

3.5. CONJUGATE BEAM METHOD

Conjugate beam is an imaginary beam of length equal to that of the original beam but for which the load diagram is the $\frac{M}{EI}$ diagram(i.e., the load at any pint on the conjugate beam is equal to the B.M at that point divided by EI).

1. The slope at any section of the given beam is equal to the shear force at the corresponding section of the conjugate beam.
2. The deflection at any section for the given beam is equal to the bending moment at the corresponding section of the conjugate beam.

Hence before applying the conjugate beam method, conjugate beam is constructed. The load on the conjugate beam at any point is equal to the B.M at that point divided by EI. Hence the loading on the conjugate beam is known. Then the shear force at any point on the conjugate beam gives the slope at the corresponding point of actual beam. And the B.M at any point on the conjugate beam gives the deflection at the corresponding point of the actual beam.

3.5.1 SLOPE AND DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH A POINT LOAD AT CENTRE

Fig. shows a simply supported beam AB of length L carrying a point load W at the centre C.

Since the beam is symmetrically loaded,

$$R_A = R_B = \frac{\text{Total load}}{2} = \frac{W}{2}$$

BM at the ends A and B= 0 (since A and B are simply supported ends)

$$\text{BM at Centre, } C = R_A \cdot \frac{L}{2} = \frac{W}{2} \cdot \frac{L}{2} = \frac{WL}{4}$$

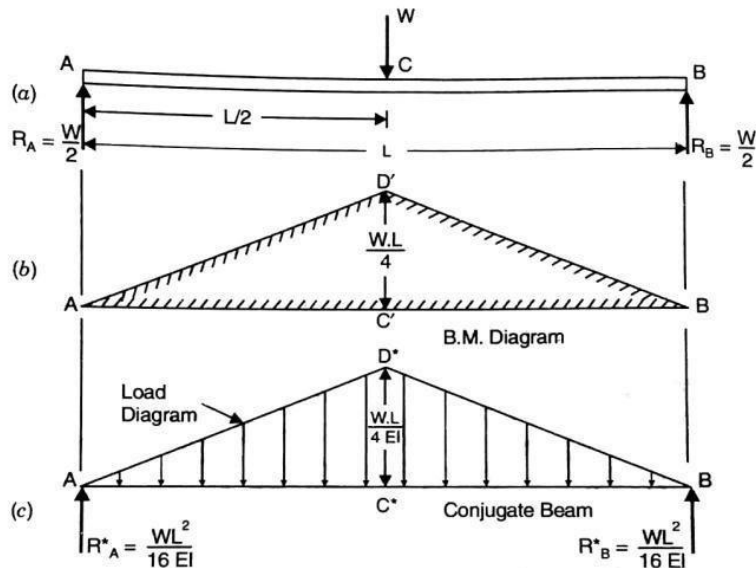
The B.M. diagram is shown in Fig.

Now the conjugate beam AB can be constructed. The load on the conjugate beam will be obtained by dividing the B.M at that point by EI.

The shape of the loading on the conjugate beam will be same as of B.M diagram.

$$WL$$

The ordinate of loading on conjugate beam will be equal to $\frac{M}{EI} = \frac{\frac{WL}{4}}{EI} = \frac{WL}{4EI}$



Let R_A^* = Reaction at A for conjugate beam

R_B^* = Reaction at B for conjugate beam

Total load on the conjugate beam

= Area of the load diagram

$$= \frac{1}{2} \times AB \times C'D^* = \frac{1}{2} \times L \times \frac{WL}{4EI}$$

$$= \frac{WL^2}{8EI}$$

Reaction at each support for the conjugate beam will be half of the total load

$$R_A^* = \frac{1}{2} \times \frac{WL^2}{8EI} = \frac{WL^2}{16EI} \quad R_B^* = \frac{WL^2}{16EI}$$

