1.8 GAUSS'S LAW AND ITS APPLICATIONS

ELECTRIC FLUX ($\boldsymbol{\varkappa}$)

If the test charge is moved towards the charge $\boldsymbol{Q}$,the test charge will experience force due to the main charge $\boldsymbol{Q}$.The lines of force can be designated as electric flux which is equal to the charge itself. The electric flux $(\boldsymbol{\varkappa})$ eminates from electric charge$\boldsymbol{Q}$.

$$\boldsymbol{\varkappa = Q}$$

ELECTRIC FLUX DENSITY($\boldsymbol{D}$):

Electric flux density or Displacement density is defined as the electric flux per unit area

$$\boldsymbol{D = \frac{Q}{A} \quad Coulomb/metre^2}$$

For sphere surface area

$$\boldsymbol{A = 4\pi r^2}$$

Substitute $\boldsymbol{A}$ in $\boldsymbol{D}$

$$\boldsymbol{D = \frac{Q}{4\pi r^2}}$$

But

$$\boldsymbol{E = \frac{Q}{4\pi \varepsilon r^2}}$$

Substitute $\boldsymbol{D}$ in $\boldsymbol{E}$

$$\boldsymbol{E = \frac{D}{\varepsilon}}$$

$$\boldsymbol{D = \varepsilon E}$$

GAUSS'S LAW:

The gauss law states that the electric flux passing through any closed surface is equal to the total charge enclosed by the surface.

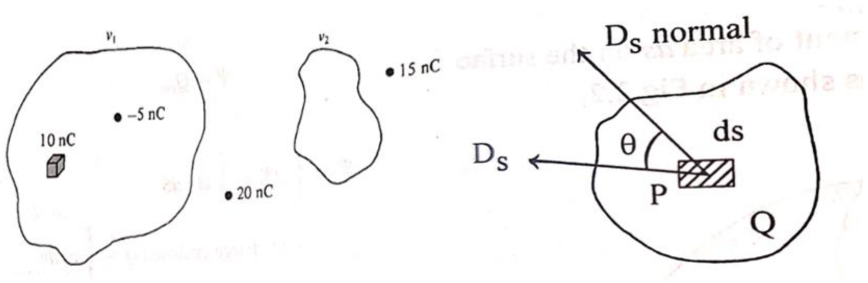

Figure 1.8.1 Illustration of Gauss's law

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-128]

Consider a small element of area $\boldsymbol{ds}$ in a plane surface having a charge $\boldsymbol{Q}$ and $\boldsymbol{P}$ be a point in the element. At every point on the surface the electric flux density $\boldsymbol{D}$ will have value $\boldsymbol{D_s}$. Let $\boldsymbol{D_s}$ make an angle $\boldsymbol{\theta}$ with $\boldsymbol{ds}$ as shown in figure 1.8.1.The flux crossing $\boldsymbol{ds}$ is the product of the normal component $\boldsymbol{D_s}$ and $\boldsymbol{ds}$.

$$dx = D_{s\,normal} \,.\, ds$$

$$D_{s\,normal} = D_s \cos\theta$$

$$dx = D_s \cos\theta\, ds$$

For dot product

$$dx = D_s \,.\, ds$$

a small element of area $\boldsymbol{ds}$ can also be written as $\boldsymbol{dA}$

$$dx = D_s \,.\, dA$$

The total flux passing through the closed surface is given by

$$x = \int dx = \oint_s D_s \,.\, dA$$

$$x = D_s A$$

Substitute $\boldsymbol{D_s}$ value in above equation

$$D_s = \frac{Q}{A}$$

$$x = \frac{Q}{A} A$$

$$x = Q$$

Proof:

Consider a charge Q at the origin of a spherical co-ordinate system, whose co-ordinates as r, θ, φ as shown in figure1.8.2.The radius of the sphere is r.

The electric flux density due to the charge Q is

$$D = \frac{Q}{A} \quad Coulomb/metre^2$$

For sphere surface area

$$A = 4\pi r^2$$

Substitute A in D

$$D = \frac{Q}{4\pi r^2}$$

But

$$E = \frac{Q}{4\pi\varepsilon r^2}$$

Substitute D in E

$$E = \frac{D}{\varepsilon}$$

$$D = \varepsilon E$$

$$E = \frac{Q}{4\pi\varepsilon r^2}$$

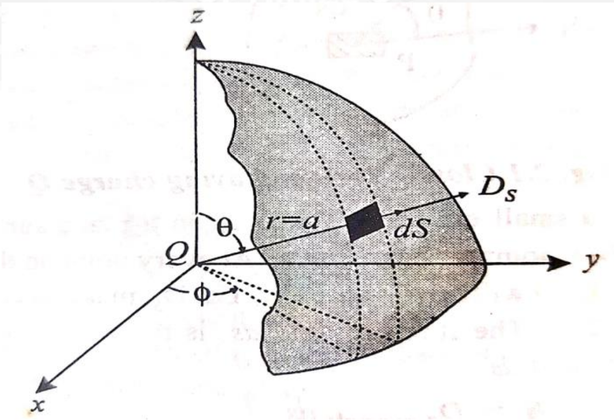

Figure 1.8.2 Surface element ds in the spherical co-ordinate

[Source: "Electromagnetic Theory" by Dr.P.Dananjayan, page-2.3]

Consider a small element of area $\boldsymbol{ds}$ on the surface of the sphere at a distance $\boldsymbol{r}$ from the origin as shown in figure. The sides of spherical co-ordinate system are $\boldsymbol{dr, rd\theta}$ ***and*** $\boldsymbol{r \sin\theta d\varphi}$.

