ANALYTIC FUNCTIONS – NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYTICITY IN CARTESIAN AND POLAR CO-ORDINATES

Analytic [or] Holomorphic [or] Regular function

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire Function: [Integral function]

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$.

Example: $e^z, \sin z, \cos z, sinhz, \cosh z$

Example: Show that $f(z) = log\ z$ analytic everywhere except at the origin and find its derivatives.

Solution:

Let $z = re^{i\theta}$

$$f(z) = log\ z$$

$$= \log(re^{i\theta}) = logr + \log(e^{i\theta}) = logr + i\theta$$

But, at the origin, $r = 0$. Thus, at the origin,

$$f(z) = log0 + i\theta = -\infty + i\theta$$

So, $f(z)$is not defined at the origin and hence is not differentiable there.

Note : $e^{-\infty} = 0$ $\log e^{-\infty} = \log 0\,; -\infty = log0$

At points other than the origin, we have

$u(r,\theta) = log\ r$	$v(r,\theta) = \theta$
$u_r = \frac{1}{r}$ $u_\theta = 0$	$v_r = 0$ $v_\theta = 1$

So, $logz$ satisfies the C−R equations.

Further $\frac{1}{r}$ is not continuous at $z = 0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z = 0$. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin. The derivative is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to 'r', we get

$$(i.e.)\ e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

Example: Check whether $w = \overline{z}$ is analytics everywhere.

Solution:

Let $w = f(z) = \overline{z}$

u+$iv = x - iy$

$u = x$	$v = -y$
$u_x = 1$ $u_y = 0$	$v_x = 0$ $v_y = -1$

$u_x \neq v_y$ at any point p(x,y)

Hence, C−R equations are not satisfied.

∴The function $f(z)$ is nowhere analytic.

Example: Test the analyticity of the function $w = \sin z$.

Solution:

Let $w = f(z) = sinz$

$u + iv = \sin(x + iy)$

$u + iv = \sin x \cos iy + \cos x \sin iy$

$u + iv = \sin x \cosh y + i \cos x \sin hy$

Equating real and imaginary parts, we get

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$ $u_y = \sin x \sinh y$	$v_x = -\sin x \sinh y$ $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

C −R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Solution:

Let $f(z) = 2xy + i(x^2 - y^2)$

$(i.e.)$

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not an analytic function.

Example: Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Solution:

Given $f(z) = \cosh z = \cos(iz) = \cos[i(x + iy)]$

$= \cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$

$u + iv = \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y$	$v = \sinh x \sin y$
u_x =sinh x cos y $u_y = -\cosh x \sin y$	v_x =cosh x sin y v_y =sinh x cos y

$\therefore u_x, u_y, v_x$ and v_y exist and are continuous.

$u_x = v_y$ and $u_y = -v_x$

C−R equations are satisfied.

$\therefore f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \sinh(x + iy) = \sinh z$

Example: If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$, and prove that $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$.

Solution:

Let $w = u(x,y) + iv(x,y)$

As $f(z)$ is analytic, we have $u_x = v_y, u_y = -v_x$

Now, $\frac{\boldsymbol{dw}}{\boldsymbol{dz}} = f'(z) = u_x + iv_x = v_y - iu_y = i(u_y + iv_y)$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}(u + iv) = -i\frac{\partial}{\partial y}(u + iv)$$

$$= \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

We know that, $\frac{\partial w}{\partial z} = 0$

$$\therefore \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$$

Also $\frac{\partial^2 w}{\partial \bar{z}\, \partial z} = 0$

Example: Prove that every analytic function $\boldsymbol{w = u(x,y) + iv(x,y)}$ can be expressed as a function of z alone.

Proof:

Let $z = x + iy$ and $\bar{z} = x - iy$

$x = \frac{z+\bar{z}}{2}$ and $y = \frac{z+\bar{z}}{2i}$

Hence, u and v and also w may be considered as a function of z and $\bar{z}$

Consider $\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$

$$= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right)$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$

$= 0$ by C−R equations as w is analytic.

This means that *w* is independent of $\bar{z}$

$(i.e.)$ *w* is a function of *z* alone.

This means that if $w = u(x,y) + iv(x,y)$ is analytic, it can be rewritten as a function of $(x + iy)$.

Equivalently a function of $\bar{z}$ cannot be an analytic function of *z*.

Example: Find the constants a, b, c if $\mathbf{f(z)} = (\boldsymbol{x + ay}) + \mathbf{i}(\boldsymbol{bx + cy})$ is analytic.

Solution:

$$f(z) = u(x,y) + iv(x,y)$$

$= (x + ay) + i(bx + cy)$

$u = x + ay$	$v = bx + cy$
$u_x = 1$ $u_y = a$	$v_x = b$ $v_y = c$

Given $f(z)$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ and $a = -b$

Example: Examine whether the following function is analytic or not $f(z) = e^{-x}(\cos y - i \sin y)$.

Solution:

Given $f(z) = e^{-x}(\cos y - i \sin y)$

$\Rightarrow u + iv = e^{-x} \cos y - ie^{-x} \sin y$

$u = e^{-x} \cos y$	$v = -e^{-x} \sin y$
$u_x = -e^{-x} \cos y$ $u_y = -e^{-x} \sin y$	$v_x = e^{-x} \sin y$ $v_y = -e^{-x} \cos y$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: Test whether the function $f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}\left(\frac{y}{x}\right)$ is analytic or not.

Solution:

Given $f(z) = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$(i.e.) u + iv = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$u = \frac{1}{2}\log(x^2 + y^2)$	$v = \tan^{-1}\left(\frac{y}{x}\right)$

$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x)$	$v_x = \frac{1}{1+\frac{y^2}{x^2}}\left[-\frac{y}{x^2}\right]$
$= \frac{x}{x^2+y^2}$	$= \frac{-y}{x^2+y^2}$
$u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y)$	$v_y = \frac{1}{1+\frac{y^2}{x^2}}\left[\frac{1}{x}\right]$
$= \frac{y}{x^2+y^2}$	$= \frac{x}{x^2+y^2}$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: Find where each of the following functions ceases to be analytic.

(i) $\frac{z}{(z^2-1)}$ **(ii)** $\frac{z+i}{(z-i)^2}$

Solution:

(i) Let $f(z) = \frac{z}{(z^2-1)}$

$$f'(z) = \frac{(z^2-1)(1)-z(2z)}{(z^2-1)^2} = \frac{-(z^2+1)}{(z^2-1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist.

$(i.e.)\ f'(z) \to \infty$

$(i.e.)(z^2-1)^2 = 0$

$(i.e.)\quad z^2 - 1 = 0$

$z = 1$

$z = \pm 1$

$\therefore f(z)$ is not analytic at the points $z = \pm 1$

(ii) Let $f(z) = \frac{z+i}{(z-i)^2}$

$$f'(z) = \frac{(z-i)^2(1)(z+i)[2(z-i)]}{(z-i)^4} = \frac{(z+3i)}{(z-i)^3}$$

$f'(z) \to \infty, at\ z = i$

$\therefore f(z)$ is not analytic at $z = i$.