According to Conjugate beam method,

Slope at supports, θ_A = Shear force at A for the conjugate beam = R_A^*

$$= \frac{WL^2}{16EI} = \theta_B$$

And

y_C = B.M at C for the conjugate beam

= $R_A^* \times \frac{L}{2}$ - Load corresponding to AC^*D^* X Distance of C.G. of AC^*D^* from C

$$= \frac{WL^2}{16EI} \times \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4EI} \right) \times \left(\frac{1}{3} \times \frac{L}{2} \right)$$

$$= \frac{WL^3}{32EI} - \frac{WL^3}{96EI}$$

$$= \frac{WL^3}{48EI}$$

3.5.2.SLOPE AND DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH UNIFORMLY DISTRIBUTED LOAD

Fig. shows a simply supported beam AB of length L carrying a UDL w/m over its entire length

Since the beam is symmetrically loaded,

$$R_A = R_B = \frac{\text{Total load}}{2} = \frac{wL}{2}$$

BM at the ends A and B = 0 (since A and B are simply supported ends)

BM at Centre, $C = \frac{WL^2}{8}$

The B.M. diagram is shown in Fig.

Now the conjugate beam AB can be constructed. The load on the conjugate beam will be obtained by dividing the B.M at that point by EI.

The shape of the loading on the conjugate beam will be same as of B.M diagram.

WL^2

The ordinate of loading on conjugate beam will be equal to $\frac{M}{EI} = \frac{\frac{WL^2}{8}}{EI} = \frac{WL^2}{8EI}$

Let R_A^* = Reaction at A for conjugate beam

R_B^* = Reaction at B for conjugate beam

Total load on the conjugate beam

= Area of the load diagram

$$\begin{aligned} &= \frac{2}{3} \times L \times \frac{WL^2}{8EI} \\ &= \frac{WL^3}{12EI} \end{aligned}$$

Reaction at each support for the conjugate beam will be half of the total load

$$\begin{aligned} R_A^* &= R_B^* = \frac{1}{2} \times \frac{WL^3}{12EI} \\ &= \frac{WL^3}{24EI} \end{aligned}$$

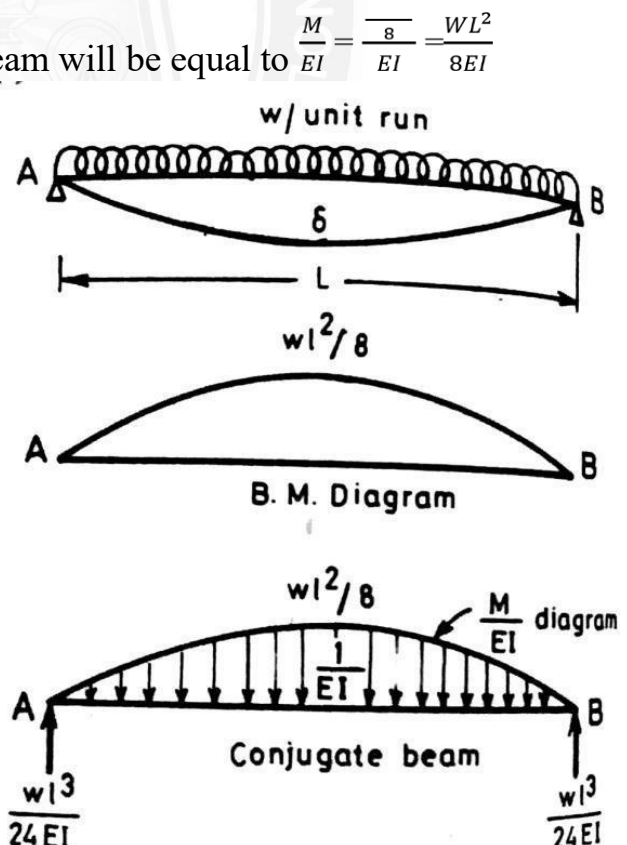
According to Conjugate beam method,

Slope at supports, θ_A = Shear force at A for the conjugate beam = R_A^*

$$= \frac{WL^3}{24EI} = \theta_B$$

And

y_c = B.M at C for the conjugate beam



$$\begin{aligned}
 &= R_A * \times \frac{L}{2} - \text{Load corresponding to AC*D* X Distance of} \\
 &\quad \text{C.G. of AC*D* from C} \\
 &= \frac{WL^3}{24EI} \times \frac{L}{2} - \left(\frac{2}{3} \times \frac{L}{2} \times \frac{WL^2}{8EI} \right) \times \left(\frac{3}{8} \times \frac{L}{2} \right) \\
 &= \frac{WL^4}{48EI} - \frac{6WL^4}{768EI} = \frac{5WL^4}{384EI}
 \end{aligned}$$

