

Area enclosed by plane curves

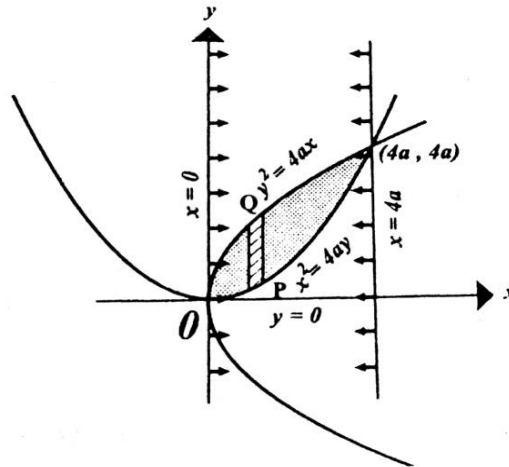
Area enclosed by plane curves (Cartesian coordinates)

$$\text{Area} = \int \int dy \, dx \quad (\text{or}) \quad \text{Area} = \int \int dx \, dy$$

Example:

Find the area between the parabola $y^2 = 4ax$ and $x^2 = 4ay$

Solution:



$$\text{Area} = \int \int dy \, dx$$

$$y : \frac{x^2}{4a} \rightarrow 2\sqrt{ax}$$

$$x : 0 \rightarrow 4a$$

$$= \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy \, dx$$

$$= \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} (2\sqrt{ax} - \frac{x^2}{4a}) dx$$

$$= \left[\frac{2\sqrt{a} x^{3/2}}{3/2} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \frac{4}{3} \sqrt{a} (4a)^{3/2} - \frac{(4a)^3}{12a}$$

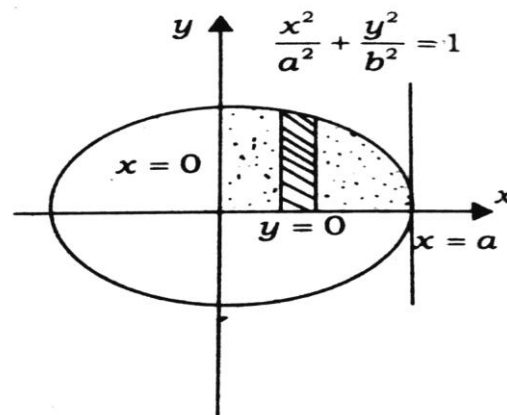
$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Example:

Find the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:



$$\text{Area} = 4 \iint dx dy$$

$$x : 0 \rightarrow \frac{a}{b} \sqrt{b^2 - y^2}$$

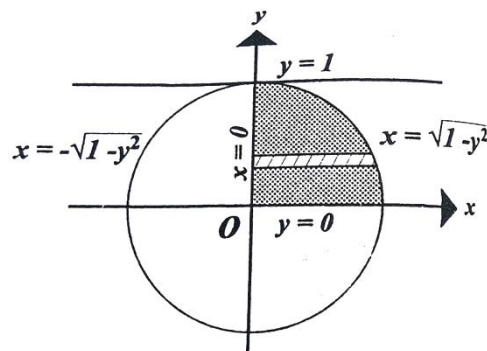
$$y : 0 \rightarrow b$$

$$\begin{aligned} \text{Area} &= 4 \int_0^b \int_0^{\frac{a}{b} \sqrt{b^2 - y^2}} dy dx \\ &= 4 \int_0^b [x]_0^{\frac{a}{b} \sqrt{b^2 - y^2}} dy \\ &= 4 \int_0^b \left[\frac{a}{b} \sqrt{b^2 - y^2} - 0 \right] dy \\ &= \frac{4a}{b} \left[\frac{b^2}{2} \sin^{-1} \left(\frac{y}{b} \right) + \frac{y}{2} \sqrt{b^2 - y^2} \right]_0^b \\ &= \frac{4a}{b} \left[\left(\frac{b^2}{2} \frac{\pi}{2} + 0 \right) - 0 \right] \\ &= \frac{4ab}{b} \frac{b^2}{2} \frac{\pi}{2} \\ &= \pi ab \end{aligned}$$

