DISCRETE FOURIER TRANSFORM

The discrete Fourier transform converts a finite sequence of equally-spaced samples of a function into a same-length sequence of equally-spaced samples of the discrete-time Fourier transform, which is a complex-valued function of frequency.

The N point DFT of x (n) can be expressed as

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

For k=0, 1, 2 ….N-1

Example-1

Compute the DFT of the sequence is given by

x (n)= {0,1,2,1}

Soln:

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

The given signal x (n) is 4 point signal. Let us compute 4 point DFT.

$X(K) = \sum_{n=0}^{4-1} x(n) e^{\frac{-j2\pi kn}{4}}$

$=\sum_{n=0}^{3} x(n) e^{\frac{-j\pi kn}{2}}$

$=x(0)e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{\frac{-j3\pi k}{2}}$

$= 0 + e^{\frac{-j\pi k}{2}} + 2\, e^{-j\pi k} + e^{\frac{-j3\pi k}{2}}$

$=\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + 2(\cos \pi k - j \sin \pi k) + \cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}$

$= (\cos\frac{\pi k}{2} + 2\cos \pi k + \cos\frac{3\pi k}{2}) - j(\sin\frac{\pi k}{2} + \sin\frac{3\pi k}{2})$

K=0, 1, 2, 3

When k=0;

X (0) = (cos 0+2 cos 0+cos 0)-j(sin 0+sin 0)

= (1+2+1)-j (0+0) =4

When k=1

$X(1) = (\cos\frac{\pi}{2} + 2\cos\pi + \cos\frac{3\pi}{2}) - j(\sin\frac{\pi}{2} + \sin\frac{3\pi}{2})$

= (0-2+0)-j (1-1) = -2

When k=2

$X(2) = (\cos\pi + 2\cos 2\pi + \cos 3\pi) - j(\sin\pi + \sin 3\pi)$

= (-1+2-1)-j(0+0)

=0

When k=3

$X(3) = (\cos\frac{3\pi}{2} + 2\cos 3\pi + \cos\frac{9\pi}{2}) - j(\sin\frac{3\pi}{2} + \sin\frac{9\pi}{2})$

= (0-2+0)-j (-1+1) =-2

Answer:

X(0)=4,X(1)=2,X(2)=0,X(3)=-2

PROBLEMS:

1. **Determine the 4-point DFT of the sequence x(n) =(1,0,1,0)**

DFT of the sequence is defined as

$$X(k) = \sum_{n=o}^{N-1} x(n)e^{-j2\pi\frac{kn}{N}} \quad k = 0, 1, \ldots (N-1)$$

$N = 4$

$$X(k) = \sum_{n=o}^{3} x(n)e^{-j2\pi\frac{kn}{4}} \quad = \sum_{n=o}^{3} x(n)e^{-j\pi\frac{kn}{2}}$$

$$X(k) = x(0)e^{0} + x(1)e^{-j\pi\frac{k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}}$$

Substitute $x(0), x(1), x(2)$ and $x(3)$ values

$$= 1 + 0 + 1e^{-j\pi k} + 0$$

$$X(k) = 1 + e^{-j\pi k}$$

Sub $k = 0$ in equation(1)

$$X(0) = 1 + e^{-j\pi(0)} = 1 + 1 = 2$$

Sub $k = 0$ in equation(1)

$$X(1) = 1 + e^{-j\pi} = 1 + \cos\pi - j\sin\pi$$
$$= 1 - 1 - j(0) = 0$$

Sub $k = 2$ in equation(1)

$$X(2) = 1 + e^{-j2\pi} = 1 + \cos 2\pi - j\sin 2\pi$$
$$= 1 + 1 - j(0) = 2.$$

Sub $k = 3$ in equation(1)

$$X(3) = 1 + e^{-j3\pi} = 1 + \cos 3\pi - j\sin 3\pi$$
$$= 1 + [-1 - j(0)] = 1 - 1 = 0$$

The output sequence is $\boxed{X(k) = \{2, 0, 2, 0\}}$

2. Find the 4-point DFT of the sequence x(n)=[1,1,-1,-1].(MAY/JUNE 2013)

$X_N=[W_N]\, x_N$

N=4

$X_4=[W_4]\, x_4$

$$W_4=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}1\\ 1\\ -1\\ -1\end{bmatrix}=\begin{bmatrix}1+1-1-1\\ 1-j+1-j\\ 1-1-1+1\\ 1+j+1+j\end{bmatrix}$$

$$X_4=\begin{bmatrix}0\\ 2-2j\\ 0\\ 2+2j\end{bmatrix}$$

3. Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

SOLN:

$$X(k)=\sum_{n=0}^{N-1} x(n)\, W_N^{nk}=\sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk}=\sum_{n=0}^{3} x(n)\, e^{-j(\pi/2)nk},\quad k=0,1,2,3$$

$$X(0)=\sum_{n=0}^{3} x(n)\, e^{0}=x(0)+x(1)+x(2)+x(3)=1-1+2-2=0$$

$$X(1)=\sum_{n=0}^{3} x(n)e^{-j(\pi/2)n}=x(0)+x(1)e^{-(j\pi/2)}+x(2)e^{-j\pi}+x(3)e^{-j(3\pi/2)}$$

$$=1+(-1)(0-j)+2(-1-j0)-2(0+j)$$

$$=-1-j$$

$$X(2)=\sum_{n=0}^{3} x(n)e^{-j\pi n}=x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}$$

$$=1-1(-1-j0)+2(1-j0)-2(-1-j0)=6$$

$$X(3)=\sum_{n=0}^{3} x(n)e^{-j(3\pi/2)n}=x(0)+x(1)e^{-j(3\pi/2)}+x(2)e^{-j3\pi}+x(3)e^{-j(9\pi/2)}$$

$$=1-1(0+j)+2(-1-j0)-2(0-j)=-1+j$$

$$X(k)=\{0,\ -1-j,\ 6,\ -1+j\}$$

4. Compute the DFT of the 3-point sequence $x(n) = \{2, 1, 2\}$. Using the same sequence, compute the 6-point DFT and compare the two DFTs.

