UNIT - 2

DIFFERENTIAL CALCULUS

Introduction:

Differential Calculus is the most important and very essential branch of mathematics in modern mathematical science and also in applied science, engineering and technology.

In this chapter, the concept of functions and how one can define a function have been introduced. Later the concepts like limits continuity and differentiability have been introduced in a way such that the concept may easily understood by students. The differentiation of various functions like algebraic, trigonometric, exponential and logarithmic are thoroughly discussed. Plenty of examples have been worked out and there are many problems related to the above concepts. The concept of maxima and minima which is also an application of differential calculus has also been introduced. A large number of exercises on various methods are also given.

2.1 Representation of functions:

Functions:

A function is a rule that assigns to each element x in a set A to exactly one element called $f(x)$ in a set B.

The set A of all possible input values is called the domain of the function. The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

A symbol that represents an arbitrary number in the domain of a function f is called an independent variable. A symbol that represents a number in the range of f is called a dependent variable. There are four ways to represent a function:

(i) Function represented Verbally

(ii) Function represented Visually

(iii) Function represented Numerically

(iv) Function represented Algebraically (ie, a function is represented by an explicit function)

Real valued functions:

A function, whose domain and co-domain are subsets of the set of all real numbers, is known as real-valued function.

Explicit functions:

If x and y be so related that y can be expressed explicity in terms of x, then y is called

explicit function of x. **Example:** $y = 2x^2 - 4x + 2$

Implicit functions:

If x and y be so related that y cannot be expressed explicity in terms of x, then y is called implicit function of x. **Example:** $x^3 + y^3 - 3xy = 0$

Piecewise function:

A piecewise function is a function which is defined by multiple sub functions, each sub function applying to a certain interval of the main function domain.

Odd and Even functions:

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function. **Example:** $cosx$, x^2, x^4, $|x|$ are even functions.

If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function. **Example:** $sin\,x$, x, x^3 are odd functions.

Example:

Classify the following functions as odd or even functions:

(i) $\boldsymbol{x^3 \cos 2x}$ **(ii)** $\boldsymbol{\frac{x^2 cosx}{1+x^4}}$

Solutions:

(i) Let $f(x) = x^3 \cos 2x$

$f(-x) = (-x)^3 \cos 2(-x)$

$= -x^3 \cos 2x = -f(x)$

$\therefore f(x) = x^3 \cos 2x$ is an odd function

(ii) Let $f(x) = \frac{\mathrm{x}^2 \mathrm{cosx}}{1+\mathrm{x}^4}$

$$f(-x) = \frac{(-x)^2\cos(-x)}{1+(-x)^4}$$

$$= \frac{x^2 \cos x}{1+x^4} = f(x)$$

$\therefore f(x) = \frac{\mathrm{x}^2 \mathrm{cosx}}{1+\mathrm{x}^4}$ is an even function.

Graph of functions:

If f is a function with domain D, then its graph is the set of ordered pair $\{(x, f(x))/x\epsilon D\}$.

Domain, Co-domain, Range and Image:

Let: $A \to B$, then the set A is called the domain of the function and set B is called Co-domain.

The set of all the images of all the elements of A under the function f is called the range of f and it is denoted by $f(A)$.

Range of f is $f(A) = \{ f(x): x\epsilon A \}$

clearly $f(A) \subseteq B$

If $x \in A , y \in B \; and \; y = f(x)$ then y is called the image of x under f.

Example:

Find the domain and range of the function:

(i) $f(x) = \frac{1}{x^2-x}$(ii) $f(x) = \frac{4}{3-x}$(iii) $f(x) = \sqrt{5x+10}$ (iv) $f(x) = 1 + x^2$(v) $f(x) = \sqrt{x+2}$

