

System of Simultaneous Linear Differential equations with constant coefficients

Simultaneous Linear equations

Linear differential equations in which there are two or more dependent variables and a single independent variable such equations are known as simultaneous linear equations.

Consider the simultaneous equation in two dependent variables x and y and one independent variable t .

$$f_1(D)x + g_1(D)y = h_1(t) \dots (1)$$

$$f_2(D)x + g_2(D)y = h_2(t) \dots (2)$$

Where f_1, f_2, g_1 and g_2 are polynomials in the operator D

Example:

Solve $\frac{dy}{dx} + x = t^2; \frac{dx}{dt} - y = t$

Solution:

$$x + Dy = t^2 \dots (1)$$

$$Dx - y = t \dots (2)$$

Eliminate ' x '

$$(1) \times D \Rightarrow Dx + D^2y = D(t^2)$$

$$Dx + D^2y = 2t \dots (3)$$

$$(2) \Rightarrow Dx - y = t \dots (4)$$

$$(3) - (2) \Rightarrow \frac{D^2y + y = t}{(D^2 + 1)y = t}$$

$$(D^2 + 1)y = t$$

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$\alpha = 0, \beta = 1$$

$$C.F = e^{at}[A \cos \beta t + B \sin \beta t]$$

$$= A \cos t + B \sin t$$

$$P.I = \frac{1}{D^2 + 1}(t)$$

$$= \frac{1}{1 + D^2}(t)$$

$$\begin{aligned}
 &= (1 + D^2)^{-1}t \\
 &= (1 - D^2)t \\
 &= t - D^2(t) \\
 &= t
 \end{aligned}$$

$$y = A\cos t + B\sin t + t$$

$$Dy = -A\sin t + B\cos t + 1$$

$$\begin{aligned}
 (1) \Rightarrow x &= t^2 - Dy \\
 &= t^2 - [-A\sin t + B\cos t + 1] \\
 &= t^2 + A\sin t - B\cos t - 1 \\
 x &= t^2 + A\sin t - B\cos t - 1 \\
 y &= A\cos t + B\sin t + t
 \end{aligned}$$

Example:

Solve the simultaneous differential equations

$$\frac{dx}{dt} + 2y = \sin 2t \quad ; \quad \frac{dy}{dt} - 2x = \cos 2t$$

Solution:

$$\frac{dx}{dt} + 2y = \sin 2t$$

$$Dx + 2y = \sin 2t \dots (1)$$

$$\frac{dy}{dt} - 2x = \cos 2t$$

$$-2x + Dy = \cos 2t \dots (2) \quad \text{Eliminate 'x'}$$

$$(1) \times 2 \Rightarrow 2Dx + 4y = 2\sin 2t \dots (3)$$

$$(2) \times D \Rightarrow -2Dx + D^2y = -2\sin 2t \dots (4)$$

$$(3) + (4) \Rightarrow D^2y + 4y = 0$$

$$(D^2 + 4)y = 0$$

$$\text{Auxiliary Equation is } m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i$$

$$\alpha = 0, \beta = 2$$

$$C.F = e^{\alpha t} [A\cos\beta t + B\sin\beta t]$$

$$y = A\cos 2t + B\sin 2t$$

$$\text{To find } x, \quad (5) \Rightarrow Dy = -2A\sin 2t + 2B\cos 2t$$

$$\begin{aligned}
 (2) \Rightarrow 2x &= Dy - \cos 2t \\
 &= -2A \sin 2t + 2B \cos 2t - \cos 2t \\
 x &= -A \sin 2t + B \cos 2t - \frac{\cos 2t}{2}
 \end{aligned}$$

Example:

Solve $\frac{dx}{dt} + 2x + 3y = 2e^{2t}$; $\frac{dy}{dt} + 3x + 2y = 0$

Solution:-

Given $Dx + 3x + 2y = 0$

$$(D + 2)x + 3y = 2e^{2t} \dots (1)$$

$$Dy + 3x + 2y = 0$$

$$3x + (D + 2)y = 0 \dots (2)$$

Eliminate x

$$(2) \times (D + 2) \Rightarrow 3(D + 2)x + (D + 2)^2 y = 0 \dots (3)$$

$$(1) \times (3) \Rightarrow 3(D + 2)x + 9y = 6e^{2t} \dots (4)$$

$$(3) - (4) \Rightarrow \frac{[(D + 2)^2 - 9]y = -6e^{2t}}$$

$$(D^2 + 4D - 5)y = -6e^{2t}$$

Auxiliary Equation is $m^2 + 4m - 5 = 0$

$$m = -5, 1$$

$$C.F = Ae^{-5t} + Be^t$$

$$\begin{aligned}
 P.I &= \frac{1}{D^2 + 4D - 5} (-6e^{2t}) \quad \text{Replace D by 2} \\
 &= \frac{-6}{7} e^{2t}
 \end{aligned}$$