Example:

Evaluate $\iint xy \, dx \, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$

Solution:



$$x : 0 \rightarrow \sqrt{1-y^2}$$

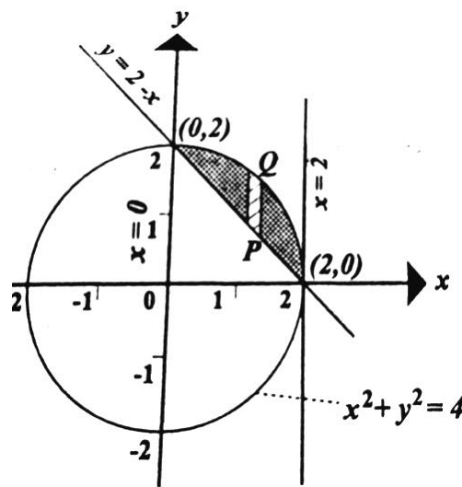
$$y : 0 \rightarrow 1$$

$$\begin{aligned} \iint xy \, dx \, dy &= \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy \\ &= \int_0^1 \left[\frac{x^2 y}{2} \right]_0^{\sqrt{1-y^2}} dy \\ &= \frac{1}{2} \int_0^1 (\sqrt{1-y^2})^2 y \, dy \\ &= \frac{1}{2} \int_0^1 (1-y^2) y \, dy \\ &= \frac{1}{2} \int_0^1 (y - y^3) \, dy \\ &= \frac{1}{2} \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 \\ &= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{1}{2} \left[\frac{1}{4} \right] = \frac{1}{8} \end{aligned}$$

Example:

Find the smaller of the area bounded by $y = 2 - x$ and $x^2 + y^2 = 4$

Solution:



$$\text{Area} = \iint dy \, dx$$

$$y : 2 - x \rightarrow \sqrt{4 - x^2}$$

$$x : 0 \rightarrow 2$$

$$\begin{aligned} &= \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy \, dx \\ &= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}} dx \\ &= \int_0^2 [\sqrt{4-x^2} - (2-x)] dx \\ &= \left[\frac{x}{2} \sqrt{4-x^2} + \frac{2^2}{2} \sin^{-1} \left(\frac{x}{2} \right) - 2x + \frac{x^2}{2} \right]_0^2 \end{aligned}$$

$$= 0 + \frac{4}{2} \left(\frac{\pi}{2} \right) - 4 + \frac{4^2}{2}$$

$$= \pi - 2 \text{ square unit}$$

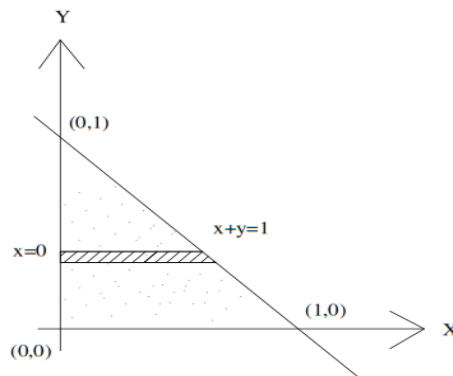
Example:

Evaluate $\iint xy \, dx \, dy$ over the positive quadrant for which $x + y \leq 1$

Solution:

$$x : 0 \rightarrow 1 - y$$

$$y : 0 \rightarrow 1$$

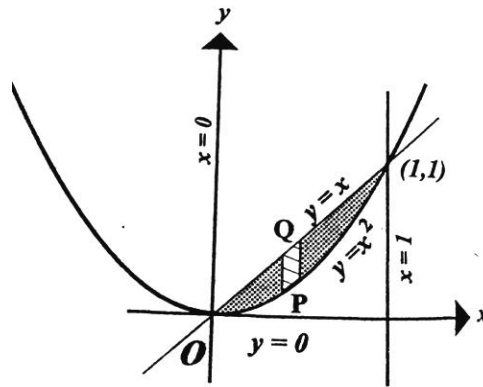


$$\begin{aligned} \iint xy \, dx \, dy &= \int_0^1 \int_0^{1-y} xy \, dx \, dy \\ &= \int_0^1 \left(\frac{x^2 y}{2} \right)_0^{1-y} dy \\ &= \int_0^1 \frac{1-y^2}{2} dy \\ &= \frac{1}{2} \int_0^1 (y^2 - 2y^2 + y^3) dy \\ &= \frac{1}{2} \left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1 \\ &= \frac{1}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{1}{2} \left[\frac{16-8+3}{12} \right] = \frac{1}{24} \end{aligned}$$

Example:

Using double integral find the area bounded by $y = x$ and $y = x^2$

Solution:



$$\text{Area} = \iint dy \, dx$$

$$y : x^2 \rightarrow x$$

$$x : 0 \rightarrow 1$$

$$= \int_0^1 \int_{x^2}^x dy \, dx$$

$$= \int_0^1 [y]_{x^2}^x \, dx$$

$$= \int_0^1 (x - x^2) \, dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example:

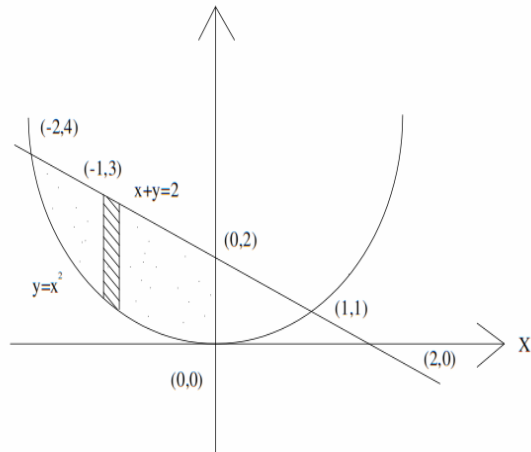
Find the area enclosed by the curves $y = x^2$ and $x + y - 2 = 0$

Solution:

$$\text{Given } y = x^2 \text{ and } x + y - 2 = 0$$

x	0	1	2	-1	-2
Y=2-x	2	1	0	3	4

x	0	1	-1	2	-2
$y = x^2$	0	1	1	4	4



$$\text{Area} = \iint dy \, dx$$

$$y : x^2 \rightarrow 2 - x$$

$$x : -2 \rightarrow 1$$

$$\begin{aligned} \int_{-2}^1 \int_{x^2}^{2-x} dy \, dx &= \int_{-2}^1 [y]_{x^2}^{2-x} dx \\ &= \int_{-2}^1 (2 - x - x^2) dx \\ &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^1 \\ &= \left[2 - \frac{1}{2} - \frac{1}{3} \right] - \left[-4 - \frac{4}{2} + \frac{8}{3} \right] = \frac{27}{6} \end{aligned}$$

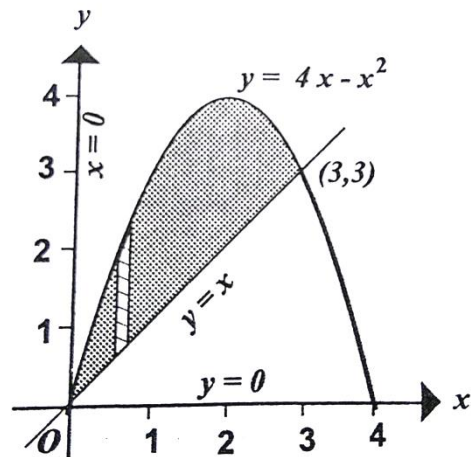
Example:

Find by double integration the area lying between the parabola $y=4x-x^2$ and the line $y=x$