(i) $f(x) = \frac{1}{x^2-x}$

Solution:

$$x^2 - x = 0 \Longrightarrow x(x-1) = 0$$

$$\Longrightarrow x = 0 \, , \; x - 1 = 0 \Rightarrow x = 1$$

Domain is $(-\infty, 0) \cup (0,1) \cup (1, \infty)$

Range is $(0, \infty)$

(ii) $\boldsymbol{f(x) = \frac{4}{3-x}}$

Solution:

$$3 - x = 0 \Longrightarrow x = 3$$

Domain is $(-\infty, 3) \cup (3, \infty)$

Range is $(-\infty, 0) \cup (0, \infty)$

(iii) $\boldsymbol{f(x) = \sqrt{5x + 10}}$

Solution:

Since square root of a negative number is not defined, $5x + 10 \geq 0$

$$\Longrightarrow 5x \geq -10 \Longrightarrow x \geq -2$$

Domain is $[-2\,, \infty)$

Range is $[0, \infty)$

(iv)$\boldsymbol{f(x) = 1 + x^2}$

Solution:

ie, $y = 1 + x^2 \Rightarrow \ \ y - 1 = x^2$

Here $x^2 \geq 0 \Longrightarrow y - 1 \geq 0 \Longrightarrow y \geq 1$

Domain is $[-\infty\,, \infty)$

Range is $[1, \infty)$

(v) $\boldsymbol{f(x) = \sqrt{x + 2}}$

Solution:

Since square root of a negative number is not defined, $x + 2 \geq 0 \Longrightarrow x \geq -2$

Domain is $[-2\,, \infty)$

Range is $[0, \infty)$

Example:

Find the domain and sketch the graph of the function $\boldsymbol{f(x) = \begin{cases} x + 2 & if\ x < 0 \\ 1 - x & if\ x \geq 0 \end{cases}}$

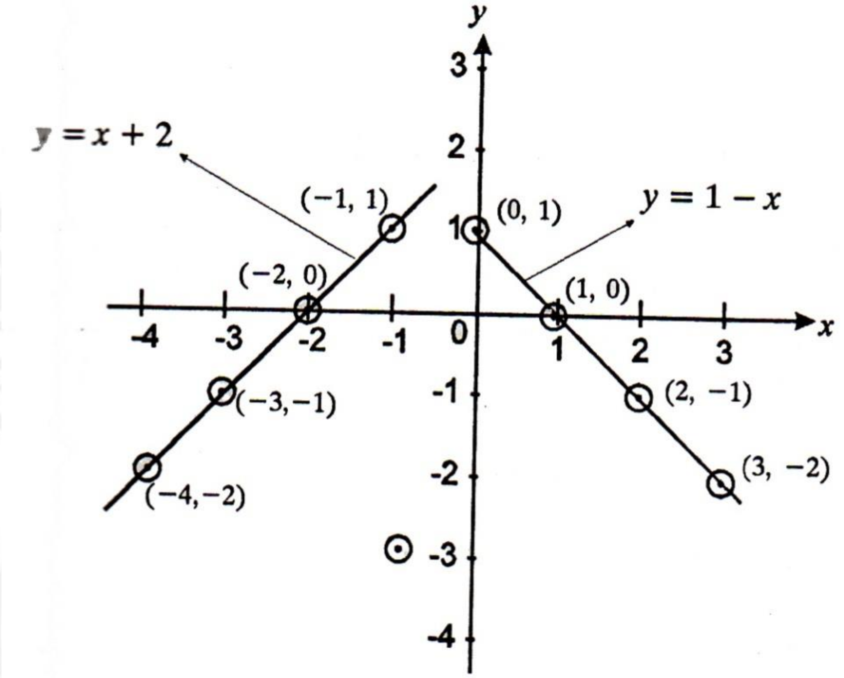
Solution:

Given $f(x) = \begin{cases} x + 2 & if\ x < 0 \\ 1 - x & if\ x \geq 0 \end{cases}$

ie, $y = x + 2,\ x < 0 \qquad y = 1 - x\ , x \geq 0$

x<0	-1	-2	-3	-4	…
y = x+2	1	0	-1	-2	…

x ≥0	0	1	2	3	…
y = 1-x	1	0	-1	-2	…



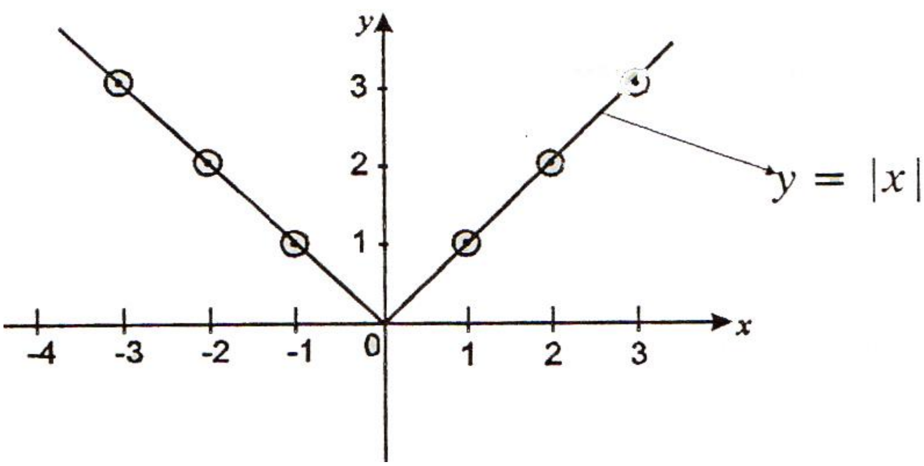
Domain is $(-\infty, \infty)$

Example:

Sketch the graph of the absolute value function $f(x) = |x|$

Solution:

Let $y = f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



Definition:

Vertical line test for a function:

A Curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

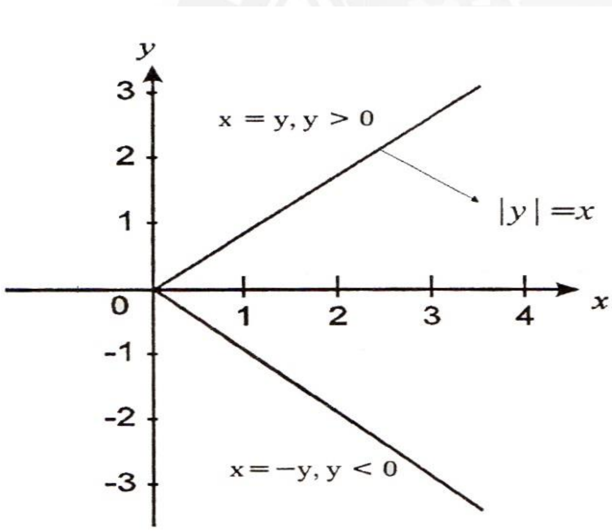
Note:

A circle cannot be the graph of a function. Since some vertical lines intersect the circle twice.

Example:

Graph the following equations and explain they are not graph of functions of x.

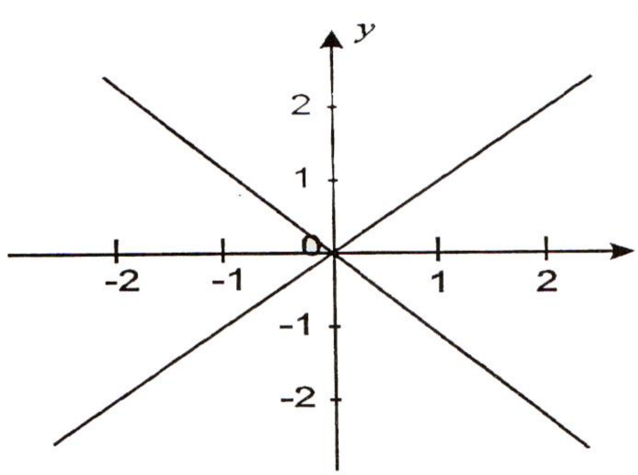
a)$|y| = x$**b)** $y^2 = x^2$



Solution:

a) $|y| = x$

For each positive value of x, there are two values of y.



b) $y^2 = x^2$

For each value of $x \neq 0$, there are two values of y.

Increasing and Decreasing functions:

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be increasing on I.

If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is said to be decreasing on I.

