

Maxima and Minima for Functions of Two Variables

(a) Maximum value

$f(a, b)$ is a maximum value of $f(x, y)$, if there exists some neighbourhood of the point (a, b) such that for every point $(a + h, b + k)$ of the neighbourhood.

$$f(a, b) > f(a + h, b + k)$$

(b) Minimum value

$f(a, b)$ is a minimum value of $f(x, y)$, if there exists some neighborhood of the point (a, b) such that for every point $(a + h, b + k)$ of the neighborhood.

$$f(a, b) < f(a + h, b + k)$$

(c) Extremum value

$f(a, b)$ is said to be an extremum value of $f(x, y)$ if it is either a maximum or minimum.

(d) Necessary conditions for a maximum or a minimum.

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

$$\text{Notations: } f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

(e) Sufficient conditions:

If $f_x(a, b) = 0$ and $f_y(a, b) = 0$ and $f_{xx}(a, b) = A$, $f_{xy}(a, b) = B$, $f_{yy}(a, b) = C$, then

- (i) $f(a, b)$ is maximum value if $AC - B^2 > 0$ and $A < 0$ or $B < 0$
- (ii) $f(a, b)$ is minimum value if $AC - B^2 > 0$ and $A > 0$ or $B > 0$
- (iii) $f(a, b)$ is not an extremum (saddle) if $AC - B^2 < 0$
- (iv) if $AC - B^2 = 0$ then the test is inconclusive.

(f) Stationary value

A function $f(x, y)$ is said to be stationary at (a, b) or $f(a, b)$ is said to be Stationary value of $f(x, y)$ if $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Note:

Every extremum value is a stationary value but a stationary value need not be an extremum value

Example:

Find the extreme values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

Solution:

$$\text{Given } f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

$$f_x = 3x^2 - 3 \quad ; \quad f_y = 3y^2 - 12$$

$$f_{xx} = 6x = A, \quad f_{xy} = 0 = B, \quad f_{yy} = 6y = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$3x^2 - 3 = 0$	$3y^2 - 12 = 0$
$x^2 - 1 = 0$	$y^2 - 4 = 0$
$x = \pm 1$	$y = \pm 2$

$\therefore$ Stationary points are $(1, 2), (1, -2), (-1, 2), (-1, -2)$

	$(1, 2)$	$(1, -2)$	$(-1, 2)$	$(-1, -2)$
$A = 6x$	$6 > 0$	$6 > 0$	$-6 < 0$	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min. point	Saddle point	Saddle point	Max. point

$\therefore$ Maximum value of $f(x, y)$ is

$$\begin{aligned} f(-1, -2) &= (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20 \\ &= -1 - 8 + 3 + 24 + 20 = 38 \end{aligned}$$

Minimum value of $f(x, y)$ is

$$f(1, 2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20 = 2$$

Example:

A flat circular plate is heated so that the temperature at any point (x, y) is $u(x, y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.

Solution:

$$u(x, y) = x^2 + 2y^2 - x$$

$$u_x = 2x - 1 \quad u_y = 4y$$

$u_x = 0$	$u_y = 0$
$\Rightarrow 2x - 1 = 0$	$4y = 0$
$\Rightarrow x = \frac{1}{2}$	$y = 0$

$$A = u_{xx} = 2 ; C = u_{yy} = 4 \quad B = u_{xy} = 0$$

$$\Delta = AC - B^2 > 0$$

U is minimum at $\left(\frac{1}{2}, 0\right)$ and its minimum value is $-\frac{1}{4}$

Example:

Find the maxima and minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ [A.U Jan 2010]

Solution:

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$f_x = 4x^3 - 4x + 4y \quad ; \quad f_y = 4y^3 + 4x - 4y$$

$$f_{xx} = 12x^2 - 4 = A, \quad f_{xy} = 4 = B, \quad f_{yy} = 12y^2 - 4 = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$4x^3 - 4x + 4y = 0$	$4y^3 + 4x - 4y = 0$
$x^3 - x + y = 0 \dots (1)$	$y^3 + x - y = 0 \dots (2)$

$$(1) + (2) \Rightarrow x^3 + y^3 = 0 \Rightarrow x^3 = -y^3 \Rightarrow y = -x$$

$$(1) \Rightarrow x^3 - x - x = 0 \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$$

$$\Rightarrow x = 0 \text{ (or) } (x^2 - 2) = 0$$

$$\Rightarrow x = 0 \text{ (or) } x = \pm\sqrt{2}$$

$\therefore$ The stationary points are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

