

Method of Undetermined Coefficient

The given differential equation is $F(D)y = f(x)$

To find the particular integral (P.I) of the given equation, we have to assume a trial solution that contains unknown constants. This unknown constants are to be determined by substitution in the given equation and the trial solution depends on the given function $f(x)$.

Sl.No	Function $f(x)$	Choice of P.I
1	ke^{px}	Ce^{px}
2	$k\sin(ax + b)$ (or) $k\cos(ax + b)$	$C_1\sin(ax + b) + C_2\cos(ax + b)$
3	$ke^{px}\sin(ax + b)$ (or) $ke^{px}\cos(ax + b)$	$C_1e^{px}\sin(ax + b) + C_2e^{px}\cos(ax + b)$
4	kx^m where $m = 0, 1, 2, \dots$	$C_0 + C_1x + C_2x^2 + \dots \dots C_mx^m$

Straight Case:

If the R.H.S function $f(x)$ is not a member of the solution set, then choose, P.I, (y_p) from the above table depending on the nature of $f(x)$

Sum Case:

When the R.H.S $f(x)$ is a Combination (sum) of the functions in column "2" of the table, then P.I is chosen as a Combination of the corresponding function in third column and proceed as in straight case.

Modified Case:

When any term of $f(x)$ is a member of the solution set S, then the method fails. If we choose y_p from the table. In such cases the choice from the table should be modified as follows.

a) If a term u of $f(x)$ is also a term of the C.F then the choice from the table corresponding to u should be multiplied by

* x if u corresponds to a simple root of C.F

* x^2 if u corresponds to a double root of C.F

* x^s if u corresponds to a s-fold root of C.F

b) Suppose $x^r u$ is a term $f(x)$ and u is a term of C.F corresponding to an S-fold root then the choice from the table corresponding to $x^r u$ should be multiplied by x^s .

Type I : Straight Case:

Example :

$$\text{Solve } (D^2 - 3D + 2)y = 6e^{3x}$$

Solution:

$$\text{Given } y'' - 3y' + 2y = 6e^{3x} \dots (1)$$

$$\text{Auxiliary Equation is } m^2 - 3m + 2 = 0$$

$$m = 1, 2$$

$$C.F = Ae^x + Be^{2x} \dots (2)$$

$$\text{Here the Solution Set } S = \{e^x, e^{2x}\}$$

R.H.S of (1) is not a member of S

$$P.I \quad y_p = Ce^{3x} \dots (3)$$

$$y_p' = 3Ce^{3x}$$

$$y_p'' = 9Ce^{3x}$$

$$(1) \Rightarrow 9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x} = 6e^{3x}$$

$$2Ce^{3x} = 6e^{3x}$$

$$2C = 6$$

$$C = 3$$

$$(3) \Rightarrow y_p = 3e^{3x}$$

The general solution is $y = y_c + y_p$

$$y = Ae^x + Be^{2x} + 3e^{3x}$$

Type II : Sum Case:

Example :

$$\text{Solve } (D^2 + 2D + 5)y = 2x^2 + 3e^{-x}$$

Solution:

$$\text{Given } y'' + 2y' + 4y = 2x^2 + 3e^{-x} \dots (1)$$

$$\text{Auxiliary Equation is } m^2 + 2m + 4 = 0$$

$$m = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= -1 \pm \sqrt{3}i$$

$$C.F = e^{-x} [A \cos \sqrt{3}x + B \sin \sqrt{3}x] \dots (2)$$

Here the Solution Set $S = \{e^{-x} \cos \sqrt{3}x, e^{-x} \sin \sqrt{3}x\}$

R.H.S of (1) is not a member of S

Choose $P.I$ $y_p = C_0 + C_1x + C_2x^2 + C_3e^{-x} \dots (3)$

$$y_p' = C_1 + 2C_2x - C_3e^{-x}$$

$$y_p'' = 2C_2 + C_3e^{-x}$$

$$(1) \Rightarrow 2C_2 + C_3e^{-x} + 2C_1 + 4C_2x - 2C_3e^{-x} + 4C_0 + 4C_1x + 4C_2x^2 + 4C_3e^{-x}$$