Example :

(1) $f(x) = x^2$ is decreasing in $(-\infty, 0]$ and increasing in $[0, \infty)$

(2) $f(x) = -x^3$ is decreasing in $(-\infty, \infty)$

Single Valued function:

A function is said to be single valued function if there corresponds only one value for $y = f(x)$ for all x. **Example:** $y = f(x) = 2x + 1$

x	0	1	2
$y = f(x)$	1	3	5

Multi Valued function:

A function is said to be multi valued function if there corresponds more than one value for

$y = f(x)$, for all x.

Example: $y^2 = x^2 + 1$

$\Longrightarrow y = \pm\sqrt{x^2 + 1}$

Exercise:

I. Determine the following functions is even, odd or neither.

1. $f(x) = x^3 + x$ **Ans:** Odd function

2. $f(x) = 1 - x^4$ **Ans:** Even function

3. $f(x) = \frac{x}{x^2+1}$ **Ans:** Odd function

4. $f(x) = x|x|$ **Ans:** Odd function

5. $f(x) = x + 1$ **Ans:** Neither even nor odd

6. $f(x) = \frac{x}{x+1}$ **Ans:** Neither even nor odd

7. $f(x) = e^{x^2}$ **Ans:** Even function

II. Find the domain and range of the function:

1. $f(x) = \frac{x+4}{x^2-9}$ **Ans:** Domain $(-\infty, -3) \cup (-3,3) \cup (3, \infty)$ Range $(-\infty, \infty)$

2. $f(t) = \sqrt[3]{2t-1}$ **Ans:** Domain$(-\infty, \infty)$ Range $[0, \infty)$

3. $f(x) = \frac{1}{\sqrt[4]{x^2-5x}}$ **Ans:** Domain $(-\infty, 0) \cup (0,5) \cup (5, \infty)$ Range $(0, \infty)$

4. $f(x) = \sqrt{5x - 15}$ **Ans:** Domain $[3, \infty)$ Range $[0, \infty)$

5. $f(x) = \sqrt{-2x + 10}$ **Ans:** Domain $(-\infty, 5]$ Range $[0, \infty)$

III. Sketch the graph and find the domain and range of each functions:

1. $f(x) = 2x - 1$ **Ans:** Domain $(-\infty, \infty)$ Range $(-\infty, \infty)$

2. $f(x) = x^2$ **Ans:** Domain $(-\infty, \infty)$ Range $[0, \infty)$

Limit of a function:

Definition:

Suppose $f(x)$ is defined when x is near the number a, Then we write $\lim_{x \to a} f(x) = L$ and say the limit of $f(x)$, as x approaches a, equals L.

The above definition says that the value of $f(x)$ approach as x approaches a. In other words, the value of $f(x)$ tend to get closer and closer to L as x gets closer and closer to a from either side of a but $x \neq a$. The alternate notation for $\lim_{x \to a} f(x) = L$ is $f(x) \to L \ as \ x \to a$.

One-sided Limits:

Left-hand limit of $f(x)$:

Suppose $f(x)$ is defined when x is near the number from left hand side of a, Then we write $\lim_{x \to a^-} f(x) = L$ and say the left-hand limit of $f(x)$, as x approaches a.

Right-hand limit of $f(x)$:

Suppose $f(x)$ is defined when x is near the number from right hand side of a, Then we write $\lim_{x \to a^+} f(x) = L$ and say the right-hand limit of $f(x)$, as x approaches a.

Definition:

Suppose $f(x)$ is defined when x is near the number a. Then we write $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Infinite Limits:

Suppose $f(x)$ is defined on both sides of ‘a’ except possibly at‘a’ itself. Then

(i) $\lim_{x \to a} f(x) = \infty$ means that the value of $f(x)$, can be made arbitrarily large by taking x to be sufficiently close to ‘a’ but not equal to a.

(ii)$\lim_{x \to a} f(x) = -\infty$ means that the value of $f(x)$, can be made arbitrarily large negative by taking x to be sufficiently close to ‘a’ but not equal to a.

