

Triple Integrals

Triple integration in cartesian co-ordinates is defined over a region R is defined by

$$\iiint_R f(x, y, z) dx dy dz \text{ or } \iiint_R f(x, y, z) dV \text{ or } \iiint_R f(x, y, z) d(x, y, z).$$

Example:

Evaluate $\int_0^a \int_0^b \int_0^c (x + y + z) dz dy dx$

Solution:

$$\begin{aligned} \int_0^a \int_0^b \int_0^c (x + y + z) dz dy dx &= \int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2} \right]_0^c dy dx \\ &= \int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2} \right) dy dx \\ &= \int_0^a \left[xcy + \frac{y^2 c}{2} + \frac{c^2 y}{2} \right]_0^b dx \\ &= \int_0^a \left(xbc + \frac{b^2 c}{2} + \frac{bc^2}{2} \right) dx \\ &= \left[\frac{x^2 bc}{2} + \frac{b^2 cx}{2} + \frac{bc^2 x}{2} \right]_0^a \\ &= \left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right] \\ &= \frac{abc}{2} (a + b + c) \end{aligned}$$

Example:

Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x dz dx dy$

Solution:

$$\begin{aligned} \int_0^1 \int_{y^2}^1 \int_0^{1-x} x dz dx dy &= \int_0^1 \int_{y^2}^1 [xz]_0^{1-x} dx dy \\ &= \int_0^1 \int_{y^2}^1 x(1-x) dx dy \\ &= \int_0^1 \int_{y^2}^1 (x - x^2) dx dy \\ &= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 dy \\ &= \int_0^1 \left(\frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy \\ &= \left[\frac{y}{2} - \frac{y}{3} - \frac{y^5}{10} + \frac{y^7}{21} \right]_0^1 \\ &= \frac{1}{2} - \frac{1}{3} - \frac{1}{10} + \frac{1}{21} \\ &= \frac{105 - 70 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35} \end{aligned}$$

Example:

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dx dy dz$

Solution:

$$\begin{aligned}
 \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dx dy dz &= \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dz dy dx \\
 &= \int_0^1 \int_0^{1-x} [e^z]_0^{x+y} dy dx \\
 &= \int_0^1 \int_0^{1-x} (e^{x+y} - 1) dy dx \\
 &= \int_0^1 [e^{x+y} - y]_0^{1-x} dx \\
 &= \int_0^1 (e^{x+1-x} - 1 + x - e^x) dx \\
 &= \int_0^1 (e - 1 + x - e^x) dx \\
 &= \left[ex - x + \frac{x^2}{2} - e^x \right]_0^1 \\
 &= e - 1 + \frac{1}{2} - e - 0 + 0 - 0 + 1 \\
 &= \frac{1}{2}
 \end{aligned}$$

Example:

Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz dx dy$

Solution:

$$\begin{aligned}
 \int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz dx dy &= \int_0^a \int_0^{\sqrt{a^2-y^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dx dy \\
 &= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} dx dy \\
 &= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(\sqrt{a^2-y^2})^2 - x^2} dx dy \\
 &= \int_0^a \left[\frac{x}{2} \sqrt{(a^2-y^2) - x^2} + \right. \\
 &\quad \left. \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy \\
 &= \int_0^a \left(0 + \frac{a^2-y^2}{2} \sin^{-1} 1 - 0 - 0 \right) dy \\
 &= \int_0^a \left(\frac{a^2-y^2}{2} \right) \frac{\pi}{2} dy \\
 &= \frac{\pi}{4} \int_0^a (a^2 - y^2) dy \\
 &= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a \\
 &= \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} - 0 \right]
 \end{aligned}$$

$$= \frac{\pi}{4} \left(\frac{2a^3}{3} \right)$$

$$= \frac{\pi a^3}{6}$$

Example:

Evaluate $\int_0^{\log a} \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$