The differential area $\boldsymbol{d_s = dr.rd\theta = r\ dr\ d\theta}$

$$\boldsymbol{d_s = rd\theta.r\sin\theta d\varphi = r^2 \sin\theta\ d\theta\ d\varphi}$$

$$\boldsymbol{d_s = r\sin\theta d\varphi\,.dr = r\sin\theta d\varphi\ \ dr}$$

For dot product

$$\boldsymbol{dx = D_s\,.ds}$$

Integrate the above equation on both sides

$$\int \boldsymbol{dx} = \int_s \boldsymbol{D_s\,.ds}$$

$$\int \boldsymbol{dx} = \iint \boldsymbol{D_s\,.ds}$$

Substitute $\boldsymbol{D_s}$ in above equation

$$\boldsymbol{D_s = \frac{Q}{4\pi r^2}}$$

$$\int \boldsymbol{dx} = \iint_s \boldsymbol{\frac{Q}{4\pi r^2}.ds}$$

Substitute $\boldsymbol{ds}$ in above equation

$$\boldsymbol{d_s = r^2\sin\theta\ d\theta\ d\varphi}$$

$$\int \boldsymbol{dx} = \iint_s \boldsymbol{\frac{Q}{4\pi r^2}\,.r^2\sin\theta\ d\theta\ d\varphi}$$

$$\boldsymbol{x} = \iint_s \boldsymbol{\frac{Q}{4\pi r^2}\,.r^2\sin\theta\ d\theta\ d\varphi}$$

$$x = \iint_{s} \frac{Q}{4\pi} \,.\sin\theta \, d\theta \, d\varphi$$

The limit for θ is $0 \ to \ \pi$

The limit for φ is $0 \ to \ 2\pi$

First integrate with respect to θ

$$x = \iint_{s} \frac{Q}{4\pi} \,.\sin\theta \, d\theta \, d\varphi$$

$$x = \frac{Q}{4\pi} \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta \, d\theta \, d\varphi$$

$$x = \frac{Q}{4\pi} \int_{\varphi=0}^{\varphi=2\pi} [-\cos\theta]_{0}^{\pi} \quad d\varphi$$

$$x = \frac{Q}{4\pi} \int_{\varphi=0}^{\varphi=2\pi} [\,(-\cos\pi) - (-\cos 0)] \quad d\varphi$$

$$x = \frac{Q}{4\pi} \int_{\varphi=0}^{\varphi=2\pi} [-(-1) - (-1)] \quad d\varphi$$

$$x = \frac{Q}{4\pi} \int_{\varphi=0}^{\varphi=2\pi} [\,(1) + (1)] \quad d\varphi$$

$$x = \frac{Q}{4\pi} \int_{\varphi=0}^{\varphi=2\pi} [\,(2)] \quad d\varphi$$

$$x = \frac{Q}{2\pi} \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

$$x = \frac{Q}{2\pi} \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

Next integrate with respect to $\boldsymbol{\varphi}$

$$x = \frac{Q}{2\pi} \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

$$x = \frac{Q}{2\pi} [\varphi]_0^{2\pi}$$

$$x = \frac{Q}{2\pi} [(2\pi) - (0)]$$

$$x = \frac{Q}{2\pi} [(2\pi)]$$

$$x = Q$$

The electric flux crossing the surface is equal to the charge enclosed by the surface.

The gauss's law can be stated in the point form as follows.

The divergence of electric flux density is equal to the volume charge density.

$$\nabla . D = \rho_v$$

APPLICATION OF GAUSS'S LAW:

The surface over which the gauss law is applied is called as Gaussian surface**.**

The gauss law is applied to the surface in following condition.

- The surface is closed
- The electric flux density $\boldsymbol{D}$ is either normal to the surface at each other.
- The electric flux density $\boldsymbol{D}$ is constant over the part of the surface.

INFINITE LINE CHARGE:

The infinite line of uniform charge $\boldsymbol{\rho_l\, c/m}$ lies along the $\boldsymbol{Z}$-axis**.** The line charge act on $\boldsymbol{x, y}$ – axis is zero. To determine $\boldsymbol{D}$ at a point $\boldsymbol{P}$**.**Choose a cylindrical surface containing $\boldsymbol{P}$ to satisfy the symmetry condition as shown in figure 1.8.3.The electric flux density $\boldsymbol{D}$ is constant on and normal to the cylindrical Gaussian surface.