Example:

Evaluate $\lim_{x \to 2} x^2 - x + 2$

Solution:

Let $f(x) = x^2 - x + 2$

X	f(x)	x	f(x)
1.9	3.71	2.1	4.31
1.99	3.9701	2.01	4.0301

1.999	3.997001	2.001	4.003001
1.9999	3.99970001	2.0001	4.00030001

$x<2$ $x>2$

From the table, $\lim_{x\to 2} x^2 - x + 2 = 4$

Example:

Find the value of $\lim_{x\to 1}\frac{x^2-1}{x-1}$

Solution:

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)}$$

$$= x+1, x \neq 1$$

X	f(x)	x	f(x)
0.9	1.9	1.1	2.1
0.99	1.99	1.01	2.01
0.999	1.999	1.001	2.001

$x < 1$ $x > 1$

We can say f(x) approaches the limit 2 as x approaches 1.

$$\therefore \lim_{x\to 1}\frac{x^2-1}{x-1} = 2$$

Example:

Find the value of $\lim_{x\to 0}\frac{\sin x}{x}$

Solution:

$$f(x) = \frac{\sin x}{x} \quad , x \neq 0$$

X	f(x)	x	f(x)
± 1.0	0.8415	± 0.1	0.9983
± 0.5	0.9589	± 0.05	0.9996
± 0.4	0.9735	± 0.01	0.9999
± 0.3	0.9851	± 0.005	0.99999
± 0.2	0.9933	± 0.001	0.999999

From the table $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

Example:

Investigate $\boldsymbol{\lim_{x\to 0} \sin\frac{\pi}{x}}$

Solution:

Let $f(x) = \sin\frac{\pi}{x}$

x	1	1/3	0.1	1/2	1.4	0.01
f(x)	0	0	0	0	0	0

Our guess $\lim_{x\to 0} \sin\frac{\pi}{x} = 0$ is wrong.

$\therefore f\left(\frac{1}{n}\right) = \sin n\pi = 0$ for any integer n.

$\therefore f\left(\frac{1}{n}\right) = 0$ which is not possible.

$\therefore$ The limit does not exists.

Example:

Use a table of values to estimate the value of the limit $\boldsymbol{\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}}$

Solution:

Let $f(x) = \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

x	f(x)	x	f(x)
-1	0.2679	1	0.2361
-0.5	0.2583	0.5	0.2426
-0.1	0.2516	0.1	0.2485
-0.05	0.2508	0.05	0.2492
-0.01	0.2502	0.01	0.2498
-0.001	0.25	0.001	0.25

$$\therefore \lim_{x\to 0}\frac{\sqrt{x+4}\ -2}{x} = 0.25 = \frac{1}{4}$$

Exercise:

I. Using the table guess the value of the limits if it exists:

1. $\lim_{x\to 2} x+3$ **Ans:** 5
2. $\lim_{x\to 2}\frac{x^2-4}{x-2}$ **Ans:** 4
3. $\lim_{x\to\infty}\frac{x^2-6x+7}{4x^2+2x+1}$ **Ans:** $\frac{1}{4}$
4. $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$ **Ans:** $\frac{1}{2}$
5. $\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$ **Ans:** $\frac{1}{6}$
6. $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$ **Ans:** Limit does not exists.
7. $\lim_{x\to 0}\frac{tan\ 3x}{tan\ 5x}$ **Ans:** $\frac{3}{5}$
8. $\lim_{x\to 0}\left(\frac{1}{x^2}\right)$ **Ans:** Limit does not exists.

Calculating limits using limit laws:

In this topic, we use the properties of limits, called the limit laws and some of the well known limits, to calculate the limits.

Limit laws:

Suppose that C is a constant, $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then

(i) $\lim_{x\to a}[C] = C$

(ii) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(iv) $\lim_{x\to a}[Cf(x)] = C\lim_{x\to a} f(x)$

(v) $\lim_{x\to a}[f(x)\, g(x)] = \lim_{x\to a} f(x)\text{x}\lim_{x\to a} g(x)$

(vi) $\lim_{x\to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\ if\ \lim_{x\to a} g(x) \neq 0$

(vii) $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, n is a positive integer.

