1.9 Exponential Distribution

The definition of exponential distribution is the probability distribution of the time between the events in a Poisson Process.

If you think about it, the amount of time until the event occurs means during the waiting period, not a single event has happened.

This is, in other words Poisson (X = 0).

$$\text{Poisson} \quad P(X=k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$$

$$\text{"} \quad P(X=0) = \frac{\lambda^0 \cdot e^{-\lambda}}{0!} = e^{-\lambda}$$

Why did we have to invent Exponential Distribution?

To predict the amount of waiting time until the next event (i.e., success, failure, arrival, etc.).

For example, we want to predict the following:

- The amount of time until the customer finishes browsing and actually purchases something in your store (success).
- The amount of time until the hardware on AWS EC2 fails (failure).
- The amount of time you need to wait until the bus arrives (arrival).

Relationship between a Poisson and an Exponential Distribution:

If the number of events per unit time follows a Poisson distribution, then the amount of time between events follows the exponential distribution. Assuming that the time between events is not affected by the times between previous events (i.e., they are independent), then the number of events per unit time follows a Poisson distribution with the rate $\lambda = 1/\mu$.

Who else has Memoryless property?

The exponential distribution is the only continuous distribution that is memoryless (or with a constant failure rate). Geometric distribution, its discrete counterpart, is the only discrete distribution that is memory less.

Find the MGF of Exponential distribution and hence find Mean and variance.

Sol. Let X follows the exponential distribution.

By definition, $f(x) = \lambda e^{-\lambda x}; x \geq 0$

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-t)x}dx = \lambda\left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^\infty$$

$$= \frac{-\lambda}{\lambda-t}[e^{-\infty} - e^0] = \frac{-\lambda}{\lambda-t}[0-1]$$

$$= \frac{\lambda}{\lambda-t}$$

To find the mean and variance:

$$M_X(t) = \frac{\lambda}{\lambda\left(1-\frac{t}{\lambda}\right)} = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots$$

Coefficient of $t = \frac{1}{\lambda}$; Coefficient of $t^2 = \frac{1}{\lambda^2}$

$$E(X) = 1! \times \text{ coefficient of } t = \frac{1}{\lambda}$$

$$E(X^2) = 2! \times \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$$

Variance $= E(X^2) - [E(X)]^2$

$$= \frac{2}{\lambda^2} - \left[\frac{1}{\lambda}\right]^2$$

Variance $= \frac{1}{\lambda^2}$

Problems based on Exponential Distribution:

1. The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$ (i) What is the probability that the required time exceeds 2 hours. (ii) What is the conditional probability that the repair takes at least 11 hours given that its duration exceeds 8 hours.

Solution:

Given X is exponentially distributed with parameter $\lambda = \frac{1}{2}$

The exponential distribution is given by $f(x) = \lambda e^{-\lambda x};\ \ x \geq 0$

$$f(x) = \frac{1}{2} e^{-\frac{x}{2}};\ \ x \geq 0$$

(i) P(repair time exceeds 2 hours) = $P(X > 2)$

$$= \int_2^\infty f(x)dx$$

$$= \frac{1}{2}\int_2^\infty e^{-\frac{x}{2}}dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{\frac{-1}{2}}\right]_2^\infty$$

$$= -\left[e^{-\frac{x}{2}}\right]_2^\infty$$

$$= -[0 - e^{-1}]$$

$$= e^{-1}$$

(ii) P(time required atleast 11 hours / exceeds 8 hours)= $P(X \geq 11/X > 8)$

$$= P(X > 3)$$

$$= \int_3^\infty f(x)dx$$

$$= \frac{1}{2}\int_3^\infty e^{-\frac{x}{2}}dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{\frac{-1}{2}}\right]_3^\infty$$

$$= -\left[e^{-\frac{x}{2}}\right]_3^\infty$$

$$= -\left[0 - e^{-\frac{3}{2}}\right]$$

$$= e^{-\frac{3}{2}}$$

2. The length of time a person speaks over phone follows exponential distribution with mean 6 minutes. What is the probability that the person will talk for (i) more than 8 minutes (ii) between 4 and 8 minutes.

Solution:

Given X is exponentially distributed with parameter $\lambda = \frac{1}{6}$

The exponential distribution is given by $f(x) = \lambda e^{-\lambda x}; \; x \geq 0$

$$f(x) = \frac{1}{6} e^{-\frac{x}{6}}; \; x \geq 0$$

(i))$P(X > 8) \; = \int_8^\infty f(x)dx$

$$= \frac{1}{6}\int_2^\infty e^{-\frac{x}{6}} dx$$

$$= \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{\frac{-1}{6}}\right]_8^\infty$$

$$= -\left[e^{-\frac{x}{6}}\right]_8^\infty$$

$$= -\left[0 - e^{-\frac{8}{6}}\right]$$

$$= e^{-\frac{8}{6}}$$

(ii) P(between 4 and 8 minutes) = $P(4 < X < 8)$

$$= \int_4^8 f(x)dx$$

$$= \frac{1}{6}\int_4^8 e^{-\frac{x}{6}} dx$$

$$= \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{\frac{-1}{6}}\right]_4^8$$

$$= -\left[e^{-\frac{8}{6}} - e^{-\frac{4}{6}}\right] = 0.25$$