Example.3.5.1. A beam ABCD is simply supported at it's A and D over a span of 30 m. It is made up of three portions AB, BC and CD each 10 metres in length. The moments of inertia of the section of these portions are I, 3I and 2I respectively. Where $I = 2 \times 10^{10} \text{mm}^4$. The beam carries a point load of 150 kN at B and a point load of 300 kN at C. Neglecting the weight of the beam calculate the slopes and deflections at A, B, C and D. Take $E = 200 \text{ kN/mm}^2$.

Solution:

Let R_A and R_D be the reactions at the supports. Taking moment about D, We have,

$$R_D \times 30 - 150 \times 10 - 300 \times 20 = 0$$

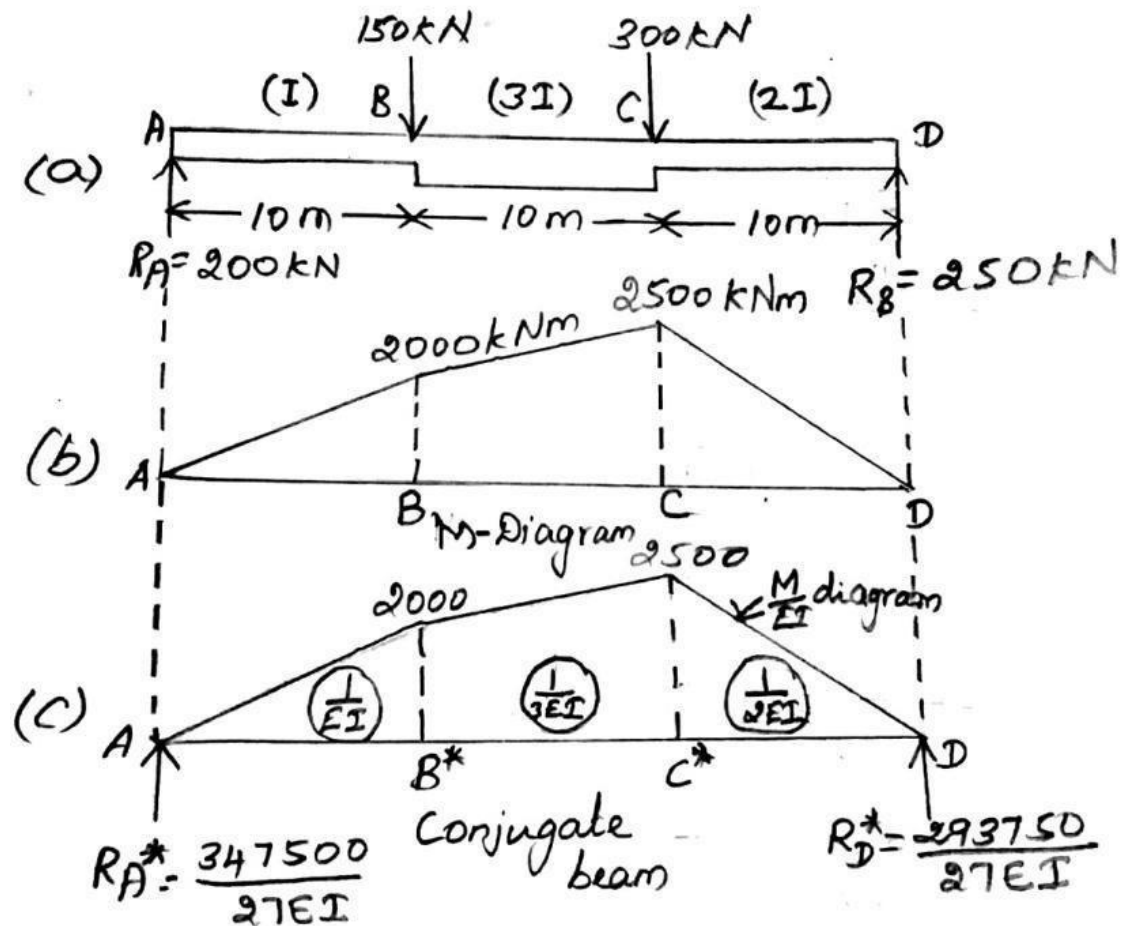
$$R_D \times 30 = 150 \times 10 + 300 \times 20$$