(viii) $\lim_{x\to a}[\sqrt[n]{f(x)}] = \left[\sqrt[n]{\lim_{x\to a} f(x)}\right]$, n is a positive integer.

Some well known results:

(i) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

(ii) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

(iii) $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

(iv) $\lim_{x\to 0}\frac{a^x-1}{x} = \log a\ ,\ \ a > 0$

(v) $\lim_{x\to 1}\frac{\log x}{x-1} = 1$

Example:

Evaluate the limit and justify each step for the following:

(i) $\lim_{x\to -1}(x^4 - 3x)(x^2 + 5x + 3)$

(ii) $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$

(iii) $\lim_{u\to -2} \sqrt{u^4+3u+6}$

Solution:

(i) $\lim_{x\to -1}(x^4-3x)(x^2+5x+3) = \lim_{x\to -1}(x^4-3x) \lim_{x\to -1}(x^2+5x+3)$

$$= [\lim_{x\to -1} x^4 - 3\lim_{x\to -1} x]\text{x}[\lim_{x\to -1} x^2 + 5\lim_{x\to -1} x + \lim_{x\to -1} 3]$$

$$= [(-1)^4 - 3(-1)][(-1)^2 + 5(-1) + 3]$$

$$= 4(-1) = -4$$

(ii) $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$

$$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}$$

$$= \frac{-8+8-1}{5+6} = \frac{-1}{11}$$

(iii) $\lim_{u\to -2} \sqrt{u^4+3u+6} = \sqrt{\lim_{u\to -2}(u^4+3u+6)}$

$$= \sqrt{(-2)^4+3(-2)+6}$$

$$= \sqrt{16-6+6} = \sqrt{16} = 4$$

Theorem : 1

Limits of Polynomials:

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$, then $\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$.

Theorem : 2

Limit of Rational Functions:

If $P(x)$ and $Q(x)$ are polynomials and Q(c) ≠ 0 then $\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$

Eliminating zero denominators algebraically:

If the denominator of the rational function is not zero at the limit point c then theorem: 2 is applied.

If the denominator is zero, cancelling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer at c.

Example:

Evaluate $\lim_{h\to 0}\frac{(3+h)^2-9}{h}$

Solution:

$$\lim_{h\to 0}\frac{(3+h)^2-9}{h} = \lim_{h\to 0}\frac{9+6h+h^2-9}{h}$$

$$= \lim_{h\to 0}\frac{\mathrm{h}(6+\mathrm{h})}{\mathrm{h}}$$

$$= \lim_{h\to 0} 6+h = 6$$

Example:

Evaluate $\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4}$

Solution:

$$\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4}\frac{(x+1)(x+4)}{(x-1)(x+4)}$$

$$= \lim_{x\to -4}\frac{x+1}{x-1}$$

$$= \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}$$

Example:

Evaluate the limit if it exists $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$

Solution:

$$\lim_{x\to 1}\frac{x^4-1}{x^3-1} = \lim_{x\to 1}\frac{(x-1)(x^3+x^2+x+1)}{(x-1)(x^2+x+1)}$$

$$= \lim_{x\to 1}\frac{(x^3+x^2+x+1)}{(x^2+x+1)}$$

$$= \frac{1+1+1+1}{1+1+1} = \frac{4}{3}$$

Example:

Evaluate $\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

Solution:

$$\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t} \text{ x } \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}$$

$$= \lim_{t\to 0}\frac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0}\frac{1+\mathrm{t}-(1-\mathrm{t})}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \lim_{t\to 0}\frac{2\mathrm{t}}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{2} = 1$$

Example:

Evaluate $\lim\limits_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

Solution:

$$\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \lim_{x\to -4}\frac{\frac{x+4}{4x}}{4+x}$$

$$= \lim_{x\to -4}\frac{1}{4x} = \frac{1}{-16}$$

Example:

Evaluate the limit if it exists $\lim\limits_{x\to 2}\frac{x^2-x+6}{x-2}$

Solution:

$$\lim_{x\to 2}\frac{x^2-x+6}{x-2} = \frac{8}{0} = \infty$$

So the limit does not exists.