$$y = Ae^{-5t} + Be^t - \frac{6}{7} e^{2t} \dots (5)$$

To find x (5) $\Rightarrow Dy = -5Ae^{-5t} + Be^t - \frac{12}{7} e^{2t}$

$$(2) \Rightarrow 3x = -(D + 2)y$$

$$= -Dy - 2y$$

$$= 5Ae^{-5t} - Be^t + \frac{12}{7} e^{2t} - 2Ae^{-5t} - 2Be^t + \frac{12}{7} e^{2t}$$

$$3x = 3Ae^{-5t} - 3Be^t + \frac{24}{7} e^{2t}$$

$$x = Ae^{-5t} - Be^t + \frac{8}{7} e^{2t}$$

Example:

$$\text{Solve } \frac{dx}{dt} + 2x - 3y = t \quad ; \quad \frac{dy}{dt} - 3x + 2y = e^{2t}$$

Solution:

$$\text{Given } \frac{dx}{dt} + 2x - 3y = t \dots (1)$$

$$Dx + 2x - 3y = t$$

$$(D + 2)x - 3y = t \dots (2)$$

$$\text{Given } \frac{dy}{dt} - 3x + 2y = e^{2t} \dots (3)$$

$$Dy - 3x + 2y = e^{2t}$$

$$-3x + (D + 2)y = e^{2t} \dots (4)$$

$$(2) \times (3) \Rightarrow 3(D + 2)x - 9y = 3t$$

$$(4) \times (D + 2) \Rightarrow -3(D + 2)x + (D + 2)^2 y = (D + 2)e^{2t}$$

$$\frac{-9y + (D + 2)^2 y = 3t + (D + 2)e^{2t}}{(-9 + D^2 + 4D + 4)y = 3t + 4e^{2t}}$$

$$(-9 + D^2 + 4D + 4)y = 3t + 4e^{2t}$$

$$(D^2 + 4D - 5)y = 3t + 4e^{2t}$$

$$\text{Auxiliary Equation is } m^2 + 4m - 5 = 0$$

$$m = -5, 1$$

$$C.F = Ae^t + Be^{-5t}$$

$$P.I_1 = \frac{1}{D^2 + 4D - 5} (3t)$$

$$= \frac{3}{-5 \left[1 - \frac{D^2 + 4D}{5} \right]} t$$

$$= \frac{3}{-5 \left[1 - \frac{D^2 + 4D}{5} \right]} t$$

$$= \frac{-3}{5} \left[1 - \left(\frac{D^2 + 4D}{5} \right) \right]^{-1} (t)$$

$$= \frac{-3}{5} \left[1 + \left(\frac{D^2 + 4D}{5} \right) + \left(\frac{D^2 + 4D}{5} \right)^2 + \dots \right] (t)$$

$$= \frac{-3}{5} \left[t + \frac{4}{5} \right]$$

$$= \frac{-3}{5} t - \frac{12}{25}$$

$$P.I_2 = \frac{1}{D^2 + 4D - 5} 4e^{2t}$$

$$= \frac{4}{4-8-5} e^{2t} \quad \text{Replace D by 2}$$

$$= \frac{4}{7} e^{2t}$$

$$y = C.F + P.I_1 + P.I_2$$

$$y = Ae^t + Be^{-5t} - \frac{3}{5}t - \frac{12}{25} + \frac{4}{7}e^{2t}$$

$$Dy = Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t}$$

$$(3) \Rightarrow 3x = \frac{dy}{dt} + 2y - e^{2t}$$

$$= \left[Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t} \right] + 2 \left[Ae^t + Be^{-5t} - \frac{3}{5}t - \frac{12}{25} + \frac{4}{7}e^{2t} \right] - e^{2t}$$

$$= Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t} + 2Ae^t + 2Be^{-5t} - \frac{6}{5}t + \frac{24}{25} + \frac{8}{7}e^{2t} - e^{2t}$$

$$3x = 3Ae^t - 3Be^{-5t} - \frac{6}{5}t - \frac{39}{25} + \frac{9}{7}e^{2t}$$

$$x = Ae^t - Be^{-5t} - \frac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

$$\therefore x = Ae^t - Be^{-5t} - \frac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

$$y = Ae^t + Be^{-5t} - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$$