	$(0,0)$	$(\sqrt{2}, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$
$A = 12x^2 - 4$	$-4 < 0$	$20 > 0$	$20 > 0$
$B = 4$	4	4	4
$C = 12y^2 - 4$	-4	20	20
$AC - B^2$	0	$384 > 0$	$384 > 0$

Conclusion	Cannot be an extreme point	Minimum point	Minimum point
------------	----------------------------	---------------	---------------

Minimum at $(\sqrt{2}, -\sqrt{2})$

$$\begin{aligned}
 &= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 \\
 &= 4 + 4 - 4 - 8 - 4 \\
 &= -8
 \end{aligned}$$

Minimum at $(-\sqrt{2}, \sqrt{2})$

$$\begin{aligned}
 &= (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})\sqrt{2} - 2(\sqrt{2})^2 \\
 &= 4 + 4 - 8 - 4 - 4 = -8
 \end{aligned}$$

Example:

Find the maxima and minima values of $x^2 - xy + y^2 - 2x + y$

Solution:

$$\begin{aligned}
 f(x, y) &= x^2 - xy + y^2 - 2x + y \\
 f_x &= 2x - y - 2 \quad ; \quad f_y = -x + 2y + 1 \\
 f_{xx} &= 2 = A, \quad f_{xy} = -1 = B, \quad f_{yy} = 2 = C
 \end{aligned}$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$2x - y - 2 = 0 \dots (1)$	$-x + 2y + 1 = 0 \dots (2)$

$$(1) \Rightarrow 2x - y = 2$$

$$(1) \times 2 \Rightarrow -2x + 4y = -2$$

$$3y = 0$$

$$\Rightarrow y = 0$$

Substitute in (1), we get $2x - 2 = 0$

$$x - 1 = 0$$

$$x = 1$$

$\therefore$ Stationary point is $(1, 0)$

Now, $(AC - B^2)_{(1,0)} = 3 > 0$

Also, $A > 0, B < 0$

$\therefore (1, 0)$ is a minimum point. $\therefore$ Minimum value of $f(x, y)$ is -1 .

Example:

Find the extreme values of $f(x, y) = x^3y^2(1 - x - y)$.

Solution:

$$\begin{aligned}\text{Given } f(x, y) &= x^3y^2(1 - x - y) \\ &= x^3y^2 - x^4y^2 - x^3y^3 \\ f_x &= 3x^2y^2 - 4x^3y^2 - 3x^2y^3 \\ f_y &= 2x^3y - 2x^4y - 3x^3y^2 \\ f_{xx} &= 6xy^2 - 12x^2y^2 - 6xy^3 = A \\ f_{xy} &= 6x^2y - 8x^3y - 9x^2y^2 = B \\ f_{yy} &= 2x^3 - 2x^4 - 6x^3y = C\end{aligned}$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$	$2x^3y - 2x^4y - 3x^3y^2 = 0$
$x^2y^2(3 - 4x - 3y) = 0$	$x^3y(2 - 2x - 3y) = 0$
$\Rightarrow x = 0, y = 0, 4x + 3y = 3$	$\Rightarrow x = 0, y = 0, 2x + 3y = 2$

$$4x + 3y = 3 \dots (1) \quad 2x + 3y = 2 \dots (2)$$

$$(1) - (2) \Rightarrow 2x = 1 \quad ; \quad x = \frac{1}{2}$$

$$(1) - (2) \times 2 \Rightarrow -3y = -1 \quad ; \quad y = \frac{1}{3}$$

$\therefore$ Stationary points are $(0,0)$, $(\frac{1}{2}, \frac{1}{3})$, $(0,1)$, $(0, \frac{2}{3})$, $(\frac{3}{4}, 0)$ and $(1,0)$

Since Put $x = 0$ in (1), we get $3y = 3 \Rightarrow y = 1$, i.e., the point is $(0,1)$