$$= 2x^2 + 3e^{-x}$$

Equating the coefficients of

$$x^2 : 4C_2 = 2$$

$$C_2 = \frac{2}{4} = \frac{1}{2}$$

$$e^{-x} : 4C_3 + C_3 - 2C_3 = 3$$

$$3C_3 = 3$$

$$C_3 = 1$$

$$x : 4C_2 + 4C_1 = 0$$

$$4\left(\frac{1}{2}\right) + 4C_1 = 0$$

$$4C_1 = \frac{-4}{2}$$

$$C_1 = \frac{-4}{8} = \frac{-1}{2}$$

Constant: $2C_2 + 2C_1 + 4C_0 = 0$

$$2\left(\frac{1}{2}\right) + 2\left(\frac{-1}{2}\right) + 4C_0 = 0$$

$$C_0 = 0$$

$$(3) \Rightarrow y_p = \frac{-1}{2}x + \frac{1}{2}x^2 + e^{-x}$$

$$= \frac{-x}{2} + \frac{x^2}{2} + e^{-x}$$

The general solution is $y = C.F + P.I$

$$y = e^{-x} [A \cos \sqrt{3}x + B \sin \sqrt{3}x] - \frac{x}{2} + \frac{x^2}{2} + e^{-x}$$

Example :

Solve $(D^2 + D - 2)y = x + \sin x$

Solution:

Given $y'' + y' - 2y = x + \sin x \dots (1)$

Auxiliary Equation is $m^2 + m - 2 = 0$

$$m = 1, -2$$

$$C.F = Ae^x + Be^{-2x} \dots (2)$$

Here the Solution Set $S = \{e^x, e^{-2x}\}$

R.H.S of (1) is not a member of S

choose P.I $y_p = C_0 + C_1x + C_2\sin x + C_3\cos x \dots (3)$

$$y_p' = C_1 + C_2\cos x - C_3\sin x$$

$$y_p'' = -C_2\sin x - C_3\cos x$$

$$(1) \Rightarrow -C_2\sin x - C_3\cos x + C_1 + C_2\cos x - C_3\sin x - 2C_0 - 2C_1x - 2C_3\sin x - 2C_3\cos x = x + \sin x$$

Equating the coefficients of

$x:$ $-2C_1 = 1$

$$C_1 = -\frac{1}{2}$$

Constant: $C_1 - 2C_0 = 0$

$$C_0 = -\frac{1}{4}$$

$\sin x:$ $-C_2 - C_3 - 2C_2 = 1$

$$-3C_2 - C_3 = 1 \dots (4)$$

$\cos x:$ $-C_3 + C_2 - 2C_3 = 0$

$$C_2 - 3C_3 = 0 \dots (5)$$

Solving (4) & (5) we get $C_2 = -\frac{3}{10}$ $C_3 = -\frac{1}{10}$

$$(3) \Rightarrow y_p = -\frac{1}{4} - \frac{1}{2}x - \frac{3}{10}\sin x - \frac{1}{10}\cos x$$

The general solution is $y = C.F + P.I$

$$y = Ae^x + Be^{-2x} - \frac{1}{4} - \frac{1}{2}x - \frac{3}{10}\sin x - \frac{1}{10}\cos x$$

