

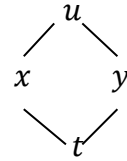
Total Derivative

Differentiation of composite functions

Type-I

If $u = f(x, y)$ is a function of x and y where $x = f(t)$ and $y = g(t)$ then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \dots (1)$$



In the differential form, (i) can be written as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

du is called the total differential of u .

Example:

If $u = x^2 y^3$, where $x = \log t$, $y = e^t$ find $\frac{du}{dt}$

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \dots (1)$$

given $u = x^2 y^3$, $x = \log t$, $y = e^t$

$$\frac{\partial u}{\partial x} = 2xy^3, \quad \frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = e^t$$

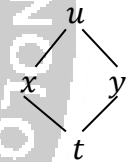
$$\frac{\partial u}{\partial y} = 3x^2 y^2$$

$$(1) \Rightarrow \frac{du}{dt} = 2xy^3 \cdot \frac{1}{t} + 3x^2 y^2 \cdot e^t$$

$$\begin{aligned} \text{Put } x &= \log t, y = e^t \\ &= 2 \log t (e^t)^3 \cdot \frac{1}{t} + 3 (\log t)^2 (e^t)^2 e^t \end{aligned}$$

$$= \frac{2 \log t \cdot e^{3t}}{t} + 3 (\log t)^2 e^{3t}$$

$$\frac{du}{dt} = e^{3t} \log t \left(\frac{2}{t} + 3 \log t \right)$$



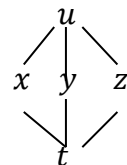
Example:

If $u = xy + yz + zx$, where $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$ find $\frac{du}{dt}$

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} \dots (1)$$

Given $u = xy + yz + zx$, $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$



$$\frac{\partial u}{\partial x} = y + z, \quad \frac{dx}{dt} = -\frac{1}{t^2}, \quad \frac{dy}{dt} = e^t, \quad \frac{dz}{dt} = -e^{-t}$$

$$\frac{\partial u}{\partial y} = x + z$$

$$\frac{\partial u}{\partial z} = y + x$$

$$(1) \Rightarrow \frac{du}{dt} = (y + z) \left(-\frac{1}{t^2} \right) + (x + z)e^t + (y + x)(-e^{-t})$$

$$\text{Put } x = \frac{1}{t}, y = e^t, z = e^{-t}$$

$$= (e^t + e^{-t}) \left(-\frac{1}{t^2} \right) + \left(\frac{1}{t} + e^{-t} \right) e^t + \left(e^t + \frac{1}{t} \right) (-e^{-t})$$

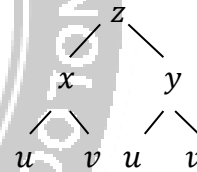
$$\frac{du}{dt} = \frac{-(e^t + e^{-t})}{t^2} + \left(\frac{1}{t} + e^{-t} \right) e^t + \left(e^t + \frac{1}{t} \right) (-e^{-t})$$

Type-II

If $z = f(x, y)$ is a function of x and y where x and y are the functions of u and v such that

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$



Example:

If $g(x, y) = \psi(u, v)$ where $u = x^2 - y^2$ and $v = 2xy$ then prove that

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2 + y^2) \left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} \right)$$

Solution:

$$\text{Given } u = x^2 - y^2, v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2y$$

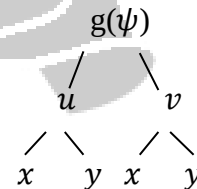
$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v} \frac{\partial v}{\partial x} \dots (1)$$

$$\frac{\partial g}{\partial y} = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \psi}{\partial v} \frac{\partial v}{\partial y} \dots (2)$$

$$(1) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u} 2x + \frac{\partial \psi}{\partial v} 2y$$

$$\frac{\partial g}{\partial x} = 2 \left(x \frac{\partial \psi}{\partial u} + y \frac{\partial \psi}{\partial v} \right)$$



$$\begin{aligned}\frac{\partial^2 g}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial x} \right) = 2 \left(x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v} \right) 2 \left(x \frac{\partial \psi}{\partial u} + y \frac{\partial \psi}{\partial v} \right) \\ &= 4 \left[x^2 \frac{\partial^2 \psi}{\partial u^2} + 2xy \frac{\partial^2 \psi}{\partial u \partial v} + y^2 \frac{\partial^2 \psi}{\partial v^2} \right] \dots (3)\end{aligned}$$