Put $x = 0$ in (2), we get $3y = 2 \Rightarrow y = \frac{2}{3}$, i.e., the point is $(0, \frac{2}{3})$

Put $y = 0$ in (1), we get $4x = 3 \Rightarrow x = \frac{3}{4}$, i.e., the point is $(\frac{3}{4}, 0)$

Put $y = 0$ in (2), we get $2x = 2 \Rightarrow x = 1$, i.e., the point is $(1,0)$

Let $6xy^2 - 12x^2y^2 - 6xy^3 = A$

$6x^2y - 8x^3y - 9x^2y^2 = B$

$2x^3 - 2x^4 - 6x^3y = C$

	$(0,0)$	$(\frac{1}{2}, \frac{1}{3})$	$(0,1)$	$(0, \frac{2}{3})$	$(\frac{3}{4}, 0)$	$(1,0)$
--	---------	------------------------------	---------	--------------------	--------------------	---------

A	0	$\frac{-1}{9} < 0$	0	0	0	0
B	0	$\frac{-1}{12}$	0	0	0	0
C	0	$\frac{-1}{8}$	0	0	$\frac{27}{128}$	0
$AC - B^2$	0	$\frac{1}{144} > 0$	0	0	0	0
Conclusion	In conclusive	Max.point	In conclusive	In conclusive	In conclusive	In conclusive

Thus, $\left(\frac{1}{2}, \frac{1}{3}\right)$ is a maximum point

$$\therefore \text{Maximum value } f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left[1 - \frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{1}{432}$$

Example:

Find the extreme values of $f(x, y) = x^3 y^2 (12 - x - y)$.

Solution:

$$\text{Given } f(x, y) = x^3 y^2 (12 - x - y)$$

$$= 12x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$f_x = 36x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$f_y = 24x^3 y - 2x^4 y - 3x^3 y^2$$

$$f_{xx} = 72xy^2 - 12x^2 y^2 - 6xy^3$$

$$f_{xy} = 72x^2 y - 8x^3 y - 9x^2 y^2 = B$$

$$f_{yy} = 24x^3 - 2x^4 - 6x^3 y = C$$

To find the stationary points.

$f_x = 0$	$f_y = 0$
$36x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$	$24x^3 y - 2x^4 y - 3x^3 y^2 = 0$
$x^2 y^2 (36 - 4x - 3y) = 0$	$x^3 y (24 - 2x - 3y) = 0$

$\Rightarrow x = 0, y = 0, 4x + 3y = 36$	$\Rightarrow x = 0, y = 0, 2x + 3y = 24$
--	--

$$4x + 3y = 36 \dots (1) \quad 2x + 3y = 24 \dots (2)$$

$$(1) - (2) \Rightarrow 2x = 12 ; \therefore x = 6$$

$$\therefore (1) \Rightarrow (4)(6) + 3y = 36$$

$$24 + 3y = 36$$

$$3y = 12$$

$$y = 4$$

$\therefore$ The Stationary points are $(0,0), (6,4)$

	$(0,0)$	$(6,4)$
$72xy^2 - 12x^2y^2 - 6xy^3 = A$	0	$-2304 < 0$
$72x^2y - 8x^3y - 9x^2y^2 = B$	0	$-1728 < 0$
$24x^3 - 2x^4 - 6x^3y = C$	0	$-2592 < 0$
$AC - B^2$	0	$2985984 > 0$
	inconclusive	Max. point

$$A = (72)(6)(16) - (12)(36)(16) - (6)(6)(64)$$

$$= 6912 - 6912 - 2304 = -2304$$

$$B = (72)(36)(4) - 8(216)(4) - 9(36)(16)$$

$$= 10368 - 6912 - 5184 = -1728$$

$$C = 24(216) - 2(1296) - 6(216)(4)$$

$$= 5184 - 2592 - 5184 = -2592$$

$$AC - B^2 = (-2304)(-2592) - (-1728)^2$$

$$= 5971968 - 2985984 = 2985984 > 0$$

Thus $(6, 4)$ is a maximum point

$$\therefore \text{Maximum value } f(x, y) = f(6, 4) = (6)3(4)^2(12 - 6 - 4)$$

$$= (216)(16)(2) = 6912.$$