$$R_D = 250 \text{ kN}$$

$$R_A + R_D = 150 + 300$$

$$R_A + 250 = 450$$

$$R_A = 450 - 250 = 200 \text{ kN.}$$



B.M at A = 0

B.M at B = $200 \times 10 = 2000 \text{ kNm}$

B.M at C = $200 \times 20 - 150 \times 10 = 2500 \text{ kNm}$.

B.M at D = 0

Fig. (b) shows the B.M diagram for the given beam.

Fig (c) shows the $\frac{M}{EI}$ diagram which is the loading on the conjugate beam. The thickness on the diagram is $\frac{1}{EI}$ for the portion AB, $\frac{1}{3EI}$ for the portion BC and $\frac{1}{2EI}$ for the portion CD. The properties of the load on the conjugate beam are given below:

| Load component | Magnitude | Distance From A | Moment About A |
|---|--------------------|-----------------|----------------------|
| Load on AB* = $\frac{1}{2} \times 2000 \times \frac{1}{EI}$ | $\frac{10000}{EI}$ | $\frac{20}{3}$ | $\frac{200000}{3EI}$ |

| | | | |
|---|---------------------|----------------|-----------------------|
| $\text{Load on BC}^* = 2000 \times 10 \times 500 \times \frac{1}{3EI}$ $\frac{1}{2} \times 10 \times 500 \times \frac{1}{3EI}$ | $\frac{20000}{3EI}$ | 15 | $\frac{100000}{EI}$ |
| | $\frac{2500}{3EI}$ | $\frac{50}{3}$ | $\frac{125000}{9EI}$ |
| $\text{Load on CD}^* = \frac{1}{2} \times 10 \times 2500 \times \frac{1}{2EI}$ | $\frac{6250}{EI}$ | $\frac{70}{3}$ | $\frac{437500}{3EI}$ |
| Total | $\frac{71250}{3EI}$ | | $\frac{2937500}{9EI}$ |

Let R_A^* and R_D^* be the reactions at A and D for the conjugate beam.

Taking moments about A, we have,

$$R_D^* \times 30 = \frac{2937500}{9EI}$$

$$\therefore R_D^* = \frac{2937500}{270EI} = \frac{293750}{27EI}$$

$$R_A^* = \frac{71250}{3EI} - \frac{293750}{27EI} = \frac{347500}{27EI}$$

Now we can easily determine the slopes and deflections at A,B,C,D for the given beam

Slope at A for the given beam = S.F. at A for the conjugate beam

$$= \frac{347500}{27EI} = \frac{347500 \times 10^9}{27 \times 200 \times 10^3 \times 2 \times 10^{10}}$$

$$= \mathbf{0.003218 \text{ radians}}$$

Slope at B for the given beam = S.F. at B for the conjugate beam

$$= \frac{347500}{27EI} - \frac{1000}{EI} = \frac{77500}{27EI}$$

$$= \frac{77500 \times 10^9}{27 \times 200 \times 10^3 \times 2 \times 10^{10}} = \mathbf{0.0007176 \text{ radians}}$$

Slope at C for the given beam = S.F. at C for the conjugate beam

$$= \frac{293750}{27EI} - \frac{6250}{EI} = \frac{125000}{27EI}$$

$$= \frac{125000 \times 10^9}{27 \times 200 \times 10^3 \times 2 \times 10^{10}} = \mathbf{0.001157 \text{ radians}}$$

Slope at D for the given beam = S.F. at D for the conjugate beam

$$= \frac{293750}{27EI}$$

$$= \frac{293750 \times 10^9}{27 \times 200 \times 10^3 \times 2 \times 10^{10}} = \mathbf{0.00272 \text{ radians}}$$

Deflection at A = B.M at A for the conjugate beam = 0

(Since A is simply supported)