Consider a cylindrical surface .The axis of cylindrical are $(\boldsymbol{x, y, z})$**.**The line charge act on the cylinder is$\boldsymbol{\rho_l}$**.**

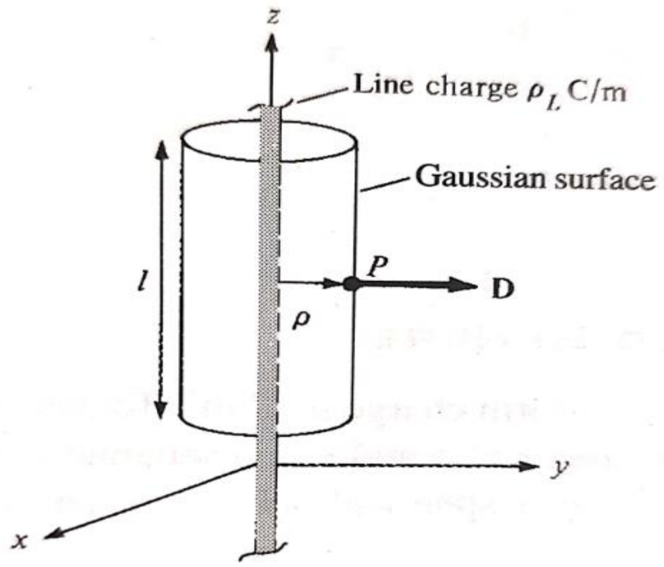

Figure 1.8.3 Gaussian surface about infinite line charge

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-129]

The electric flux density

$$\boldsymbol{D = \frac{Q}{A}}$$

$$\boldsymbol{Q = DA}$$

Surface area can be written as

$$\boldsymbol{A = \iint ds}$$

Substitute $\boldsymbol{A}$ in $\boldsymbol{Q}$

$$\boldsymbol{Q = \iint D\, ds}$$

Substitute

$$D = D_s$$

$$Q = \iint D_s \, ds$$

$$Q = \iint D_s \, ds + \iint D_s \, ds + \iint D_s \, ds$$

(Side) (Top) (Front)

The electric flux density in x *and* y *direction* is zero.

The electric flux density only occur in the side of the cylinder

$$Q = \iint D_s \, ds + 0 + 0$$

$$Q = \iint D_s \, ds$$

The sides of cylindrical co-ordinate systems are $dr, rd\varphi$ and dz

The differential area $d_s = dr \, . rd\varphi = rdrd\varphi$

$$d_s = rd\varphi dz$$

$$d_s = dz \, dr = drdz$$

Consider

$$d_s = rd\varphi dz$$

Substitute d_s in Q

$$Q = \iint D_s rd\varphi dz$$

$$Q = D_s \iint rd\varphi dz$$

The limit for the Z *is* 0 *to* l

The limit for the φ *is* 0 *to* 2π

$$Q = D_s \int_{z=0}^{z=l} \int_{\varphi=0}^{\varphi=2\pi} r d\varphi dz$$

$$Q = D_s r \int_{z=0}^{z=l} \int_{\varphi=0}^{\varphi=2\pi} d\varphi dz$$

$$Q = D_s r \int_{z=0}^{z=l} [\varphi]_0^{2\pi} dz$$

$$Q = D_s r \int_{z=0}^{z=l} [(2\pi) - (0)] dz$$

$$Q = D_s r \int_{z=0}^{z=l} [(2\pi)] dz$$

$$Q = D_s r \int_{z=0}^{z=l} 2\pi dz$$

$$Q = 2\pi D_s r \int_{z=0}^{z=l} dz$$

$$Q = 2\pi D_s r [z]_0^l$$

$$Q = 2\pi D_s r [(l) - (0)]$$

$$Q = 2\pi D_s r [(l)]$$

$$Q = 2\pi r l D_s$$

$$D_s = \frac{Q}{2\pi\, rl}$$

Area of Cylinder $\quad A = 2\pi rl$

The line charge density

$$\rho_l = \frac{Q}{l} \quad Coulomb/meter(c/m)$$

$$\rho_l = \frac{Q}{l}$$

$$Q = \rho_l l$$

Substitute $\boldsymbol{Q}$ value in $\boldsymbol{D_s}$ equation

$$D_s = \frac{Q}{2\pi\, rl}$$

$$D_s = \frac{\rho_l l}{2\pi\, rl}$$

$$D_s = \frac{\rho_l}{2\pi\, r}$$

The electric field for infinite line charge

$$D = \varepsilon E$$

Substitute $\quad D = D_s$

$$D_s = \varepsilon E$$

$$E = \frac{D_s}{\varepsilon}$$

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$E = \frac{D_s}{\varepsilon_0 \varepsilon_r}$$

$$Relative\ Permitivity\ (\varepsilon_r) = 1$$

$$E = \frac{D_s}{\varepsilon_0 \times 1}$$

$$E = \frac{D_s}{\varepsilon_0}$$

Substitute D_s value in E equation

$$E = \frac{\frac{\rho_l}{2\pi r}}{\varepsilon_0}$$

$$E = \frac{\rho_l}{\varepsilon_0 \times 2\pi r}$$

$$E = \frac{\rho_l}{2\pi r \varepsilon_0}$$

COAXIAL CYLINDER

Consider the two coaxial cylindrical conductors forming a coaxial cable. The radius of the inner cylinder is a while the radius of the outer cylinder is b.The coaxial cable is shown in figure 1.8.4.The length of cable is L.

The line charge density of inner cylinder is ρ_l. The line charge density of inner cylinder is$-\rho_l$.