Solution:

$$\begin{aligned} \int_0^{\log a} \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx &= \int_0^{\log a} \int_0^x [e^{x+y+z}]_0^{x+y} dy dx \\ &= \int_0^{\log a} \int_0^x (e^{2(x+y)} - e^{x+y}) dy dx \\ &= \int_0^{\log a} \left[\frac{e^{2(x+y)}}{2} - e^{x+y} \right]_0^x dx \\ &= \int_0^{\log a} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right) dx \\ &= \int_0^{\log a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \right) dx \\ &= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{8} + e^x \right]_0^{\log a} \\ &= \frac{e^{4 \log a}}{8} - \frac{3}{8} e^{2 \log a} + e^{\log a} - \frac{1}{8} + \frac{3}{8} - 1 \\ &= \frac{e^{\log a^4}}{8} - \frac{3}{8} e^{\log a^2} + a + \left(\frac{-1+6-8}{8} \right) \\ &= \frac{a^4}{8} - \frac{3a^2}{8} + a - \frac{6}{8} \quad [\because e^{\log x} = x] \end{aligned}$$

Example:

Express the region $x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 1$ **by triple integration.**

Solution:

For the given region, z varies from 0 to $\sqrt{1-x^2-y^2}$

y varies from 0 to $\sqrt{1-x^2}$

x varies from 0 to 1

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx$$

Example:

Evaluate $\iiint x^2 y z dx dy dz$ **taken over the tetrahedron bounded by the planes**

$x = 0, y = 0, z = 0$ **and** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution:

$$\text{Given } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Limits are , z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

x varies from 0 to a

$$\begin{aligned}
 \iiint x^2 y z dx dy dz &= \int_0^a \int_0^{b\left(1 - \frac{x}{a}\right)} \int_0^{c\left(1 - \frac{x}{a} - \frac{y}{b}\right)} x^2 dz dy dx \\
 &= \int_0^a \int_0^{b\left(1 - \frac{x}{a}\right)} \left[x^2 y \frac{z^2}{2} \right]_0^{c\left(1 - \frac{x}{a} - \frac{y}{b}\right)} dy dx \\
 &= \int_0^a \int_0^{b\left(1 - \frac{x}{a}\right)} \left(\frac{x^2 y c^2 \left(1 - \frac{x}{a} - \frac{y}{b}\right)^2}{2} \right) dy dx \\
 &= \frac{c^2}{2} \int_0^a \int_0^{bk} x^2 y \left(k - \frac{y}{b}\right)^2 dy dx \quad \left[\because k = 1 - \frac{x}{a} \right] \\
 &= \frac{c^2}{2} \int_0^a \int_0^{bk} x^2 y \left(k^2 + \frac{y^2}{b^2} - \frac{2ky}{b}\right) dy dx \\
 &= \frac{c^2}{2} \int_0^a \int_0^{bk} x^2 \left(yk^2 + \frac{y^3}{b^2} - \frac{2ky^2}{b}\right) dy dx \\
 &= \frac{c^2}{2} \int_0^a x^2 \left[\frac{k^2 y^2}{2} + \frac{y^4}{4b^2} - \frac{2ky^3}{3b} \right]_0^{bk} dx \\
 &= \frac{c^2}{2} \int_0^a x^2 \left(\frac{b^2 k^4}{2} + \frac{b^4 k^4}{4b^2} - \frac{2b^3 k^4}{3b} \right) dx \\
 &= \frac{c^2}{2} \int_0^a x^2 \left(\frac{b^2 k^4}{2} + \frac{b^2 k^4}{4} - \frac{2b^2 k^4}{3} \right) dx \\
 &= \frac{b^2 c^2}{2} \int_0^a k^4 x^2 \left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right) dx \\
 &= \frac{b^2 c^2}{24} \int_0^a x^2 \left(1 - \frac{x}{a}\right)^4 dx \\
 &\quad \left[\because (1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \dots \right] \\
 &= \frac{b^2 c^2}{24} \int_0^a x^2 \left(1 - \frac{4x}{a} + \frac{4 \times 3}{2!} \times \frac{x^2}{a^2} - \frac{4 \times 3 \times 2}{3!} \times \frac{x^3}{a^3} + \frac{4 \times 3 \times 2 \times 1}{4!} \times \frac{x^4}{a^4}\right) dx \\
 &= \frac{b^2 c^2}{24} \int_0^a \left(x^2 - \frac{4x^3}{a} + \frac{6x^4}{a^2} - \frac{4x^5}{a^3} + \frac{x^6}{a^4}\right) dx \\
 &= \frac{b^2 c^2}{24} \left[\frac{x^3}{3} - \frac{4x^4}{4a} + \frac{6x^5}{5a^2} - \frac{4x^6}{6a^3} + \frac{x^7}{7a^4} \right]_0^a \\
 &= \frac{b^2 c^2}{24} \left[\frac{a^3}{3} - \frac{a^4}{a} + \frac{6a^5}{5a^2} - \frac{2a^6}{3a^3} + \frac{a^7}{7a^4} \right] \\
 &= \frac{b^2 c^2}{24} \left[\frac{a^3}{3} - a^3 + \frac{6a^3}{5} - \frac{2a^3}{3} + \frac{a^3}{7} \right] \\
 &= \frac{a^3 b^2 c^2}{24} \left[\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7} \right] \\
 &= \frac{a^3 b^2 c^2}{24} \left(\frac{35 - 105 + 126 - 70 + 15}{105} \right) \\
 &= \frac{a^3 b^2 c^2}{24} \left(\frac{1}{105} \right)
 \end{aligned}$$