Deflection at B = B.M at B for conjugate beam

$$= \frac{347500}{27EI} \times 10 - \frac{10000}{EI} \times \frac{10}{2} = \frac{2575000}{27EI}$$

$$= \frac{2575000 \times 10^{12}}{27 \times 200 \times 10^3 \times 2 \times 10^{10}} = \mathbf{23.84 \text{ mm}}$$

Deflection at C = B.M at C for conjugate beam

$$= \frac{293750}{27EI} \times 10 - \frac{6250}{EI} \times \frac{10}{3}$$

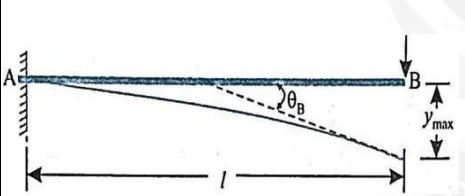
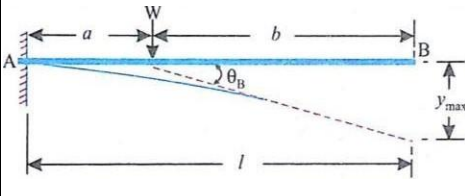
$$= \frac{2375000}{27EI}$$

$$= \frac{2375000 \times 10^{12}}{27 \times 200 \times 10^3 \times 2 \times 10^{10}} = \mathbf{21.99 \text{ mm}}$$

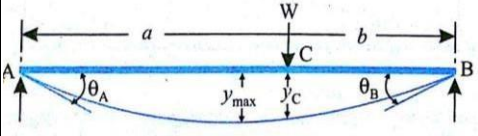
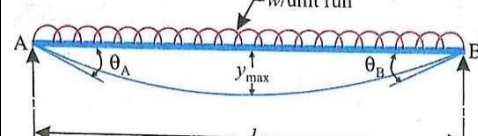
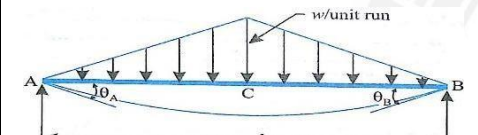
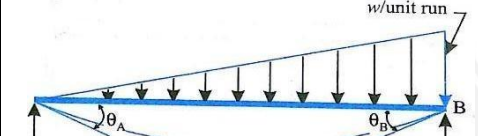
Deflection at D = B.M at D for the conjugate beam = 0

(Since D is simply supported)

IMPORTANT TERMS

| DESCRIPTION | SLOPE | DEFLECTION | MAX. BM |
|--|---|--|------------|
|  | $\theta_B = \frac{Wl^2}{2EI}$ | $y_{BMax} = \frac{Wl^3}{3EI}$ | $M_A = Wl$ |
|  | $\theta_B = \theta_C$ $\frac{Wa^2}{2EI}$ | $y_C = \frac{Wa^3}{3EI}$ and $y_{BMax} = \frac{Wa^3}{3EI} + \frac{Wa^2}{EI}(l - a)$ | $M_A = Wa$ |

| | | | |
|--|---|---|---|
| | $\theta_B = \frac{wl^3}{6EI}$ | $y_{BMax} = \frac{wl^4}{8EI}$ | $M_A = \frac{wl^2}{2}$ |
| | $\theta_B = \theta_C$ $y_C = \frac{wa^4}{8EI}$ $y_{BMax} = \frac{wa^4}{8EI} + \frac{w(l-a)^3}{6EI}$ | $y_{BMax} = \frac{wa^4}{8EI} + \frac{w(l-a)^3}{6EI}$ | $M_A = \frac{wa^2}{2}$ |
| | $\theta_B = \frac{wl^3}{6EI} - \frac{w(l-a)^3}{6EI}$ | $y_{BMax} = \frac{wl^4}{8EI} - \left[\frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI}a \right]$ | $M_A = w(l-a) \left[\frac{l-a}{2} + a \right]$ |
| | $\theta_B = \frac{wl^3}{24EI}$ | $y_{BMax} = \frac{wl^4}{30EI}$ | $M_A = \frac{wl^2}{6}$ |
| | $\theta_B = \frac{Ml}{EI}$ | $y_{BMax} = \frac{Ml^2}{2EI}$ | $M_A = M$ |
| | $\theta_A = \theta_B$ $y_C = \frac{Wl^3}{16EI}$ | $y_{BMax} = \frac{Wl^3}{48EI}$ | $M_C = \frac{Wl}{4}$ |

