

TRIGONOMETRIC INTEGRALS

(I) Products of powers of sines and cosines

Evaluating $\int \sin^m x \cos^n x dx$

Case (i) If n is odd ($n = 2k + 1$), then

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx\end{aligned}$$

Here, substitute $u = \sin x$

Case (ii) If m is odd ($m = 2k + 1$), then

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x dx &= \int (\sin^2 x)^k \cos^n x \sin x dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x dx\end{aligned}$$

Here, substitute $u = \cos x$

Note: If both m and n are odd apply case (i) or case (ii)

Case (iii) If both m and n are even, use half-angle identities

$$\begin{aligned}\sin^2 x &= \frac{1}{2}(1 - \cos 2x), & \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \\ \int_0^{\pi/2} \sin^m x \cos^n x dx &= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \frac{1}{1+n} \\ &\quad \text{(if } m \text{ is odd, } n \text{ may be even or odd)} \\ &= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1 \\ &\quad \text{(if } m \text{ is even, } n \text{ is odd)} \\ &= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{1}{2} \frac{\pi}{2} \\ &\quad \text{(if } m \text{ is even, } n \text{ is even)}\end{aligned}$$

(II) Products of powers of $\sec x$ and $\tan x$

Evaluating $\int \tan^m x \sec^n x dx$

Case (i) If m is odd ($m = 2k + 1$), then

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx\end{aligned}$$

Here, substitute $u = \sec x$

Case (ii) If n is even ($n = 2k$), then

$$\begin{aligned}\int \tan^m x \sec^{2k} x dx &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x dx \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx\end{aligned}$$

Here, substitute $u = \tan x$

(III) Products of sines and cosines of multiples of x

Evaluating $\int \sin mx \sin nx dx$, $\int \sin mx \cos nx dx$ and $\int \cos mx \cos nx dx$

Use the following identities

$$\sin mx \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \frac{1}{2} [\sin(m-n)x + \sin(m+n)x]$$

$$\cos mx \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

Example:

(i) Evaluate $\int \sin^6 x \cos^3 x dx$

Solution:

Given $\int \sin^6 x \cos^3 x dx$ Here $m = 6, n = 3$ (odd)

$$\begin{aligned} &= \int \sin^6 x \cos^2 x \cos x dx \\ &= \int \sin^6 x (1 - \sin^2 x) \cos x dx \dots (1) \end{aligned}$$

$$\text{Put } u = \sin x; \quad du = \cos x dx$$

$$(1) \Rightarrow \int u^6 (1 - u^2) du = \int (u^6 - u^8) du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C$$

$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C$$

(ii) Evaluate $\int \sin^2(\pi x) \cos^5(\pi x) dx$

Solution:

Given $\int \sin^2(\pi x) \cos^5(\pi x) dx$ (Here $m = 2, n = 5$ (odd))

$$\begin{aligned} &= \int \sin^2(\pi x) \cos^4(\pi x) \cos(\pi x) dx \\ &= \int \sin^2(\pi x) [1 - \sin^2(\pi x)]^2 \cos(\pi x) dx \dots (1) \end{aligned}$$

$$\text{Put } u = \sin \pi x; \quad du = \pi \cos \pi x dx$$

$$(1) \Rightarrow \int u^2 (1 - u^2)^2 \frac{du}{\pi} = \frac{1}{\pi} \int u^2 (1 - 2u^2 + u^4) du$$

$$= \frac{1}{\pi} \int (u^2 - 2u^4 + u^6) du$$

$$= \frac{1}{\pi} \left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right] + C$$

$$= \frac{1}{3\pi} \sin^3(\pi x) - \frac{2}{5\pi} \sin^5(\pi x) + \frac{1}{7\pi} \sin^7(\pi x) + C$$

Example:

Evaluate $\int \sin^5 x \cos^2 x dx$

Solution:

Given $\int \sin^5 x \cos^2 x dx$ (Here $m = 5$ (odd), $n = 2$)

$$= \int \sin^4 x \cos^2 x \sin x dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x dx \dots (1)$$

Put $u = \cos x$; $du = -\sin x dx$

$$(1) \Rightarrow \int (1 - u^2)^2 u^2 (-du) = -\int (1 - 2u^2 + u^4) u^2 du$$

$$= -\int (u^2 - 2u^4 + u^6) du$$

$$= -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$$

$$= -\frac{1}{3} \cos^3(x) + \frac{2}{5} \cos^5(x) - \frac{1}{7} \cos^7(x) + C$$

Example:

Evaluate $\int \cos^2 x \sin 2x dx$

Solution:

Given $\int \cos^2 x \sin 2x dx$

$$= \int 2 \sin x \cos x \cos^2 x dx$$

$$= \int 2 \sin x \cos^3 x dx \quad (\text{Here, } m = 1, n = 3)$$

$$= 2 \int \sin x \cos^3 x dx \dots (1)$$

Put $u = \cos x$; $du = -\sin x dx$

$$(1) \Rightarrow 2 \int u^3 (-du) = -2 \int u^3 du$$

$$= -2 \frac{u^4}{4} + C = -\frac{1}{2} \cos^4 x + C$$

Example:

Evaluate $\int \sin^2 x \cos^4 x dx$

Solution:

Given $\int \sin^2 x \cos^4 x dx$ (Here, $m = 2, n = 4$)

$$= \int \left(\frac{1 - \cos 2x}{2}\right) \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx$$

$$= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) dx \right] \dots (1)$$

$$\int \cos^2 2x dx = \int \frac{1 + \cos 4x}{2} dx = \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right)$$

$$\int \cos^3 2x dx = \int \cos^2 2x \cos 2x dx = \int (1 - \sin^2 2x) \cos 2x dx$$

Put $u = \sin 2x$; $du = 2 \cos 2x dx$

$$\therefore \int \cos^3 2x dx = \int (1 - u^2) \frac{du}{2} = \frac{1}{2} \left[u - \frac{u^3}{3} \right] = \frac{1}{2} \left[\sin 2x - \frac{1}{3} \sin^3 2x \right]$$

$$\begin{aligned}
 (1) &\Rightarrow \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \frac{1}{2} x - \frac{1}{8} \sin 4x - \frac{1}{2} \sin 2x + \frac{1}{6} \sin^3 2x \right] + C \\
 &= \frac{1}{8} \left[\frac{1}{2} x - \frac{1}{8} \sin 4x + \frac{1}{6} \sin^3 2x \right] + C \\
 &= \frac{1}{16} \left[x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right] + C
 \end{aligned}$$

Example:

(i) Evaluate $\int \tan x \sec^3 x dx$

Solution:

$$\begin{aligned}
 \text{Given } \int \tan x \sec^3 x dx & \quad (\text{Here } m=1 \text{ (odd)}) \\
 &= \int \sec^2 x (\sec x \tan x) dx \\
 \text{Put } u &= \sec x; \quad du = \sec x \tan x dx \\
 &= \int u^2 du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C
 \end{aligned}$$

(ii) Evaluate $\int_0^{\pi/3} \tan^5 x \sec^4 x dx$

Solution:

$$\begin{aligned}
 \text{Given } \int_0^{\pi/3} \tan^5 x \sec^4 x dx & \quad (\text{Here } m=5 \text{ (odd)}) \\
 &= \int_0^{\pi/3} \tan^4 x \sec^3 x (\sec x \tan x) dx \\
 &= \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x (\sec x \tan x) dx \quad \dots (1) \\
 \text{Put } u &= \sec x \quad \text{when } x=0 \Rightarrow u=1 \\
 du &= \sec x \tan x dx \quad x = \frac{\pi}{3} \Rightarrow u=2 \\
 \therefore (1) &\Rightarrow \int_1^2 (u^2 - 1)^2 u^3 du = \int_1^2 (u^4 - 2u^2 + 1) u^3 du \\
 &= \int_1^2 (u^7 - 2u^5 + u^3) du \\
 &= \left[\frac{u^8}{8} - \frac{2u^6}{6} + \frac{u^4}{4} \right]_1^2 \\
 &= \left(4 - \frac{64}{3} + 32 \right) - \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8} \right) = \frac{117}{8}
 \end{aligned}$$

Example:

(i) Evaluate $\int \tan^2 x \sec^4 x dx$

Solution:

$$\begin{aligned}
 \text{Given } \int \tan^2 x \sec^4 x dx \\
 &= \int \tan^2 x \sec^2 x \sec^2 x dx \\
 &= \int \tan^2 x (1 + \tan^2 x) \sec^2 x dx \quad \dots (1) \\
 \text{Put } u &= \tan x; \quad du = \sec^2 x dx
 \end{aligned}$$

$$\begin{aligned}
 (1) \Rightarrow \int u^2(1+u^2) du &= \int (u^2 + u^4) du \\
 &= \left[\frac{u^3}{3} + \frac{u^5}{5} \right] + C \\
 &= \frac{1}{3} \tan^3(x) + \frac{1}{5} \tan^5(x) + C
 \end{aligned}$$

(ii) Evaluate $\int \tan x \sec^2 x dx$

Solution:

$$\text{Given } \int \tan x \sec^2 x dx$$

$$\begin{aligned}
 \text{Put } u &= \tan x ; \quad du = \sec^2 x dx \\
 &= \int u du \\
 &= \left[\frac{u^2}{2} \right] + C = \frac{1}{2} \tan^2(x) + C
 \end{aligned}$$