Solution:

$$\text{Given } y = 4x - x^2 \text{ and } y = x$$

x	0	1	2	-1	-2	3
$y = 4x - x^2$	0	3	4	-5	-12	3



$$\text{Area} = \iint dy \, dx$$

$$y : x \rightarrow 4x - x^2$$

$$x : 0 \rightarrow 3$$

$$\begin{aligned} \int_0^3 \int_x^{4x-x^2} dy \, dx &= \int_0^3 [y]_x^{4x-x^2} dx \\ &= \int_0^3 (4x - x^2 - x) dx \\ &= \int_0^3 (3x - x^2) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2} \end{aligned}$$

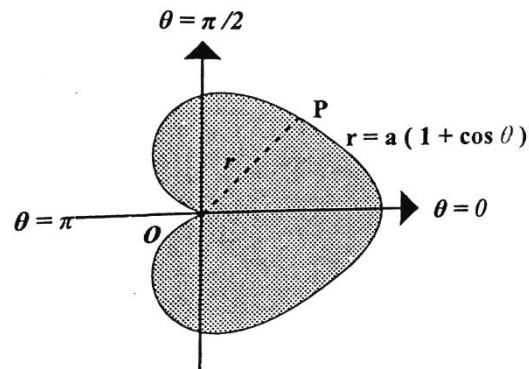
6 Area Enclosed by Plane Curves [Polar co-ordinates]

$$\text{Area} = \iint r \, dr \, d\theta$$

Example:

Find using double integral, the area of the cardioid $r = a(1 + \cos \theta)$

Solution:



$$\text{Area} = \iint r \, dr \, d\theta$$

The curve is symmetrical about the initial line

θ varies from : $0 \rightarrow \pi$

r varies from: $0 \rightarrow a(1 + \cos\theta)$

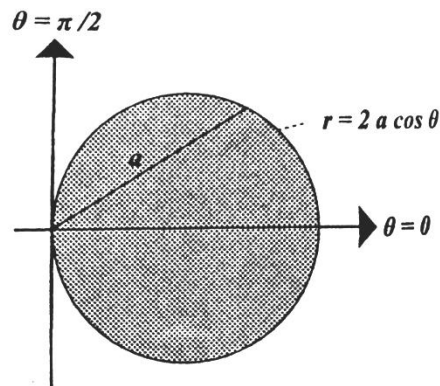
$$\begin{aligned}
 \text{Hence, required area} &= 2 \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1+\cos\theta)} r dr d\theta \\
 &= 2 \int_0^\pi \left[\frac{r^2}{2} \right]_{r=0}^{r=a(1+\cos\theta)} d\theta = \int_0^\pi [r^2]_{r=0}^{r=a(1+\cos\theta)} d\theta \\
 &= \int_0^\pi [a^2(1 + \cos\theta)^2 - 0] d\theta = a^2 \int_0^\pi [1 + \cos^2\theta + 2\cos\theta] d\theta \\
 &= a^2 \int_0^\pi \left[1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta \right] d\theta \\
 &= \frac{a^2}{2} \int_0^\pi [2 + 1 + \cos 2\theta + 4\cos\theta] d\theta = \frac{a^2}{2} \int_0^\pi [3 + \cos 2\theta + 4\cos\theta] d\theta \\
 &= \frac{a^2}{2} \left[3\theta + \frac{\sin 2\theta}{2} + 4\sin\theta \right]_0^\pi = \frac{a^2}{2} [(3\pi + 0 + 0) - (0 + 0 + 0)] \\
 &= \frac{3}{2} a^2 \pi \text{ square units.}
 \end{aligned}$$

Example:

Find the area of a circle of radius ‘a’ by double integration in polar co-ordinates.