Example:

Evaluate the limit if it exists $\lim\limits_{x\to -1}\frac{x^2-4x}{(x-4)(x+1)}$

Solution:

$$\lim_{x\to -1}\frac{x^2-4x}{(x-4)(x+1)} = \lim_{x\to -1}\frac{x(x-4)}{(x-4)(x+1)}$$

$$= \lim_{x\to -1}\frac{x}{(x+1)}$$

$$= \frac{-1}{0} = \infty$$

∴ The limit does not exists.

Example:

Prove that $\lim_{x\to 0}|x|$

Solution:

$$|x| = f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} \mathrm{x} = 0 \; for|x| = x\,, \; x > 0$$

$$\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-\mathrm{x}) = 0 \; for|x| = -x\,, \; x < 0$$

$$\therefore \lim_{x\to 0^+}|x| = 0 = \lim_{x\to 0^-}|x|$$

$$\lim_{x\to 0}|x| = 0$$

Example:

Prove that $\lim_{x\to 0}\frac{|x|}{x}$ *does not exist*.

Solution:

Let $f(x) = \frac{|x|}{x}$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\left(\frac{x}{x}\right) = \lim_{x\to 0^+}(1)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\left(\frac{-x}{x}\right) = \lim_{x\to 0^-}(-1) = -1$$

$$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$$

∴ $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Example:

Let $g(x) = \frac{x^2+x-6}{|x-2|}$ *does* $\lim_{x\to 2} g(x)$ *exist*?

Solution:

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}\frac{x^2+x-6}{-(x-2)}$$

$$= \lim_{x\to 2^-} \frac{(x-2)(x+3)}{-(x-2)}$$

$$= \lim_{x\to 2^-} -(x+3)$$

$$= -(2+3) = -5$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{x^2+x-6}{(x-2)}$$

$$= \lim_{x\to 2^+} \frac{(x-2)(x+3)}{(x-2)}$$

$$= \lim_{x\to 2^-} (2+3)$$

$$= 5$$

$$\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$$

$\therefore \lim_{x\to 2} g(x)$ does not exist.

Example:

Find the limit if it exist $\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

Solution:

$$\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{-x}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{1}{x} + \frac{1}{x}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{2}{x}\right)$$

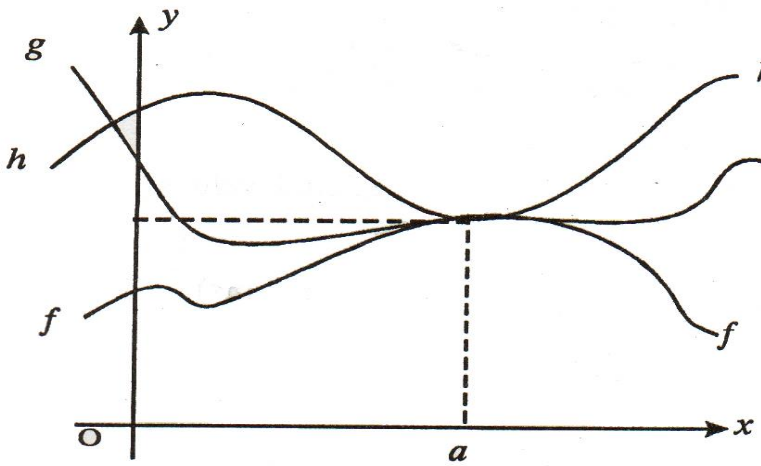
$$= \frac{2}{0} = \infty$$

$\therefore$ Limit does not exist.

Squeeze theorem (or) Sandwich theorem (or) Pinching theorem:

Statement:

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$



ie , If g(x) is squeezed in between h(x) and f(x) which have the same limit L then g(x) also forced to have the same limit L.

Example:

Show that $\lim_{x\to 0} x^2 sin\frac{1}{x} = 0$

Solution:

$$\lim_{x\to 0} x^2 sin\frac{1}{x} = \lim_{x\to 0} x^2 \lim_{x\to 0} sin\frac{1}{x}$$

Here $\lim_{x\to 0} sin\frac{1}{x}$ does not exists.