$$(2) \Rightarrow \frac{\partial g}{\partial y} = \frac{\partial \psi}{\partial u} (-2y) + \frac{\partial \psi}{\partial v} 2x$$

$$\frac{\partial g}{\partial y} = 2 \left(-y \frac{\partial \psi}{\partial u} + x \frac{\partial \psi}{\partial v} \right)$$

$$\begin{aligned}\frac{\partial^2 g}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial g}{\partial y} \right) = 2 \left(-y \frac{\partial}{\partial u} + x \frac{\partial}{\partial v} \right) 2 \left(-y \frac{\partial \psi}{\partial u} + x \frac{\partial \psi}{\partial v} \right) \\ &= 4 \left[y^2 \frac{\partial^2 \psi}{\partial u^2} - 2xy \frac{\partial^2 \psi}{\partial u \partial v} + x^2 \frac{\partial^2 \psi}{\partial v^2} \right] \dots (4)\end{aligned}$$

$$\begin{aligned}(3) + (4) \Rightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} &= 4 \left\{ (x^2 + y^2) \frac{\partial^2 \psi}{\partial u^2} + (x^2 + y^2) \frac{\partial^2 \psi}{\partial v^2} \right\} \\ &= 4(x^2 + y^2) \left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} \right)\end{aligned}$$

Hence proved.

Example:

If f is a function of x and y then $x = e^u \sin v$ and $y = e^u \cos v$, show that

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (x^2 + y^2) \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = e^{2u} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \quad (\text{OR})$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{-2u} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right)$$

Solution:

$$\text{Given } x = e^u \sin v, y = e^u \cos v$$

$$\frac{\partial x}{\partial u} = e^u \sin v, \quad \frac{\partial y}{\partial u} = e^u \cos v$$

$$\frac{\partial x}{\partial v} = e^u \cos v, \quad \frac{\partial y}{\partial v} = -e^u \sin v$$

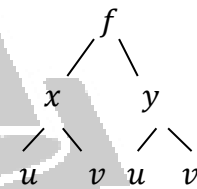
$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \dots (1)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \dots (2)$$

$$(1) \Rightarrow \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} e^u \sin v + \frac{\partial f}{\partial y} e^u \cos v$$

$$\frac{\partial f}{\partial u} = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial u^2} &= \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial u} \right) = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \\ &= \left[x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} \right] \dots (3)\end{aligned}$$



$$(2) \Rightarrow \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} e^u \cos v - \frac{\partial f}{\partial y} e^u \sin v$$

$$\frac{\partial f}{\partial v} = y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial v^2} &= \frac{\partial}{\partial v} \left(\frac{\partial f}{\partial v} \right) = \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \left(y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} \right) \\ &= \left[y^2 \frac{\partial^2 f}{\partial x^2} - 2xy \frac{\partial^2 f}{\partial x \partial y} + x^2 \frac{\partial^2 f}{\partial y^2} \right] \dots (4) \end{aligned}$$

$$\begin{aligned} (3) + (4) \Rightarrow \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} &= (x^2 + y^2) \frac{\partial^2 f}{\partial x^2} + (x^2 + y^2) \frac{\partial^2 f}{\partial y^2} \\ &= (x^2 + y^2) \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \quad [\because x^2 = e^{2u} \sin^2 v, y^2 = e^{2u} \cos^2 v] \\ &= e^{2u} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \\ \therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= e^{-2u} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right) \end{aligned}$$

Example:

If $u = e^x \sin y$ where $x = st^2$ and $y = s^2 t$ find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$

Solution:

$$\text{Given } u = e^x \sin y, \quad x = st^2, y = s^2 t$$

$$\frac{\partial u}{\partial x} = e^x \sin y, \quad \frac{\partial x}{\partial t} = 2st$$

$$\frac{\partial u}{\partial y} = e^x \cos y, \quad \frac{\partial y}{\partial t} = s^2$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \dots (1)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \dots (2)$$

$$(1) \Rightarrow \frac{\partial u}{\partial s} = e^x \sin y \times t^2 + e^x \cos y \times 2st$$

$$(2) \Rightarrow \frac{\partial u}{\partial t} = e^x \sin y \times 2st + e^x \cos y \times s^2$$