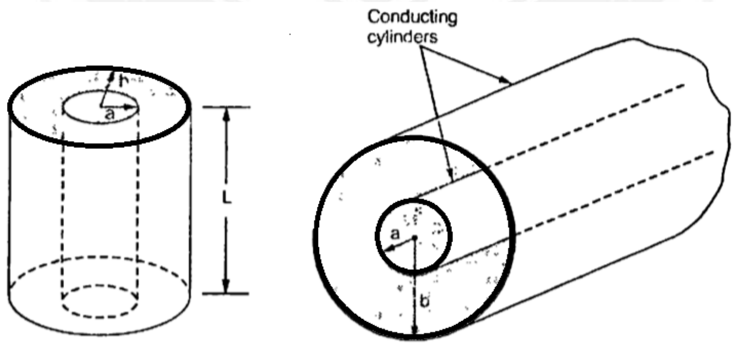

Figure 1.8.4 Coaxial Cable

[Source: "Electromagnetic Theory" by U.A.Bakshi, page-3.19]

In outer side the integral of electric flux density over a space is equal to charge.

$$\int Dd_s = Q$$

The line charge density

$$\rho_l = \frac{Q}{l} \quad Coulomb/meter(c/m)$$

$$\rho_l = \frac{Q}{l}$$

$$Q = \rho_l l$$

Substitute Q`in $\int Dd_s$

$$\int Dd_s = Q$$

$$\int Dd_s = \rho_l l$$

Substitute D`in $\int Dd_s$ equation

$$D = \varepsilon E$$

$$\int \varepsilon E\, d_s = \rho_l l$$

$$\varepsilon E \int d_s = \rho_l l$$

$$\int d_s = S = A$$

Substitute $\int d_s$`value in above equation

$$\varepsilon EA = \rho_l l$$

$$E = \frac{\rho_l l}{\varepsilon A}$$

Area of Cylinder $A = 2\pi rl$

$$E = \frac{\rho_l l}{\varepsilon 2\pi rl}$$

$$E = \frac{\rho_l}{\varepsilon 2\pi r}$$

$$E = \frac{\rho_l}{2\pi\varepsilon r}$$

INFINITE SHEET OF CHARGE

Consider an infinite sheet of uniform charge $\rho_S\ C/m^2$ lying on the $Z = 0$ plane.To determine D at point P.Choose rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in figure 1.8.5.As D is normal to the sheet.

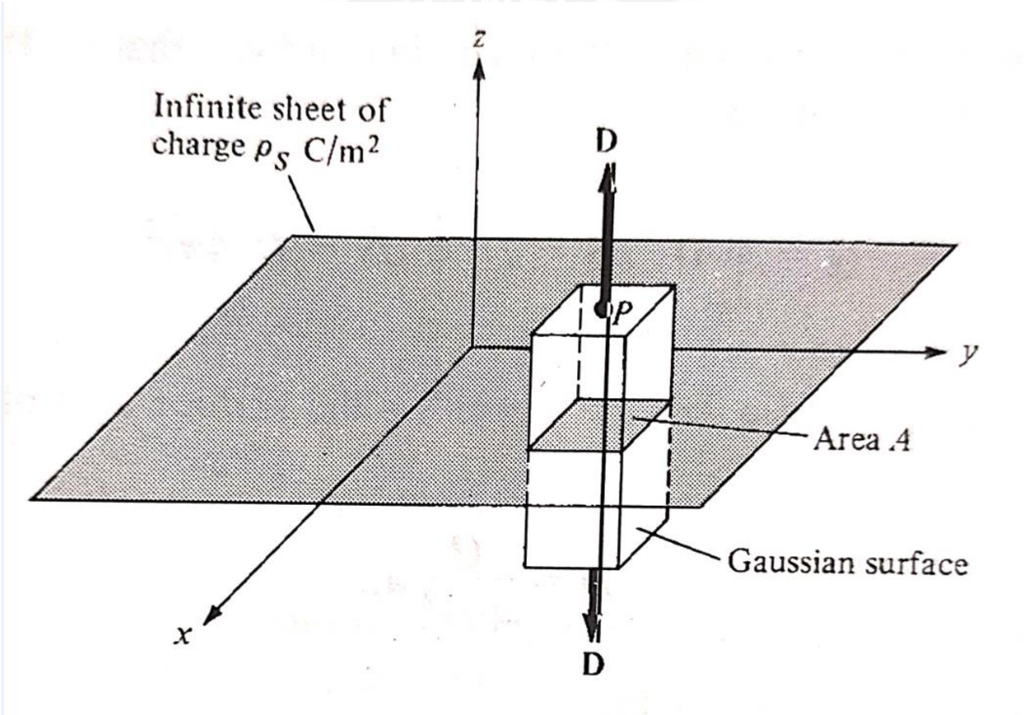

Figure 1.8.5 Gaussian surface about an infinite line sheet of charge

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-130]

$$Q = \iint D\, ds$$

Substitute

$$D = D_s$$

$$Q = \iint D_s\, ds$$

$$Q = \iint D_s\, ds + \iint D_s\, ds + \iint D_s\, ds$$

(Side) (Top) (Front)

The electric flux density in $\boldsymbol{x}\ and\ \boldsymbol{y}$ direction is zero.