$$= \frac{a^3 b^2 c^2}{2520}$$

Example:

Find the value of $\iiint xyz dx dy dz$ through the positive spherical octant for which $x^2 + y^2 + z^2 \leq a^2$

Solution:

In the positive octant, the limits are

z varies from 0 to $\sqrt{a^2 - x^2 - y^2}$

y varies from 0 to $\sqrt{a^2 - x^2}$

x varies from 0 to a

$$\begin{aligned} I &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} xyz dz dy dx \\ &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} \left[\frac{xyz^2}{2} \right]_0^{\sqrt{a^2 - x^2 - y^2}} dy dx \\ &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} xy(a^2 - x^2 - y^2) dy dx \\ &= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2 - x^2}} (a^2 xy - x^3 y - xy^3) dy dx \\ &= \frac{1}{2} \int_0^a \left[\frac{a^2 xy^2}{2} - \frac{x^3 y^2}{2} - \frac{xy^4}{4} \right]_0^{\sqrt{a^2 - x^2}} dx \\ &= \frac{1}{2} \int_0^a \left(\frac{a^2 x(a^2 - x^2)}{2} - \frac{x^3(a^2 - x^2)}{2} - \frac{x(a^2 - x^2)^2}{4} \right) dx \\ &= \frac{1}{2} \int_0^a \frac{x}{2} (a^2 - x^2) \left[a^2 - x^2 - \frac{(a^2 - x^2)}{2} \right] dx \\ &= \frac{1}{2} \int_0^a \frac{x(a^2 - x^2)(a^2 - x^2)}{4} dx \end{aligned}$$

$$= \frac{1}{8} \int_0^a x(a^2 - x^2)^2 dx$$

$$\text{Put } a^2 - x^2 = t \quad x=0 \rightarrow t=a^2$$

$$-2x dx = dt \quad x=a \rightarrow t=0$$

$$\Rightarrow I = \frac{1}{8} \int_{a^2}^0 t^2 \left(-\frac{dt}{2} \right)$$

$$= -\frac{1}{16} \int_{a^2}^0 t^2 dt$$

$$= \frac{1}{16} \int_0^{a^2} t^2 dt$$

$$= \frac{1}{16} \left[\frac{t^3}{3} \right]_0^{a^2}$$

$$= \frac{1}{16} \left(\frac{a^6}{3} \right)$$

$$= \frac{a^6}{48}$$

