t - 2e^t 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0} = \frac{2+4}{1} = 6$$

(iii) Variance $= E(X^2) - [E(X)]^2 = 6 - 4$

$$\text{Var}(X) = 2$$

PROBLEMS UNDER MGF OF DISCRETE RANDOM VARIABLE

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

1. If a random variable "X" has the MGF, $M_X(t) = \frac{2}{2-t}$, find the variance of X.

Solution:

Given $M_X(t) = \frac{2}{2-t} = 2(2-t)^{-1}$

$$M_X'(t) = -2(2-t)^{-2}(-1)$$

$$= 2(2-t)^{-2}$$

$$M_X'(t=0) = 2(2-0)^{-2} = \frac{2}{4} = \frac{1}{2}$$

$$M_X''(t) = -4(2-t)^{-3}(-1)$$

$$= 4(2-t)^{-3}$$

$$M_X''(t=0) = 4(2-0)^{-3} = \frac{4}{8} = \frac{1}{2}$$

$$Var(X) = E(X^2) - E[(X)]^2$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

2. **Let X be a RV with PDF $f(x) = ke^{-2x}, x \geq 0$. Find (i) k, (ii) MGF, (iii) Mean and (iv) variance**

Sol: Given: $f(x) = ke^{-2x}; 0 \leq x < \infty$

i) To find k:

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty ke^{-2x}dx = 1k\left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1 \Rightarrow \frac{k}{-2}(e^{-\infty} - 1)$$

$$= 1$$

$$\frac{k}{-2}(0-1) = 1 \Rightarrow \qquad \frac{k}{2} = 1$$

$$k = 2$$

(ii) $M_X(t) = E[e^{tx}]$ $\qquad = \int_0^\infty e^{tx} f(x)dx$

$$= 2\int_0^\infty e^{tx}e^{-2x}dx = 2\int_0^\infty e^{tx-2x}dx$$

$$= 2\int_0^\infty e^{-(2-t)x}dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^\infty = 2\left(0+\frac{1}{2-t}\right)$$

$$M_X(t) = \frac{2}{2-t}$$

(iii) To find Mean and Variance:

$$M_X(t) = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \cdots$$

Coefficient of $t = \frac{1}{2}$ Coefficient of $t^2 = \frac{1}{2^2}$

Mean $E(X) = \mu_1' = 1! \times$ coefficient of $t \Rightarrow E(X) = \frac{1}{2}$

$$E(X^2) = 2! \times \text{coefficient of } t^2 = 2 \times \frac{1}{2^2} = \frac{1}{2}$$

(iv) Variance $= E(X^2) - [E(X)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4}$

$$\operatorname{Var}(X) = \frac{1}{4}$$

3. Let X be a continuous RV with PDF $f(x) = Ae^{\frac{-x}{3}}; x \geq 0$. Find (i) A, (ii) MGF,(iii) Mean and (iv) variance

Sol: Given: $f(x) = Ae^{\frac{-x}{3}}; 0 \leq x \leq \infty$

(i) To find A :

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty Ae^{\frac{-x}{3}}dx = 1A\left[\frac{e^{\frac{-x}{3}}}{-\frac{1}{3}}\right]_0^\infty = 1 \Rightarrow -3A(0-1) = 1$$

$$3A = 1 \Rightarrow A = \frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3}e^{\frac{-x}{3}}; 0 \leq x \leq \infty$$

(ii) $M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx}f(x)dx = \frac{1}{3}\int_0^\infty e^{tx}e^{\frac{-x}{3}}dx$

$$= \frac{1}{3}\int_0^\infty e^{tx-\frac{x}{3}}dx = \frac{1}{3}\int_0^\infty e^{-\left(\frac{1}{3}-t\right)x}dx = \frac{1}{3}\left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)}\right]_0^\infty$$

$$= \frac{1}{3}\left[0+\frac{1}{\frac{1}{3}-t}\right] = \frac{1}{3}\frac{1}{\frac{1-3t}{3}}$$

$$= (1-3t)^{-1}$$

(iii) To find mean and variance:

$M_X(t) = (1-3t)^{-1}$

$= 1 + 3t + 9t^2 + 27t^3 + \cdots$

coefficient of $t = 3$ coefficient of $t^2 = 9$

E(X)=1! X coefficient of t in $M_X(t) = 1$ X 3

Mean = 3

E(X)=2! X coefficient of t^2 in $M_X(t)$

= 2X9 =18

(iv)Variance = $E(X^2) - [E(X)]^2$ = 18-9

Var (X) =9

4. Let X be a continuous RV with PDF $f(x)$

$$= \begin{cases} x & ; 0 < x < 1 \\ 2-x & ; 1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}$$ **. Find (i) MGF, (ii) Mean and variance.**

Sol: Since X is defined in the region $0 < x < 2, X$ is a continuous RV.

