UNIT-IV

FOURIER TRANSFORMS

Fourier Transforms and its properties

Fourier Transform

We know that the complex form of Fourier integral is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\, d\lambda.$$

Replacing λ by s, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt .$$

It follows that if

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \text{ --------------------- (1)}$$

Then, $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \text{ ----------------------(2)}$$

The function F(s), defined by (1), is called the **Fourier Transform** of f(x). The function f(x), as given by (2), is called the **inverse Fourier Transform** of F(s). The equation (2) is also referred to as the **inversion formula**.

Properties of Fourier Transforms

(1) Linearity Property

If F(s) and G(s) are Fourier Transforms of f(x) and g(x) respectively, then

$$F\{a f(x) + bg(x)\} = a F(s) + bG(s),$$

where a and b are constants.

We have
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$G(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

Therefore,

$$F\{a f(x) + b g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \{a f(x) + bg(x)\} dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$$= a F(s) + bG(s) \text{ i.e,}$$

$$F\{a f(x) + bg(x)\} = a F(s) + bG(s)$$

(2) Shifting Property

(i) If F(s) is the complex Fourier Transform of f(x), then

$$F\{f(x-a)\} = e^{isa} F(s).$$

We have
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{ ---------------------(i)}$$

Now,
$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$$

Putting x-a = t, we have

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(t+a)} f(t)\, dt .$$

$$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt .$$

$= e^{ias} . F(s).$ (by (i)).

(ii) If F(s) is the complex Fourier Transform of f(x), then

$F\{e^{iax} f(x)\} = F(s+a).$

We have $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$ ----------------------- (i)

Now, $F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} . e^{iax} f(x)\, dx.$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x}. f(x)\, dx .$

$= F(s+a)$ by (i) .

(3) Change of scale property

If F(s) is the complex Fourier transform of f(x), then

$F\{f(ax)\} = 1/a\ F(s/a),\ a \neq 0.$

We have $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$ ----------------------- (i)

Now, $F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax)\, dx.$

Put ax = t, so that dx = dt/a.

$\therefore F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\,ist/a} .f(t)\, dt/a .$

$= \frac{1}{\ } \frac{1}{\sqrt{2\pi}} . \int_{-\infty}^{\infty} e^{i(s/a)t} f(t)\, dt .a$

1

$= \frac{\quad}{} \,.\, F(s/a).$ (by (i)).a

(4) Modulation theorem.

If F(s) is the complex Fourier transform of f(x), Then

$$F\{f(x)\cos ax\} = ½\{F(s+a) + F(s-a)\}.$$

We have
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

Now,
$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}.f(x)\cos ax.\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx.}\, f(x) \frac{e^{iax} + e^{-iax}}{2}\, dx.$$

$$= \quad e^{i(s+a)x}.f(x)\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\, dx$$

$$= \frac{1}{\quad} \{ F(s+a) + F(s-a)\}2$$

(5) n^{th} derivative of the Fourier Transform

If F(s) is the complex Fourier Transform of f(x),Then

$$F\{x^n f(x)\} = (-i)^n d^n/ds^n .F(s).$$

We have
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \text{---------------------------} (i)$$

Differentiating (i) „n‟ times w.r.t „s‟, we get

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n . e^{isx} f(x)\, dx$$

$$= \frac{(i)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \{x^n f(x)\}\, dx$$

$$= (i)^n F\{x^n f(x)\}.$$

$$\Rightarrow F\{x^n f(x)\} = \frac{1}{(i)^n} . \frac{d^n F(s)}{ds^n}$$

$$\text{i.e, } F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s).$$

(6) Fourier Transform of the derivatives of a function.

If F(s) is the complex Fourier Transform of f(x), Then,

$F\{f'(x)\} = -is\, F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$.

We have $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx .$$

Now, $$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d\{f(x)\}.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx}.f(x) \right]_{-\infty}^{\infty} - is \int_{-\infty}^{\infty} f(x).\, e^{isx}\, dx. \right.$$

$$= -is \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \;, \text{ provided } f(x) = 0 \text{ as } x \to \pm \infty.$$

$$= -is\, F(s).$$

i.e, $F\{f'(x)\} = -is\, F(s)$ ------------------------------ (i)

Then the Fourier Transform of $f''(x)$,

$$\text{i.e, } F\{f''(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f''(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d\{f'(x)\}.$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{isx}.f'(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x).\, e^{isx}.(is)dx. \right]$$

$$= -is \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx \;, \text{ provided } f'(x) = 0 \text{ as } x \to \pm \infty.$$

$$= -is\, F\{f'(x).\}$$

$$= (-is).(-is)F(s). \qquad \text{by(i).}$$

$$= (-is)^2 . F(s).$$

i.e, $F\{f''(x)\} = (-is)^2 .F(s)$, Provided $f, f' \to 0$ as $x \to \pm\infty$.

In general, the Fourier transform of the n^{th} derivative of f(x) is given by $F\{f^n(x)\} = (-is)^n F(s)$,

provided the first „n-1" derivatives vanish as $x \to \pm \infty$.

Property (7)

If F(s) is the complex Fourier Transform of f(x), then $F\left[\int_a^x f(x)dx\right] = \dfrac{F(s)}{(-is)}$

Let $g(x) = \int_a^x f(x)\,dx$.

Then, $g'(x) = f(x)$. --- (i)

Now $F[g'(x)] = (-is)\,G(s)$, by property (6).

$= (-is).\,F\{g(x)\}$

$= (-is).\,F\left\{\int_a^x f(x)\,dx\right\}$.

i.e, $F\{g'(x)\} = (-is).\,F\left\{\int_a^x f(x)\,dx\right\}$.

i.e, $F\left\{\int_a^x f(x)\,dx\right\} = \frac{1}{(-is)}.\,F\{g'(x)\}$.

$= \frac{1}{(-is)}\,F\{f(x)\}$. [by (i)]

Thus, $F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{(-is)}$.

Property (8)

If F(s) is the complex Fourier transform of f(x),

Then, $F\{\overline{f(-x)}\} = \overline{F(s)}$, where bar denotes complex conjugate.

Proof

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\,dx .$$

Putting x = -t, we get

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\, e^{isx}\,dt .$$

$= F\{\overline{f(-x)}\}$.

Note: If F{f(x)} = F(s), then

(i) F{f(-x)} = F(-s).

(ii) $F\{\overline{f(x)}\}$ = F(-s).