∴ By applying $x \to 0$ Squeeze theorem,

$$-1 \le sin\frac{1}{x} \le 1$$

$$-x^2 \le sin\frac{1}{x} \le x^2$$

$\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$

By Squeeze theorem, $\lim_{x\to 0} x^2 sin\frac{1}{x} = 0$

Example:

Find $\lim_{\theta\to 0}\frac{tan\,\theta}{\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{tan\,\theta}{\theta} = \lim_{\theta\to 0}\frac{sin\,\theta}{\theta cos\theta}$$

$$= \lim_{\theta\to 0}\left(\frac{sin\,\theta}{\theta}.\frac{1}{\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right).\lim_{\theta\to 0}\left(\frac{1}{\cos\theta}\right)$$

$$= 1.1 \quad = \; 1$$

Example:

Find $\lim\limits_{\theta\to 0}\frac{1-\cos x}{x}$

Solution:

$$\lim_{\theta\to 0}\frac{1-\cos x}{x} = \lim_{\theta\to 0}\frac{2\sin^2(\frac{x}{2})}{x}$$

$$= \lim_{\theta\to 0}\frac{\sin^2(\frac{x}{2})}{\left(\frac{x}{2}\right)} \text{ x } \frac{(x/2)}{(x/2)}$$

$$= \lim_{\theta\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right)^2 \text{ x } \left(\frac{x}{2}\right)$$

$$= \lim_{\theta\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right)^2 \text{ x } \lim_{\theta\to 0}\left(\frac{x}{2}\right)$$

$$= 1 \text{ x } 0 = 0$$

Example:

Find $\lim\limits_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2}$

Solution:

$$\lim_{x\to\frac{\pi}{2}}\frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x\to\frac{\pi}{2}}\frac{2\cos^2 x}{(\pi-2x)^2}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{2\sin\left(\frac{\pi}{2}-x\right)}{2^2\left(\frac{\pi}{2}-x\right)^2}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{1}{2}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)}\right]^2$$

$$= \frac{1}{2}\lim_{\left(x-\frac{\pi}{2}\right)\to 0}\left[\frac{\sin(-)\left(x-\frac{\pi}{2}\right)}{-\left(x-\frac{\pi}{2}\right)}\right]^2 = \frac{1}{2}(1) \; = \; \frac{1}{2}$$

Example:

Find $\lim_{x\to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2}$

Solution:

$$\lim_{x\to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2} = \lim_{x\to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2\left(\frac{1}{3^2}\right)} \text{x} \left(\frac{1}{3^2}\right)$$

$$= \lim_{x\to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{\left(\frac{x}{3}\right)^2} \text{x} \left(\frac{1}{3^2}\right)$$

$$= \lim_{x\to 0} \left[\frac{\sin\left(\frac{x}{3}\right)}{\left(\frac{x}{3}\right)}\right]^2 \text{x} \lim_{x\to 0} \left(\frac{1}{9}\right) = 1 \text{ x } \frac{1}{9} = \frac{1}{9}$$

Exercise:

1. **Evaluate the limit and justify each step by indicating the appropriate limit:**

i) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$ **Ans:** 5

ii) $\lim_{x\to -3} \frac{x^2-9}{2x^2+7x+3}$ **Ans:** $\frac{6}{5}$

iii) $\lim_{x\to 2} \sqrt{\frac{2x^2+1}{3x-2}}$ **Ans:** $\frac{3}{2}$

iv) $\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$ **Ans:** $3x^2$

v) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$ **Ans:** 12

2. Prove that $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

3. Evaluate $\lim_{x\to -6} \frac{2x+12}{|x+6|}$ **Ans:** 2

4. Show that $\lim_{x\to 0} \sqrt{x^3+x^2} \sin\frac{\pi}{x} = 0$

5. Use the squeeze theorem, to show that $\lim_{x\to 0}(x^2 \cos 20\pi x)$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

VI. Using Sandwich theorem, if $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$ **Ans:** $\sqrt{5}$