Type: III Modified Case:-

Example:

$$\text{Solve } (D^2 + 9)y = \cos 3x$$

Solution:

$$\text{Given } y'' + 9y = \cos 3x \dots (1)$$

$$\text{Auxiliary Equation is } m^2 + 9 = 0$$

$$m = \pm 3i$$

$$C.F = A\cos 3x + B\sin 3x \dots (2)$$

$$\text{Here the solution set } S = \{\cos 3x, \sin 3x\}$$

R.H.S of (1) is a member of S

$$\text{Choose } P.I \quad y_p = C_1 \sin 3x + C_2 \cos 3x$$

Corresponding terms should multiplied by x

$$y_p = x[C_1 \sin 3x + C_2 \cos 3x] \dots (3)$$

$$y_p' = x[3C_1 \cos 3x - 3C_2 \sin 3x] + [C_1 \cos 3x + C_2 \sin 3x] +$$

$$y_p'' = x[-9C_1 \sin 3x - 9C_2 \cos 3x] +$$

$$[3C_1 \cos 3x - 3C_2 \sin 3x] + 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$(1) \Rightarrow -9C_1 x \sin 3x - 9C_2 x \cos 3x +$$

$$6C_1 \cos 3x - 6C_2 \sin 3x + 9C_1 x \sin x + 9C_2 x \cos 3x = \cos 3x$$

Equating the coefficients of

$\cos 3x$:

$$6C_1 = 1$$

$$C_1 = \frac{1}{6}$$

$\sin 3x$:

$$-6C_2 = 0$$

$$C_2 = 0$$

$$(3) \Rightarrow y_p = x \left[\frac{1}{6} \sin 3x \right]$$

The general solution is $y = C.F + P.I$

$$y = A\cos 3x + B\sin 3x + \frac{x}{6} \sin 3x$$

Example:

$$\text{Solve } y'' + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$$

Solution:

$$y'' + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11 \dots (1)$$

Auxiliary Equation is $m^2 + m - 6 = 0$

$$m = 2, -3$$

$$C.F = Ae^{2x} + Be^{-3x} \dots (2)$$

Here the Solution Set $S = \{e^{2x}, e^{-3x}\}$

R.H.S of (1) is not a member of S

Choose P.I $y_p = C_1e^{2x} + C_2e^{3x} + C_3x + C_4$

Corresponding terms should multiplied by x

$$y_p = x(C_1e^{2x}) + C_2e^{3x} + C_3x + C_4 \dots (3)$$

$$y_p' = C_1x2e^{2x} + C_1e^{2x} + 3C_2e^{3x} + C_3$$

$$= (2C_1x + C_1)e^{2x} + 3C_2e^{3x} + C_3$$

$$y_p'' = (2C_1x + C_1)2e^{2x} + e^{2x}(2C_1) + 9C_2e^{3x}$$

$$= 2e^{2x}[2C_1x + 2C_1] + 9C_2e^{3x}$$

$$= 4e^{2x}(x + 1)C_1 + 9C_2e^{3x}$$

$$\begin{aligned} (1) \Rightarrow & 4(x + 1)C_1e^{3x} + 9C_2e^{3x} + (2x + 1)C_1e^{3x} + 3C_2e^{3x} \\ & - 6xC_1e^{3x} - 6xC_2e^{3x} - 6C_3x - 6C_4 = 10e^{2x} - 18e^{2x} - 6x - 11 \\ & (4x + 4 + 2x + 1 - 6x)C_1e^{2x} + 6C_2e^{3x} + C_3 - 6C_3x - 6C_4 \\ & = 10e^{2x} - 18e^{3x} - 6x - 11 \end{aligned}$$

Equating the coefficients of

$$e^{2x} : 5C_1 = 10$$

$$C_1 = 2$$

$$e^{3x} : 6C_2 = 18$$

$$C_2 = -3$$

$$x : -6C_3 = -6$$

$$C_3 = 1$$

$$\text{Constant} : C_3 - 6C_4 = -11$$

$$C_4 = 2$$

$$(3) \Rightarrow y_p = 2xe^{2x} - 3e^{3x} + x + 2$$

The general solution is $y = C.F + P.I$

$$y = Ae^{2x} + Be^{-3x} + 2xe^{2x} - 3e^{3x} + x + 2$$