Solution:

$$\text{Area} = \iint r dr d\theta$$



The equation of circle with pole on the circle and diameter through the point as initial line is

$$r = 2a \cos\theta$$

Area = 2 × upper area

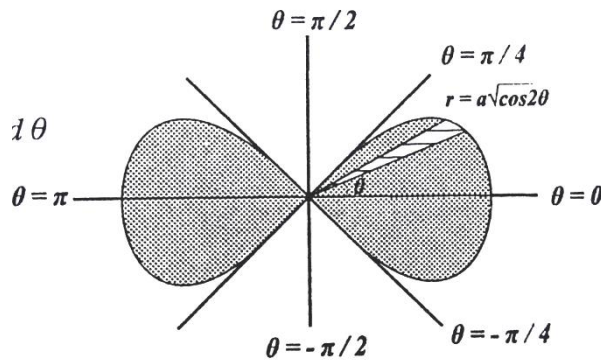
$$\begin{aligned}
 &= 2 \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=2a\cos\theta} r dr d\theta \\
 &= \int_0^{\frac{\pi}{2}} [r^2]_0^{2a\cos\theta} d\theta \\
 &= 4a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = 4a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi a^2 \text{ square units.}
 \end{aligned}$$

Example:

Find the area of the lemniscates $r^2 = a^2 \cos 2\theta$ by double integration.

Solution:

$$\text{Area} = \iint r dr d\theta$$



Area = 4 × area of upper half of one loop.

$$= 4 \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r dr d\theta$$

$$= 2 \int_0^{\pi/4} (r^2)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta d\theta$$

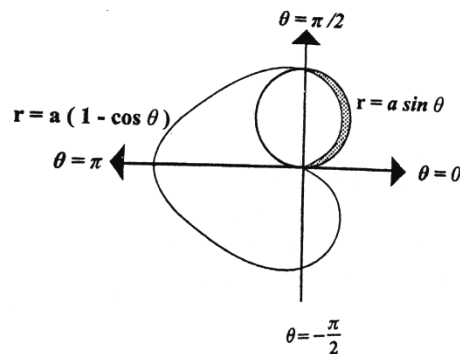
$$= 2a^2 \left(\frac{\sin 2\theta}{2} \right)_0^{\pi/4}$$

$$= a^2 \text{ square units.}$$

Example:

Find the area inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$

Solution:



From the figure, we get

θ varies from : $0 \rightarrow \frac{\pi}{2}$

r varies from: $a(1 - \cos\theta) \rightarrow a\sin\theta$

The required area = $\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=a(1-\cos\theta)}^{r=a\sin\theta} r dr d\theta$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} \right]_{r=a(1-\cos\theta)}^{r=a\sin\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{a^2 \sin^2 \theta}{2} - \frac{a^2 (1 - \cos\theta)^2}{2} \right] d\theta$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} [\sin^2 \theta - 1 - \cos^2 \theta + 2\cos\theta] d\theta$$

$$= \frac{a^2}{2} \left[\int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta - \int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta + 2 \int_0^{\frac{\pi}{2}} \cos\theta d\theta \right]$$

$$= \frac{a^2}{2} \left[-[\theta]_0^{\frac{\pi}{2}} + 2[\sin\theta]_0^{\frac{\pi}{2}} \right] \quad \because \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$= \frac{a^2}{2} \left[-\left(\frac{\pi}{2} - 0\right) + 2(1 - 0) \right] = \frac{1}{2} \frac{\pi}{2}$$

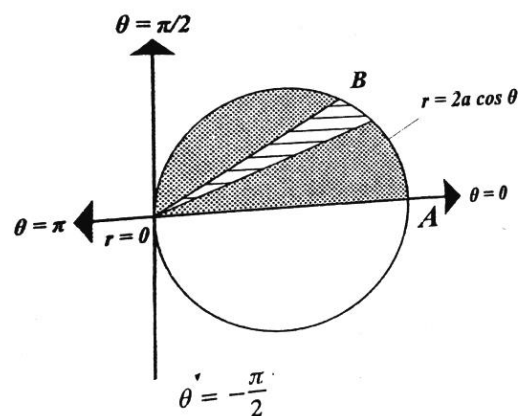
$$= \frac{a^2}{2} \left[2 - \frac{\pi}{2} \right]$$

$$= \frac{a^2}{4} (4 - \pi) \text{ square units.}$$

Example:

Evaluate $\int_R \int r^2 \sin\theta \, dr d\theta$ where R is the semi circle $r = 2a\cos\theta$ above the initial line.