Example:

(i) Evaluate $\int \sec^3 x dx$

Solution:

$$\text{Given } I = \int \sec^3 x dx = \int \sec^2 x \sec x dx$$

$$\begin{aligned}
 \text{Put } u &= \sec x \quad dv = \sec^2 x dx \\
 du &= \sec x \tan x dx \quad v = \int \sec^2 x dx = \tan x
 \end{aligned}$$

$$\int u dv = uv - \int v du$$

$$I = (\sec x) \tan x - \int \tan x (\sec x \tan x) dx$$

$$= (\sec x) \tan x - \int \tan^2 x \sec x dx$$

$$= (\sec x) \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$= (\sec x) \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$= \sec x \tan x - I + \log(\sec x + \tan x)$$

$$2I = \sec x \tan x + \log(\sec x + \tan x)$$

$$I = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log(\sec x + \tan x) + C$$

(ii) Evaluate $\int \tan^2 x \sec x dx$

Solution:

$$\text{Given } \int \tan^2 x \sec x dx = \int (\sec^2 x - 1) \sec x dx$$

$$= \int \sec^3 x dx - \int \sec x dx$$

$$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \log(\sec x + \tan x) - \log(\sec x + \tan x) + C$$

Using example (3.53(i))

$$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \log(\sec x + \tan x) + C$$

Example:

(i) Evaluate $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$

Solution:

Given $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$ (Here $m = 7$, $n = 5$)

$$\begin{aligned} \int_0^{\pi/2} \sin^m x \cos^n x \, dx &= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \frac{1}{1+n} \quad (m \text{ is odd, } n \text{ even or odd}) \\ &= \frac{7-1}{7+5} \frac{7-3}{7+5-2} \cdots \frac{2}{3+5} \frac{1}{1+5} \\ &= \left(\frac{6}{12}\right) \left(\frac{4}{10}\right) \left(\frac{2}{8}\right) \left(\frac{1}{6}\right) = \frac{1}{120} \end{aligned}$$

(ii) Evaluate $\int_0^{\pi/2} \sin^7 x \, dx$

Solution:

Given $\int_0^{\pi/2} \sin^7 x \, dx$ (Here $m = 7$ (odd), $n = 0$)

$$\begin{aligned} \int_0^{\pi/2} \sin^m x \cos^n x \, dx &= \frac{m-1}{m+n} \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \frac{1}{1+n} \quad (m \text{ is odd, } n \text{ even or odd}) \\ &= \frac{7-1}{7+0} \frac{7-3}{7+0-2} \cdots \frac{2}{3+0} \frac{1}{1+0} = \left(\frac{6}{7}\right) \left(\frac{4}{5}\right) \left(\frac{2}{3}\right) (1) = \frac{16}{35} \end{aligned}$$

Example:

i) Evaluate $\int \sin 4x \cos 5x \, dx$

Solution:

Given $\int \sin 4x \cos 5x \, dx$

$$\begin{aligned} \text{We know that, } \sin A \cos B &= \frac{1}{2} [\sin(A-B)x + \sin(A+B)x] \\ &= \frac{1}{2} \int [\sin(-x) + \sin 9x] \, dx \\ &= \frac{1}{2} \int (-\sin x + \sin 9x) \, dx \end{aligned}$$

$$= \frac{1}{2} \left[\cos x - \frac{1}{9} \cos 9x \right] + C$$

ii) Evaluate $\int \cos 3x \cos 4x \, dx$

Solution:

Given $\int \cos 3x \cos 4x \, dx$

$$\begin{aligned} \text{We know that } \cos A \cos B &= \frac{1}{2} [\cos(A-B)x + \cos(A+B)x] \\ &= \frac{1}{2} \int (\cos x + \cos 7x) \, dx \\ &= \frac{1}{2} \left[\sin x + \frac{1}{7} \sin 7x \right] + C \\ &= \frac{1}{2} \sin x + \frac{1}{14} \sin 7x + C \end{aligned}$$

iii) Evaluate $\int \sin 5x \sin x \, dx$

Solution:

Given $\int \sin 5x \sin x \, dx$

We know that $\sin A x \sin B x = \frac{1}{2} [\cos(A - B) x - \cos(A + B) x]$

$$= \frac{1}{2} \int (\cos 4x - \cos 6x) dx$$

$$= \frac{1}{2} \left[\frac{1}{4} \sin 4x - \frac{1}{6} \sin 6x \right] + C$$

$$= \frac{1}{2} \sin 4x - \frac{1}{12} \sin 6x + C$$