Solution: The given 3-point sequence is $x(n) = \{2, 1, 2\}$, $N = 3$.

$$\text{DFT } x(n) = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{2} x(n) e^{-j(2\pi/3)nk}, \quad k = 0, 1, 2$$

$$= x(0) + x(1)e^{-j(2\pi/3)k} + x(2)e^{-j(4\pi/3)k}$$

$$= 2 + \left(\cos\frac{2\pi}{3}k - j\sin\frac{2\pi}{3}k\right) + 2\left(\cos\frac{4\pi}{3}k - j\sin\frac{4\pi}{3}k\right)$$

When $k = 0$, $X(k) = X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $X(k) = X(1) = 2 + \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right)$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866)$$

$$= 0.5 + j0.866$$

When $k = 2$, $X(k) = X(2) = 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$

$$= 0.5 - j0.866$$

$\therefore$ 3-point DFT of $x(n) = X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$

To compute the 6-point DFT, convert the 3-point sequence $x(n)$ into 6-point sequence by padding with zeros.

$$x(n) = \{2, 1, 2, 0, 0, 0\}, \quad N = 6$$

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{5} x(n) e^{-j(2\pi/N)nk}, \quad k = 0, 1, 2, 3, 4, 5$$

$$= x(0) + x(1)e^{-j(2\pi/6)k} + x(2)e^{-j(4\pi/6)k} + x(3)e^{-j(6\pi/6)k} + x(4)e^{-j(8\pi/6)k} + x(5)e^{-j(10\pi/6)k}$$

$$= 2 + e^{-j(\pi/3)k} + 2e^{-j(2\pi/3)k}$$

When $k = 0$, $X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $X(1) = 2 + e^{-j(\pi/3)} + 2e^{-j(2\pi/3)}$

$= 2 + (0.5 - j0.866) + 2(-0.5 - j0.866) = 1.5 - j2.598$

When $k = 2$, $X(2) = 2 + e^{-j(2\pi/3)} + 2e^{-j(4\pi/3)}$

$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866) = 0.5 + j0.866$

When $k = 3$, $X(3) = x(0) + x(1)e^{-j(3\pi/3)} + x(2)e^{-j(6\pi/3)}$

$= 2 + (\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi)$

$= 2 - 1 + 2 = 3$

When $k = 4$, $X(4) = x(0) + x(1)e^{-j(4\pi/3)} + x(2)e^{-j(8\pi/3)}$

$$= 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)$$

$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$

$= 0.5 - j0.866$

When $k = 5$, $X(5) = x(0) + x(1)e^{-j(5\pi/3)} + x(2)e^{-j(10\pi/3)}$

$$= 2 + \left(\cos\frac{5\pi}{3} - j\sin\frac{5\pi}{3}\right) + 2\left(\cos\frac{10\pi}{3} - j\sin\frac{10\pi}{3}\right)$$

$= 2 + (0.5 - j0.866) + 2(-0.5 + j0.866) = 1.5 + j0.866$

Tabulating the above 3-point and 6-point DFTs, we have

DFT	$X(0)$	$X(1)$	$X(2)$	$X(3)$	$X(4)$	$X(5)$
3-point	5	$0.5 + j0.866$	$0.5 - j0.866$	–	–	–
6-point	5	$1.5 - j2.598$	$0.5 + j0.866$	3	$0.5 - j0.866$	$1.5 + j0.866$

5. Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.

Soln:

Given $x(n) = \{1, -2, 3, 2\}$.

Here $N = 4$, $L = 4$. The DFT of $x(n)$ is $X(k)$.

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} = \sum_{n=0}^{3} x(n)e^{-j(2\pi/4)nk} = \sum_{n=0}^{3} x(n)e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n)e^{0} = x(0) + x(1) + x(2) + x(3) = 1 - 2 + 3 + 2 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j(\pi/2)n} = x(0) + x(1)e^{-j(\pi/2)} + x(2)e^{-j\pi} + x(3)e^{-j(3\pi/2)}$$

$= 1 - 2(0 - j) + 3(-1 - j0) + 2(0 + j) = -2 + j4$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$= 1 - 2(-1 - j0) + 3(1 - j0) + 2(-1 - j0) = 4$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j(3\pi/2)n} = x(0) + x(1)e^{-j(3\pi/2)} + x(2)e^{-j3\pi} + x(3)e^{-j(9\pi/2)}$$

$= 1 - 2(0 + j) + 3(-1 - j0) + 2(0 - j) = -2 - j4$

$\therefore \quad X(k) = \{4, -2 + j4, 4, -2 - j4\}$