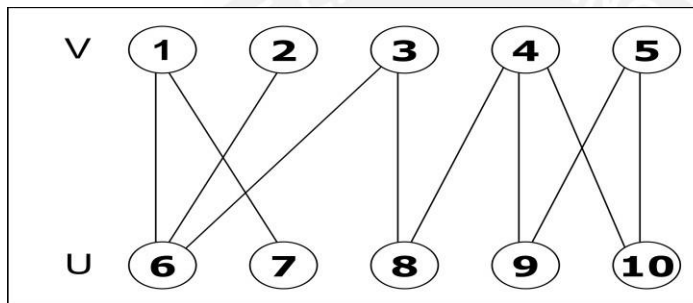


### 3. MAXIMUM MATCHING IN BIPARTITE GRAPHS

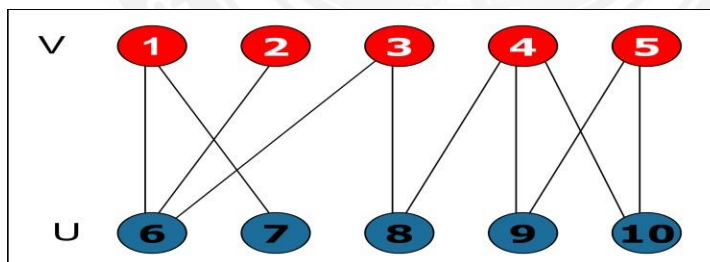
#### 4.2 Bipartite Graphs

*Bipartite graph*: a graph whose vertices can be partitioned into two disjoint sets  $V$  and  $U$ , not necessarily of the same size, so that every edge connects a vertex in  $V$  to a vertex in  $U$ .

A graph is bipartite if and only if it does not have a cycle of an odd length.

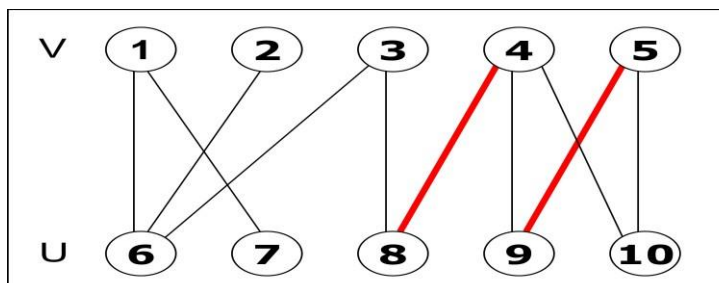


A bipartite graph is *2-colorable*: the vertices can be colored in two colors so that every edge has its vertices colored differently



#### Matching in a Graph

A *matching* in a graph is a subset of its edges with the property that no two edges



**share a vertex**

a matching in this graph  $M = \{(4,8), (5,9)\}$

*maximum (or maximum cardinality) matching* is a matching with the largest number of edges

- Always exists
- not always unique

**Free Vertices and Maximum Matching**

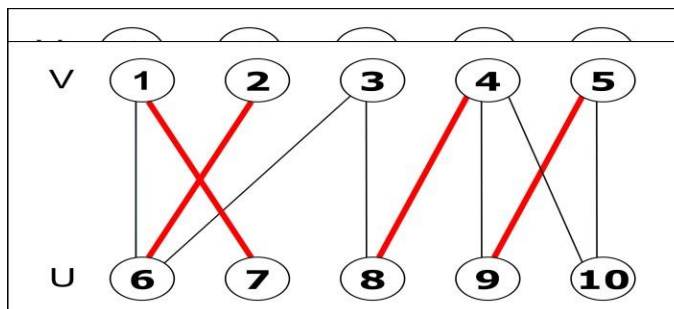
For a given matching  $M$ , a vertex is called *free* (or *unmatched*) if it is not an end point of any edge in  $M$ ; otherwise, a vertex is said to be *matched*

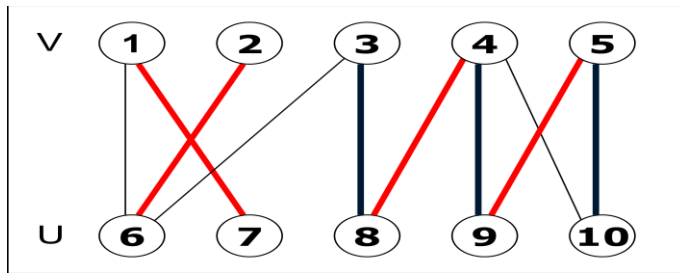
- If every vertex is matched, then  $M$  is a maximum matching
- If there are unmatched or free vertices, then  $M$  may be able to be improved
- We can immediately increase a matching by adding an edge connecting two free vertices (e.g., (1,6)above)
- Matched vertex = 4, 5, 8, 9. Free vertex = 1, 2, 3, 6, 7, 10.

**Augmenting Paths and Augmentation**

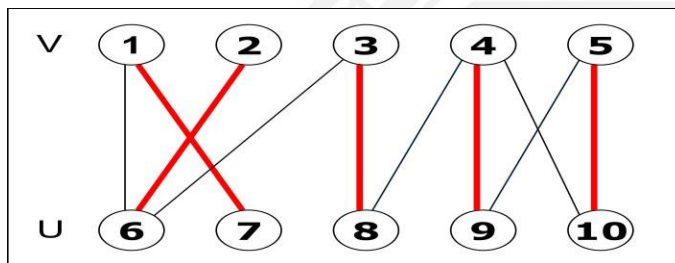
An *augmenting path* for a matching  $M$  is a path from a free vertex in  $V$  to a free vertex in  $U$  whose edges alternate between edges not in  $M$  and edges in  $M$

- The length of an augmenting path is always odd
- Adding to  $M$  the odd numbered path edges and deleting from it the even numbered path edges increases the matching size by 1(*augmentation*)
- One-edge path between two free vertices is special case of augmentin gpath





Augmentation along path 2,6,1,7



Augmentation along 3, 8, 4, 9, 5, 10

Matching on the right is maximum (*perfect matching*).

**Theorem:** A matching  $M$  is maximum if and only if there exists no augmenting path with respect to  $M$ .

#### Augmenting Path Method (template)

- Start with some initial matching . e.g., the emptyset
- Find an augmenting path and augment the current matching along that path. e.g., using breadth-first search like method
- When no augmenting path can be found, terminate and return the last matching, which is maximum