$M_X(t) = E[e^{tX}] = \int_0^2 e^{tx} f(x)dx$

$$= \int_0^1 e^{tx}x dx + \int_1^2 e^{tx}(2-x)dx = \int_0^1 x e^{tx}dx + \int_1^2 (2-x)e^{tx}dx$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) - 1\left(\frac{e^{tx}}{t^2}\right)\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[1\left(\frac{e^t}{t}\right) - 1\left(\frac{e^t}{t^2}\right) - \left(\frac{-1}{t^2}\right)\right] + \left[0 + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}\right]$$

$$= \left[\frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}\right] = \frac{1}{t^2} - \frac{2e^t}{t^2} + \frac{e^{2t}}{t^2}$$

$$M_X(t) = \frac{1 - 2e^t + e^{2t}}{t^2}$$

To find Mean and Variance:

$$M_X(t) = \frac{1}{t^2}\left[1 - 2\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right.$$
$$\left. + \left(1 + \frac{2t}{1!} + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \frac{2^4t^4}{4!} + \cdots\right)\right]$$

$$\mu_r' = r! \times \text{coefficient of } t^r$$

$$\text{Coefficient of } t = -\frac{2}{3!} + \frac{2^3}{31} = \frac{-2}{6} + \frac{8}{6} = 1$$

$$\text{Coefficient of } t^2 = -\frac{2}{4!} + \frac{2^4}{4!} = \frac{14}{24} = \frac{7}{12}$$

$$\mu_1' = 1! \times \text{coefficient of } t$$

$$\mu_1' = 1$$

Mean = 1

$$\mu_2' = 2! \times \text{coefficient of } t^2; \mu_2' = 2 \times \frac{7}{12} = \frac{7}{6}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{7}{6} - 1$$

$$\text{Var(X)} = \frac{1}{6}$$

5. **Let X be a continuous random variable with PDF $f(x)\frac{1}{2a}; -a < x < a$. Then find the M.G.F of X.**

Sol: X is a continuous random variable defined in $-a < x < a$.

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-a}^{a} e^{tx} f(x)dx$$

$$= \int_{-a}^{a} e^{tx} \frac{1}{2a} dx$$

$$= \frac{1}{2a}\left(\frac{e^{tx}}{t}\right)^{a} e^{x} - e^{-x} = 2\sin hx$$

$$= \frac{1}{2a}\left(\frac{e^{ta}-e^{-ta}}{t}\right)$$

$$= \frac{1}{2a}\frac{2\sinh at}{t}$$

$$M_X(t) = \frac{\sinh at}{at}$$

Discrete Distributions:

(i) Binomial Distribution

(ii) Poisson Distribution

(iii) Geometric Distribution

Continuous Distributions:

(i) Exponential Distribution

(ii) Uniform Distribution

(iii) Normal Distribution

1.5 Binomial Distribution

Let us consider "n" independent trails. If the successes (S) and failures (F) are recorded successively as the trials are repeated we get a result of the type

S S F F S . . . F S

Let "x" be the number of success and hence we have $(n - x)$ number of failures.

$$P(S\ S\ F\ F\ S \ldots F\ S) = P(S)\ P(S)\ P(F)\ P(F)\ P(S) \ldots P(F)\ P(S)$$

$$= p\ p\ q\ q\ p \ldots q\ p$$

$$= p\ p \ldots p \times q\ q\ q \ldots q$$

$$= x \text{ factor} \times (n - x) \text{ factors}$$

$$= p^x \cdot q^{n-x}$$

But "x" success in "n" trials can occur in nC_x ways.

Therefore the probability of "x" successes in "n" trials is given by

$$P(X = x\, successes) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots, n$$

Where $p + q = 1$

Assumptions in Binomial Distribution:

(i) There are only two possible outcomes for each trial (success or failure)

(ii) The probability of a success is the same for each trail.

(iii) There are "n" trials where "n" is constant.

(iv) The "n" trails are independent.

Mean and variance of a Binomial Distribution:

(i) Mean(μ) = np

(ii) Variance(σ^2) = npq

The variance of a Binomial Variable is always less than its mean.

$\therefore npq < np$.

Find the moment generating function of binomial distribution and hence find the mean and variance.