Solution:



θ varies from : $0 \rightarrow \frac{\pi}{2}$

r varies from: $0 \rightarrow 2a\cos\theta$

Let $I = \iint r^2 \sin\theta \, dr d\theta$

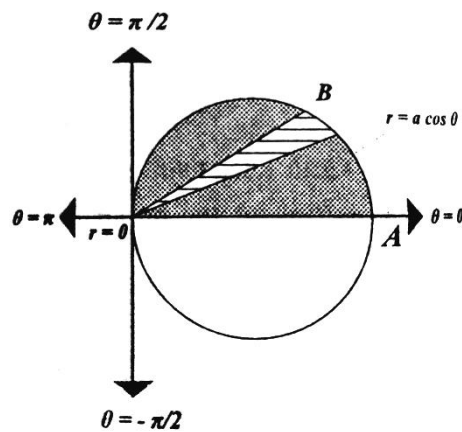
$$= \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} (r^2 \sin\theta) dr d\theta$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \sin \theta \right]_{r=0}^{r=2a \cos \theta} d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{8a^3}{3} \cos^3 \theta \sin \theta - 0 \right] d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{8a^3}{3} \cos^3 \theta \sin \theta d\theta \\
&= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3 \theta \sin \theta d\theta \\
&= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3 \theta d(\cos \theta) = \frac{8a^3}{3} \left[\frac{\cos^4 \theta}{4} \right]_0^{\frac{\pi}{2}} \\
&= \frac{8a^3}{3} \left[0 - \frac{1}{4} \right] = \frac{-2a^3}{3}
\end{aligned}$$

Example:

Evaluate $\int \int r \sqrt{a^2 - r^2} dr d\theta$ over the upper half of the circle $r = a \cos \theta$.

Solution:



θ varies from : $0 \rightarrow \frac{\pi}{2}$

r varies from: $0 \rightarrow a \cos \theta$

$$\begin{aligned}
\text{Let } I &= \int \int r \sqrt{a^2 - r^2} dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} dr d\theta
\end{aligned}$$

$ \begin{aligned} &\text{Put } a^2 - r^2 = t^2 \\ &-2r dr = 2t dt \\ &-r dr = t dt \\ &r \rightarrow 0 \quad \Rightarrow t \rightarrow a \\ &r \rightarrow a \cos \theta \quad \Rightarrow t \rightarrow \\ &\quad a \sin \theta \end{aligned} $
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$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_0^{a \sin \theta} \sqrt{t^2} (-t dt) d\theta \\
&= - \int_0^{\frac{\pi}{2}} \int_0^{a \sin \theta} t^2 dt d\theta = - \int_0^{\frac{\pi}{2}} \left[\frac{t^3}{3} \right]_{r=a}^{r=a \sin \theta} d\theta \\
&= - \frac{1}{3} \int_0^{\frac{\pi}{2}} [t^3]_a^{a \sin \theta} d\theta \\
&= - \frac{1}{3} \int_0^{\frac{\pi}{2}} [a^3 \sin^3 \theta - a^3] d\theta \\
&= - \frac{a^3}{3} \int_0^{\frac{\pi}{2}} [\sin^3 \theta - 1] d\theta \\
&= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} [1 - \sin^3 \theta] d\theta \\
&= \frac{a^3}{3} [\theta]_0^{\frac{\pi}{2}} - \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta \\
&= \frac{a^3}{3} \left[\frac{\pi}{2} - 0 \right] - \frac{a^3}{3} \left[\frac{2}{3} \cdot 1 \right] \qquad \left[\because \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta = \frac{2}{3} \cdot 1 \right] \\
&= \frac{a^3}{3} \frac{\pi}{2} - \frac{a^3}{3} \frac{2}{3} \\
&= \frac{a^3}{3} \left[\frac{\pi}{2} - \frac{2}{3} \right]
\end{aligned}$$