

Substitution Rule

Substitution Rule:

Let us see the suitable substitution to convert the given integral into a standard form.

The integrand of the form

$$(i) \int F(f(x)) f'(x) dx$$

$$(ii) \int (f(x))^n f'(x) dx$$

$$(iii) \int \frac{f'(x)}{(f(x))^n} dx$$

$$(iv) \int \frac{f'(x)}{F(f(x))} dx$$

$$(v) \int \frac{e^{f(x)}}{f'(x)} dx$$

$$(vi) \int e^{f(x)} f'(x) dx$$

Substitute $u = f(x) \therefore du = f'(x)$ and then proceed.

Algebraic functions:

Example:

(i) Evaluate $\int \sqrt{2x+1} dx$.

Solution:

$$\text{Put } u = 2x + 1 \Rightarrow du = 2dx \Rightarrow dx = \frac{du}{2}$$

$$\begin{aligned} \int \sqrt{2x+1} dx &= \int \sqrt{u} \frac{du}{2} = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right] + C \\ &= \frac{2}{2 \times 3} (u)^{3/2} + C = \frac{(2x+1)^{3/2}}{3} + C \end{aligned}$$

(ii) Evaluate $\int \frac{1}{(ax+b)^4} dx$.

Solution:

$$\text{Put } u = ax + b \Rightarrow du = a dx \Rightarrow dx = \frac{du}{a}$$

$$\begin{aligned} \int \frac{1}{(ax+b)^4} dx &= \int \frac{1}{u^4} \frac{du}{a} \\ &= \frac{1}{a} \int u^{-4} du \\ &= \frac{1}{a} \left[\frac{u^{-3}}{-3} \right] + C \\ &= \frac{-1}{3a} \left[\frac{1}{u^3} \right] + C = \frac{-1}{3a} \left[\frac{1}{(ax+b)^3} \right] + C \end{aligned}$$

(iii) Evaluate $\int x^5 \sqrt{x^2+1} dx$.

Solution:

$$\text{Put } u = x^2 + 1 \Rightarrow x^2 = u - 1; \quad du = 2x dx \Rightarrow x dx = \frac{du}{2}$$

$$\begin{aligned} \int x^5 \sqrt{x^2+1} dx &= \int x^4 \sqrt{x^2+1} x dx \\ &= \int \sqrt{u} (u-1)^2 \frac{du}{2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int \sqrt{u} (u^2 - 2u + 1) du \\
 &= \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2}) du \\
 &= \frac{1}{2} \left(\frac{u^{7/2}}{7/2} - \frac{2u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} \right) + C \\
 &= \left(\frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} - \frac{u^{3/2}}{3} \right) + C \\
 &= \left(\frac{(x^2+1)^{7/2}}{7} - \frac{2(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3} \right) + C
 \end{aligned}$$

(iv) Evaluate $\int \frac{x^2}{\sqrt{x+5}} dx$

Solution:

Given $\int \frac{x^2}{\sqrt{x+5}} dx$

Put $u = \sqrt{x+5} \Rightarrow du = \frac{1}{2\sqrt{x+5}} dx$
 $\Rightarrow 2du = \frac{1}{\sqrt{x+5}} dx$

$u^2 = x+5 \Rightarrow x = u^2 - 5 \Rightarrow x^2 = (u^2 - 5)^2 = u^4 - 10u^2 + 25$

$$\begin{aligned}
 \int \frac{x^2}{\sqrt{x+5}} dx &= \int (u^4 - 10u^2 + 25) 2 du = 2 \int (u^4 - 10u^2 + 25) du \\
 &= 2 \left[\frac{u^5}{5} - 10 \frac{u^3}{3} + 25u \right] + C \\
 &= \frac{2}{5} (x+5)^{5/2} - \frac{20}{3} (x+5)^{3/2} + 50(x+5)^{1/2} + C
 \end{aligned}$$

(v) Evaluate $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

Solution:

Given $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

Put $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx$
 $= \int \frac{1}{u^2} 2 du = 2 \int u^{-2} du = 2 \left(\frac{u^{-1}}{-1} \right) + C$
 $= -\frac{2}{u} + C$
 $= -\frac{2}{1+\sqrt{x}} + C$

(vi) Evaluate $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx$

Solution:

Given $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx$

$$\text{Put } u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx$$

$$\begin{aligned} \int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx &= \int u^{1/3} 2 du = 2 \int u^{1/3} du = 2 \frac{u^{4/3}}{(4/3)} + C \\ &= \frac{3}{2} (u)^{4/3} + C \\ &= \frac{3}{2} (1 + \sqrt{x})^{4/3} + C \end{aligned}$$

Logarithmic functions:

Example :

(i) Evaluate $\int \frac{\log x}{x} dx$

Solution:

$$\text{Given } \int \frac{\log x}{x} dx$$

$$\text{Put } u = \log x \Rightarrow du = \frac{1}{x} dx$$

$$\int \frac{\log x}{x} dx = \int u du = \frac{u^2}{2} + C = \frac{(\log x)^2}{2} + C$$

(ii) Evaluate: $\int \frac{(\log x)^2}{x} dx$

Solution:

$$\text{Given } \int \frac{(\log x)^2}{x} dx$$

$$\text{Put } u = \log x \Rightarrow du = \frac{1}{x} dx$$

$$\int \frac{(\log x)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(\log x)^3}{3} + C$$

(iii) Evaluate $\int \frac{\sin(2+\log x)}{x} dx$

Solution:

$$\text{Given } \int \frac{\sin(2+\log x)}{x} dx$$

$$\text{Put } u = 2 + \log x \Rightarrow du = \frac{1}{x} dx$$

$$\begin{aligned} \int \frac{\sin(2+\log x)}{x} dx &= \int \sin u du = -\cos u + C \\ &= -\cos(2 + \log x) + C \end{aligned}$$

(iv) Evaluate $\int \frac{dx}{x\sqrt{\log x}}$

Solution:

$$\text{Given } \int \frac{dx}{x\sqrt{\log x}}$$

$$\text{Put } u = \log x \Rightarrow du = \frac{1}{x} dx$$

$$\int \frac{dx}{x\sqrt{\log x}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{\log x} + C$$

(v) Evaluate $\int \sec x \log(\sec x + \tan x) dx$

Solution:

Given $\int \sec x \log(\sec x + \tan x) dx$

$$\text{Put } u = \log(\sec x + \tan x) \Rightarrow du = \frac{1}{(\sec x + \tan x)} (\sec x \tan x + \sec^2 x) dx$$

$$\Rightarrow du = \frac{\sec(\tan x + \sec x)}{(\sec x + \tan x)} dx$$

$$= \int u du = \frac{u^2}{2} + C$$

$$= \frac{1}{2} [\log(\sec x + \tan x)]^2 + C$$

Exponential functions

Example:

(i) Evaluate $\int e^{\cos x} \sin x dx$

Solution:

Given $\int e^{\cos x} \sin x dx$

$$\text{Put } u = e^{\cos x} \Rightarrow du = e^{\cos x} (-\sin x) dx$$

$$\int e^{\cos x} \sin x dx = \int (-du) = -\int du = -u + C = -e^{\cos x} + C$$

(ii) Evaluate $\int e^{x^3} x^2 dx$

Solution:

Given $\int e^{x^3} x^2 dx$

$$\text{Put } u = e^{x^3} \Rightarrow du = e^{x^3} 3x^2 dx \Rightarrow \frac{du}{3} = e^{x^3} x^2 dx$$

$$\int e^{x^3} x^2 dx = \int \frac{du}{3} = \frac{1}{3} \int du = \frac{1}{3} u + C = \frac{1}{3} e^{x^3} + C$$

(iii) Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution:

Given $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$\text{Put } u = e^{\sqrt{x}} \Rightarrow du = e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int 2du = 2 \int du = 2u + C = 2e^{\sqrt{x}} + C$$

(iv) Evaluate $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$

Solution:

$$\text{Given } \int \frac{e^{\tan^{-1}x}}{1+x^2} dx$$

$$\text{Put } u = \tan^{-1}x \quad du = \frac{1}{1+x^2} dx$$

$$\begin{aligned} \int \frac{e^{\tan^{-1}x}}{1+x^2} dx &= \int e^u du = e^u + C \\ &= e^{\tan^{-1}x} + C \end{aligned}$$

(v) Evaluate $\int \frac{1}{e^x + e^{-x}} dx$

Solution:

$$\text{Given } \int \frac{1}{e^x + \frac{1}{e^x}} dx = \int \frac{e^x dx}{e^{2x} + 1}$$

$$\text{Put } e^x = u \Rightarrow e^x dx = du$$

$$= \int \frac{du}{u^2 + 1}$$

$$= \tan^{-1}u + C = \tan^{-1}e^x + C$$