Sol: Binomial distribution is $p(x) = nC_x p^x q^{n-x},\ x = 0,1,2, \ldots \ldots, n$

To find Mean and Variance:

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$ $\qquad \because \sum_{x=0}^{n} nC_x a^x b^{n-x} = (a+b)^n$

$M_X(t) = (pe^t + q)^n$

Mean $E(X) = \left[\frac{d}{dt}[M_*(t)]\right]_{t=0} = \left[\frac{d}{dt}[(pe^t+q)^n]\right]_{t=0}$

$= [n(pe^t+q)^{n-1}(pe^t+0)]_{t=0}$

$= np[p+q]^{n-1}$

$E(X) = np$

$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$

$= \left[\frac{d}{dt}[n(pe^t+q)^{n-1}(pe^t)]\right]_{t=0} = np\left[\frac{d}{dt}[(pe^t+q)^{(n-1)}e^t]\right]_{t=0}$

$= np[(pe^t+q)^{n-1}e^t + e^t(n-1)(pe^t+q)^{n-2}pe^t]_{t=0}$

$= np[(p+q)^{n-1} + (n-1)(p+q)^{n-2}p]$

$= np[1+(n-1)p] \qquad = np[1+np-p]$

$= np[1-p+np] \qquad = np[q+np] \qquad = npq + n^2p^2$

$E(X^2) = (np)^2 + npq$

Variance $= E(X^2) - [E(X)]^2$

$= (np)^2 + npq - (np)^2 = n^2p^2 - n^2p^2 + npq$

Variance= $n$pq

Problems based on Binomial Distribution:

Mean = np

Variance = npq

1. Criticize the following statements " The mean of a binomial distribution is 5 and the standard deviation is 3"

Solution:

Given mean = np = 5 . . . (1)

Standard deviation = $\sqrt{npq} = 3$

⇒ Variance = npq = 9 . . . (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{9}{5} = 1.8 > 1$$

Which is impossible. Hence, the given statement is wrong.

2. If $M_X(t) = \frac{(2e^t+1)^4}{81}$, then find Mean and Variance.

Solution:

Given $M_X(t) = \frac{(2e^t+1)^4}{81}$

$\Rightarrow M_X(t) = \left(\frac{2}{3}e^t + \frac{1}{3}\right)^4$

Comparing with MGF of Binomial Distribution, $M_X(t) = (pe^t + q)^n$, we get

$p = \frac{2}{3}$ and $= \frac{1}{3}$, n = 4

(i) Mean = $np \ = 4 \times \frac{2}{3} = \frac{8}{3}$

(ii) Variance = $npq = \frac{8}{3} \times \frac{1}{3} = \frac{8}{9}$

3. Six dice are thrown 729 times. How many times do you expect atleast 3 dice to show a five or six.

Solution:

Given n = 6 and N = 729

Probability of getting (5 or 6) $p = \frac{2}{6} = \frac{1}{3}$

and $q = 1 - \frac{1}{3} = \frac{2}{3}$

$$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots, n$$

$$= 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}, x = 0, 1, 2, \ldots, 6$$

P(atleast 3 dice to show a five or six) $= P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{6-0} + 6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{6-1} + 6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{6-2}\right]$$

$$= 1 - [0.0877 + 0.2634 + 0.3292]$$

$$= 1 - 0.6803$$

$$= 0.3197$$

Number of times expecting atleast 3 dice to show 5 or 6 $= 729 \times 0.3197$

$= 233$ times

4. A machine manufacturing screw is known to produce 5% defective. In a random sample of 15 screws, what is the probability that there are (i) exactly 3 defectives (ii) not more than 3 defectives.

Solution:

Given n = 15

$p = 5\% = 0.05$

$q = 1 - p = 1 - 0.05 = 0.95$

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \dots, n$

$$= 15C_x (0.05)^x (0.95)^{15-x}, x = 0, 1, 2, \dots, 15$$

(i) P(exactly 3 defectives) = $P(X = 3)$

$$= 15C_3 (0.05)^3 (0.95)^{15-3}$$

$$= 0.056(0.95)^{12}$$

$$= 0.0307$$

(ii) P(no More than 3 defectives) = $P(X \leq 3)$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= 15C_0(0.05)^0(0.95)^{15-0} + 15C_1(0.05)^1(0.95)^{15-1}$$

$$+ 15C_3(0.05)^2(0.95)^{15-2}$$

$$+15C_3(0.05)^3(0.95)^{15-3}$$

$$= 15C_0(0.05)^0(0.95)^{15} + 15C_1(0.05)^1(0.95)^{14} + 15C_3(0.05)^2(0.95)^{13}$$

$$+15C_3(0.05)^3(0.95)^{12}$$

$$= 0.4633 + 0.3658 + 0.1348 + 0.0307$$

$$= 0.9946$$