The electric flux density only occur in the side of the infinite sheet

$$Q = \iint D_s\, ds + 0 + 0$$

$$Q = \iint D_s\, ds$$

The sides of rectangular co-ordinate systems are $\boldsymbol{dx}, \boldsymbol{dy}, \boldsymbol{dz}$

The differential area $$d_s = d_x d_y$$

$$d_s = d_y d_z$$

$$d_s = d_z d_x$$

Consider

$$d_s = d_x d_y$$

Substitute $\boldsymbol{d_s}$ in $\boldsymbol{Q}$

$$Q = \iint D_s d_x d_y$$

$$Q = D_s \iint d_x d_y$$

$$\iint d_x d_y = A$$

Substitute $\iint \boldsymbol{d_x d_y}$ in $\boldsymbol{Q}$

$$Q = D_s A$$

$$Q = \iint D_s d_x d_y + \iint D_s d_x d_y$$

(Top) **(Bottom)**

$$Q = \iint D_s d_x d_y + \iint D_s d_x d_y$$

Substitute $\iint \boldsymbol{d_x d_y}$ in $\boldsymbol{Q}$

$$\iint d_x d_y = A$$

$$Q = D_s A + D_s A$$

$$Q = 2\ D_s A$$

Surface charge density:

$$\rho_s = \frac{Q}{S} = \frac{Q}{A} \quad Coulomb/squaremeter(c/m^2)$$

$$\rho_s = \frac{Q}{S} = \frac{Q}{A}$$

$$Q = \ \rho_s S = \ \rho_s A$$

Substitute Q value in above Q equation

$$Q = 2\ D_s A$$

$$\rho_s A = 2\ D_s A$$

$$\rho_s = 2\ D_s$$

$$D_s = \frac{\rho_s}{2}$$

Substitute D_s value in E

$$D = \varepsilon E$$

$$E = \frac{D}{\varepsilon}$$

Consider

$$D = D_s$$

$$E = \frac{D_s}{\varepsilon}$$

$$E = \frac{D_s}{\varepsilon_0 \varepsilon_r}$$

$\varepsilon = \varepsilon_0 \varepsilon_r \quad \varepsilon_r = 1$

$$E = \frac{D_s}{\varepsilon_0 \times 1}$$

$$E = \frac{D_s}{\varepsilon_0}$$

Substitute $\boldsymbol{D_s}$ equation in $\boldsymbol{E}$

$$\boldsymbol{D_s = \frac{\rho_s}{2}}$$

$$\boldsymbol{E = \frac{\rho_s}{2\varepsilon_0}}$$

UNIFORMLY CHARGED SPHERE

Consider a sphere of radius $\boldsymbol{a}$ with a uniform charge $\boldsymbol{\rho_v\, C/m^3}$ .To determine $\boldsymbol{D}$ everywhere. Construct a Gaussian surfaces for case $\boldsymbol{r \le a}$ and $\boldsymbol{r \ge a}$ separately. Since charge has spherical symmetry. It is obvious that a spherical surface is an appropriate Gaussian surface.

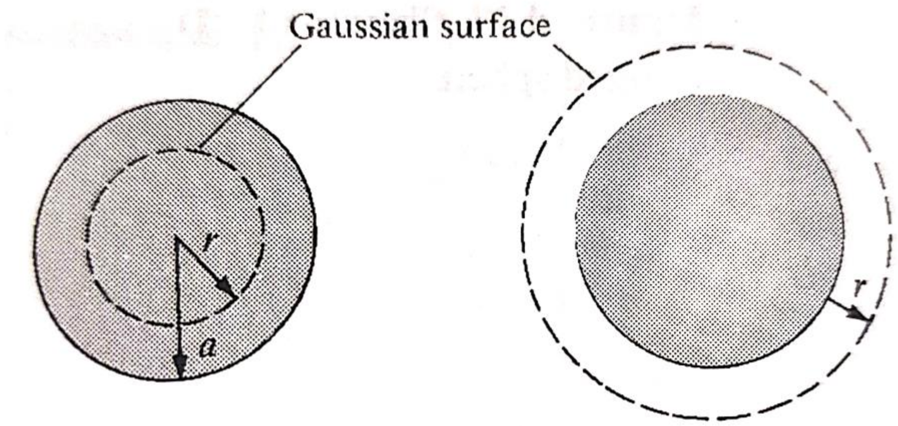

Figure 1.8.6 Gaussian surface for a uniformly charged sphere

[Source: "Elements of Electromagnetics" by Matthew N.O.Sadiku, page-131]

Case (i): The point $\boldsymbol{P}$ is outside the sphere $(\boldsymbol{r} > a)$

The Gaussian surface passing through through point $\boldsymbol{P}$ is a spherical surface of radius $\boldsymbol{r}$ The sides of spherical co-ordinate system are$\boldsymbol{dr, rd\theta\ and\ r\sin\theta d\varphi.}$

The differential area $\boldsymbol{d_s = dr.rd\theta = r\ dr\ d\theta}$

$$\boldsymbol{d_s = rd\theta. r\sin\theta d\varphi = r^2 \sin\theta\ d\theta\ d\varphi}$$

$$\boldsymbol{d_s = r\sin\theta d\varphi\,.\,dr = r\sin\theta d\varphi\ \ dr}$$

Consider differential area

$$\boldsymbol{d_s = r^2 \sin\theta\ d\theta\ d\varphi}$$

$$\boldsymbol{x = Q}$$

$$\boldsymbol{Q = \iint D\, ds}$$

$$\boldsymbol{x = Q = \iint D\, ds}$$

The limit for $\boldsymbol{\theta}$ is $\boldsymbol{0\ to\ \pi}$

The limit for $\boldsymbol{\varphi}$ is $\boldsymbol{0\ to\ 2\pi}$

First integrate with respect to $\boldsymbol{\theta}$

$$\boldsymbol{x = Q = \iint D\, ds}$$

$$\boldsymbol{Q = \iint D\, r^2 \sin\theta\, d\theta\, d\varphi}$$

$$\boldsymbol{Q = D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta\, d\varphi}$$

$$\boldsymbol{Q = D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} [-\cos\theta]_0^{\pi}\ d\varphi}$$

$$\boldsymbol{Q = D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} [(-\cos\pi) - (-\cos 0)]\ d\varphi}$$

$$\boldsymbol{Q = D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} [-(-1) - (-1)]\ d\varphi}$$

$$\boldsymbol{Q = D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} [(1) + (1)]\ d\varphi}$$

$$\boldsymbol{Q = D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} [(2)]\ d\varphi}$$

$$Q = 2D\,r^2 \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

$$Q = 2D\,r^2 \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

Next integrate with respect to $\boldsymbol{\varphi}$

$$Q = 2D\,r^2 \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

$$x = \frac{Q}{2\pi}\,[\varphi]_0^{2\pi}$$

$$Q = 2D\,r^2\,[(2\pi) - (0)]$$

$$Q = 2D\,r^2\,[(2\pi)]$$

$$Q = 4\pi D\,r^2$$

$$D = \frac{Q}{4\pi\,r^2}$$

Consider

$$D = \varepsilon E$$

$$E = \frac{D}{\varepsilon}$$

$\varepsilon = \varepsilon_0\varepsilon_r \quad \varepsilon_r = 1$

$$E = \frac{D}{\varepsilon_0\varepsilon_r}$$

$$E = \frac{D}{\varepsilon_0 \times 1}$$

$$E = \frac{D}{\varepsilon_0}$$

Substitute $\boldsymbol{D}$ in $\boldsymbol{E}$

$$\boldsymbol{D} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi\, r^2}}$$

$$\boldsymbol{E} = \frac{\frac{Q}{4\pi r^2}}{\boldsymbol{\varepsilon_0}}$$

$$\boldsymbol{E} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi\, r^2 \varepsilon_0}}$$

Total charge enclosed by volume

$$\boldsymbol{Q} = \int_{v}^{0} \boldsymbol{\rho_v\, dv}$$

The sides of spherical co-ordinate system are $\boldsymbol{dr, rd\theta\ and\ r\sin\theta d\varphi.}$

The differential Volume $\quad \boldsymbol{d_v = dr.rd\theta.r\sin\theta d\varphi}$

$$\boldsymbol{d_v = r^2 \sin\theta\, d\theta\, d\varphi\ dr}$$

Substitute $\boldsymbol{d_v}$ in $\boldsymbol{Q}$

$$\boldsymbol{Q} = \int_{v}^{0} \boldsymbol{\rho_v\, dv}$$

$$\boldsymbol{Q} = \int_{v}^{0} \boldsymbol{\rho_v\, r^2 \sin\theta\, d\theta\, d\varphi\ dr}$$

$$\boldsymbol{Q} = \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} \boldsymbol{\rho_v\, r^2 \sin\theta\, d\theta\, d\varphi\ dr}$$

The limit for $\boldsymbol{\theta}$ is $\boldsymbol{0\ to\ \pi}$

The limit for $\boldsymbol{\varphi}$ is $\boldsymbol{0\ to\ 2\pi}$

The limit for $\boldsymbol{r}$ is $\boldsymbol{0\ to\ a}$

First integrate with respect to $\boldsymbol{\theta}$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=a} [-\cos\theta]_0^{\pi} \, d\varphi \, r^2 \, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=a} [(-\cos\pi) - (-\cos 0)] \, d\varphi \, r^2 \, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=a} [[-(-1) - (-1)]] \, d\varphi \, r^2 \, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=a} [(1) + (1)] \, d\varphi \, r^2 \, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=a} [(2)] \, d\varphi \, r^2 \, dr$$

$$Q = 2\rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=a} d\varphi \, r^2 \, dr$$

Next integrate with respect to $\boldsymbol{\varphi}$

$$Q = 2\rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=a} d\varphi \, r^2 \, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=a} [\varphi]_0^{2\pi} \, r^2 \, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=a} [(2\pi) - (0)]\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=a} [(2\pi) - (0)]\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=a} [(2\pi) - (0)]\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=a} [(2\pi)]\, r^2\, dr$$

$$Q = 4\pi\rho_v \int_{r=0}^{r=a} r^2\, dr$$

Next integrate with respect to r

$$Q = 4\pi\rho_v \int_{r=0}^{r=a} r^2\, dr$$

$$Q = 4\pi\rho_v \left[\frac{r^3}{3}\right]_0^a$$

$$Q = 4\pi\rho_v \left[\left(\frac{a^3}{3}\right) - \left(\frac{0^3}{3}\right)\right]$$

$$Q = 4\pi\rho_v \left[\left(\frac{a^3}{3}\right) - 0\right]$$

$$Q = 4\pi\rho_v \left[\left(\frac{a^3}{3}\right)\right]$$

$$Q = \frac{4\pi\rho_v a^3}{3}$$

Consider electric field

$$E = \frac{Q}{4\pi\varepsilon r^2}$$

$\varepsilon = \varepsilon_0\varepsilon_r \quad \varepsilon_r = 1$

$$E = \frac{Q}{4\pi\varepsilon_0\varepsilon_r r^2}$$

$$E = \frac{Q}{4\pi\varepsilon_0 \times 1 \times r^2}$$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Substitute Q in E

$$E = \frac{\frac{4\pi\rho_v a^3}{3}}{4\pi\varepsilon_0 r^2}$$

$$E = \frac{4\pi\rho_v a^3}{3 \times 4\pi\varepsilon_0 r^2}$$

$$E = \frac{\rho_v a^3}{3\varepsilon_0 r^2}$$

Electric flux density

$$D = \varepsilon E$$

$\varepsilon = \varepsilon_0\varepsilon_r \quad \varepsilon_r = 1$

$$D = \varepsilon_0\varepsilon_r\, E$$

$$D = \varepsilon_0 \times 1 \times\; E$$

$$D = \varepsilon_0 E$$

Substitute E in D

$$E = \frac{\rho_v a^3}{3\varepsilon_0 r^2}$$

$$D = \varepsilon_0 E$$

$$D = \varepsilon_0 \frac{\rho_v a^3}{3\varepsilon_0 r^2}$$

$$D = \frac{\rho_v a^3}{3r^2}$$

Case (ii): The point P on the sphere $r = a$

Gaussian surface is same as the surface of the charged sphere

$$D = \frac{\rho_v a^3}{3r^2}$$

Substitute $r = a$ in above equation

$$D = \frac{\rho_v a^3}{3r^2}$$

$$D = \frac{\rho_v a^3}{3a^2}$$

$$D = \frac{\rho_v a}{3}$$

Case (ii): The point P is inside the sphere $r < a$

Gaussian surface is spherical surface of the r where $r < a$

The differential area $\quad d_s = dr.rd\theta = r\,dr\,d\theta$

$$d_s = rd\theta.r\sin\theta d\varphi = r^2 \sin\theta\, d\theta\, d\varphi$$

$$d_s = r\sin\theta d\varphi\,.dr = r\sin\theta d\varphi\ \ dr$$

Consider differential area

$$d_s = r^2 \sin\theta\, d\theta\, d\varphi$$

$$\boldsymbol{x} = Q$$

$$Q = \iint D\, ds$$

$$\boldsymbol{x} = \boldsymbol{Q} = \iint \boldsymbol{D}\, \boldsymbol{ds}$$

The limit for $\boldsymbol{\theta}$ is $\mathbf{0}\ \boldsymbol{to}\ \boldsymbol{\pi}$

The limit for $\boldsymbol{\varphi}$ is $\mathbf{0}\ \boldsymbol{to}\ \mathbf{2}\boldsymbol{\pi}$

First integrate with respect to $\boldsymbol{\theta}$

$$\boldsymbol{x} = \boldsymbol{Q} = \iint \boldsymbol{D}\, \boldsymbol{ds}$$

$$\boldsymbol{Q} = \iint \boldsymbol{D}\, \boldsymbol{r}^2 \sin\boldsymbol{\theta}\, \boldsymbol{d\theta}\, \boldsymbol{d\varphi}$$

$$\boldsymbol{Q} = \boldsymbol{D}\, \boldsymbol{r}^2 \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\boldsymbol{\theta}\, \boldsymbol{d\theta}\, \boldsymbol{d\varphi}$$

$$\boldsymbol{Q} = \boldsymbol{D}\, \boldsymbol{r}^2 \int_{\varphi=0}^{\varphi=2\pi} [-\cos\boldsymbol{\theta}]_{0}^{\pi}\ \boldsymbol{d\varphi}$$

$$\boldsymbol{Q} = \boldsymbol{D}\, \boldsymbol{r}^2 \int_{\varphi=0}^{\varphi=2\pi} [(-\cos\boldsymbol{\pi}) - (-\cos\mathbf{0})]\ \boldsymbol{d\varphi}$$

$$\boldsymbol{Q} = \boldsymbol{D}\, \boldsymbol{r}^2 \int_{\varphi=0}^{\varphi=2\pi} [-(-\mathbf{1}) - (-\mathbf{1})]\ \boldsymbol{d\varphi}$$

$$\boldsymbol{Q} = \boldsymbol{D}\, \boldsymbol{r}^2 \int_{\varphi=0}^{\varphi=2\pi} [(\mathbf{1}) + (\mathbf{1})]\ \boldsymbol{d\varphi}$$

$$\boldsymbol{Q} = \boldsymbol{D}\, \boldsymbol{r}^2 \int_{\varphi=0}^{\varphi=2\pi} [(\mathbf{2})]\ \boldsymbol{d\varphi}$$

$$\boldsymbol{Q} = \mathbf{2}\boldsymbol{D}\, \boldsymbol{r}^2 \int_{\varphi=0}^{\varphi=2\pi} \boldsymbol{d\varphi}$$

$$Q = 2D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

Next integrate with respect to φ

$$Q = 2D\, r^2 \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

$$x = \frac{Q}{2\pi}\, [\varphi]_0^{2\pi}$$

$$Q = 2D\, r^2\ [(2\pi) - (0)]$$

$$Q = 2D\, r^2\ [(2\pi)]$$

$$Q = 4\pi D\, r^2$$

$$D = \frac{Q}{4\pi\, r^2}$$

Consider D equation as

$$D = \varepsilon E$$

$$E = \frac{D}{\varepsilon}$$

$\varepsilon = \varepsilon_0 \varepsilon_r \quad \varepsilon_r = 1$

$$E = \frac{D}{\varepsilon_0 \varepsilon_r}$$

$$E = \frac{D}{\varepsilon_0 \times 1}$$

$$E = \frac{D}{\varepsilon_0}$$

Substitute D in E

$$D = \frac{Q}{4\pi\, r^2}$$

$$E = \frac{\frac{Q}{4\pi r^2}}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\, r^2 \varepsilon_0}$$

Electric flux density

$$D = \varepsilon E$$

$\varepsilon = \varepsilon_0 \varepsilon_r \quad \varepsilon_r = 1$

$$D = \varepsilon_0 \varepsilon_r\, E$$

$$D = \varepsilon_0 \times 1 \times\ E$$

$$D = \varepsilon_0 E$$

Substitute E value in above equation

$$D = \varepsilon_0 E$$

$$E = \frac{Q}{4\pi\, r^2 \varepsilon_0}$$

$$D = \varepsilon_0 \frac{Q}{4\pi\, r^2 \varepsilon_0}$$

$$D = \frac{Q}{4\pi\, r^2}$$

The enclosed by the sphere of radius r only and not by the entire sphere. The charge outside the Gaussian surface will not affect$D$.

Total charge enclosed by volume

$$Q = \int_v^0 \rho_v\, dv$$

The sides of spherical co-ordinate system are$dr, rd\theta\ and\ r\sin\theta d\varphi.$

The differential Volume $d_v =\ dr.rd\theta.r\sin\theta d\varphi$

$$d_v = \; r^2 \sin\theta \, d\theta \, d\varphi \; dr$$

Substitute d_v in Q

$$Q = \int_v^0 \rho_v \, dv$$

$$Q = \int_v^0 \rho_v \, r^2 \sin\theta \, d\theta \, d\varphi \; dr$$

$$Q = \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} \rho_v \, r^2 \sin\theta \, d\theta \, d\varphi \; dr$$

The limit for θ is $0 \; to \; \pi$

The limit for φ is $0 \; to \; 2\pi$

The limit for r is $0 \; to \; r$

First integrate with respect to θ

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=r} r^2 \sin\theta \, d\theta \, d\varphi \; dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=r} r^2 \sin\theta \, d\theta \, d\varphi \; dr$$

$$Q = \; \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} [-\cos\theta]_0^{\pi} \quad d\varphi \, r^2 \, dr$$

$$Q = \; \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} [\,(-\cos\pi) - (-\cos 0)] \quad d\varphi \, r^2 \, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} [\,[-(-1)-(-1)]\,] \quad d\varphi\, r^2\, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} [\,(1)+(1)] \quad d\varphi\, r^2\, dr$$

$$Q = \rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} [\,(2)] \quad d\varphi\, r^2\, dr$$

$$Q = 2\rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} d\varphi\, r^2\, dr$$

Next integrate with respect to $\boldsymbol{\varphi}$

$$Q = 2\rho_v \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=r} d\varphi\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=r} [\varphi]_0^{2\pi}\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=r} [(2\pi) - (0)]\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=r} [(2\pi) - (0)]\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=r} [(2\pi) - (0)]\, r^2\, dr$$

$$Q = 2\rho_v \int_{r=0}^{r=r} [(2\pi)]\, r^2\, dr$$

$$Q = 4\pi\rho_v \int_{r=0}^{r=r} r^2\, dr$$

Next integrate with respect to r

$$Q = 4\pi\rho_v \int_{r=0}^{r=r} r^2\, dr$$

$$Q = 4\pi\rho_v \left[\frac{r^3}{3}\right]_0^r$$

$$Q = 4\pi\rho_v \left[\left(\frac{r^3}{3}\right) - \left(\frac{0^3}{3}\right)\right]$$

$$Q = 4\pi\rho_v \left[\left(\frac{r^3}{3}\right) - 0\right]$$

$$Q = 4\pi\rho_v \left[\left(\frac{r^3}{3}\right)\right]$$

$$Q = \frac{4\pi\rho_v r^3}{3}$$

Substitute Q in D

$$D = \frac{Q}{4\pi\, r^2}$$

$$Q = \frac{4\pi\rho_v r^3}{3}$$

$$D = \frac{1}{4\pi\, r^2}\, \frac{4\pi\rho_v r^3}{3}$$

$$D = \frac{\rho_v r}{3}$$

Electric flux density

$$\boldsymbol{D = \varepsilon E}$$

$\boldsymbol{\varepsilon = \varepsilon_0 \varepsilon_r \quad \varepsilon_r = 1}$

$$\boldsymbol{D = \varepsilon_0 \varepsilon_r\, E}$$

$$\boldsymbol{D = \varepsilon_0 \times 1 \times\ E}$$

$$\boldsymbol{D = \varepsilon_0 E}$$

Substitute $\boldsymbol{D}$ value in above equation

$$\boldsymbol{D = \varepsilon_0 E}$$

$$\boldsymbol{\frac{\rho_v r}{3} = \varepsilon_0 E}$$

$$\boldsymbol{E = \frac{\rho_v r}{3\varepsilon_0}}$$