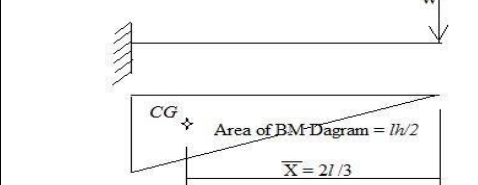
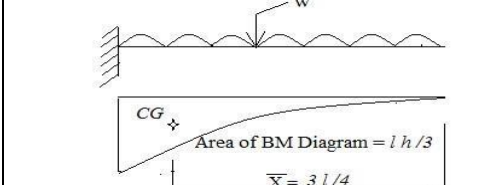
| | | | |
|--|---|--|--------------------------------|
|  | $\theta_A = \frac{Wb(l^2 - b^2)}{6EI l}$ $\theta_B = \frac{Wa(l^2 - a^2)}{6EI l}$ | $y_{max} = \frac{Wb(l^2 - b^2)^{3/2}}{9\sqrt{3}EI l}$ <p>and</p> $y_c = \frac{Wa^2a^2}{3EI l}$ | $M_c = \frac{Wab}{l}$ |
|  | $\theta_A = \theta_B = \frac{wl^3}{24EI}$ | $y_{max} = \frac{5wl^4}{384EI}$ | $M_c = \frac{wl^2}{8}$ |
|  | $\theta_A = \frac{7wl^3}{360EI}, \theta_B = \frac{4wl^3}{5EI}$ | $y_{max} = \frac{2.5wl^4}{384EI}$ | $M_c = \frac{wl^3}{9\sqrt{3}}$ |
|  | $\theta_A = \theta_B = \frac{5wl^2}{192EI}$ | $y_{max} = \frac{5wl^4}{120EI}$ | $M_c = \frac{wl^2}{12}$ |

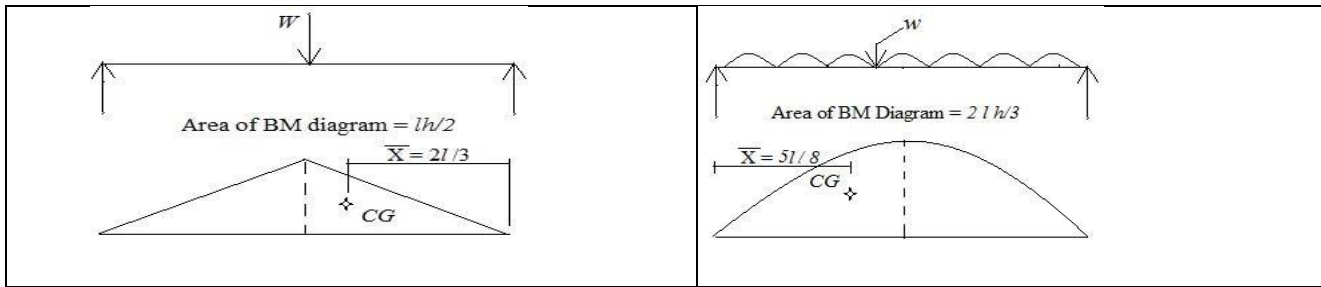
Moment Area Method

| Slope | $\theta = A/EI$ | $Y = AX/EI$ | Deflection |
|-------|-----------------|-------------|------------|
|-------|-----------------|-------------|------------|

A = Area of BM diagram; \bar{X} = Centre of gravity distance;

E= Young's modulus of beam material ; I = Moment of Inertia of beam section

| | |
|--|--|
|  |  |
|--|--|



MECAULAY'S METHOD

Step1: set the point XX at a distance of x m from free end/right support (OR) Near to fixed end/left support.

Step2: Take moment about XX

Step3: integrate the moment equation wrt x with adding of constant C_1 at the first part of equation is slope equation

Step4: integrate again wrt x with adding another constant C_2 at the first part of equation is deflection equation

Step5: put condition $x = 0$; $y = 0$; in first part of deflection eqn to find C_2 value ($C_2 = 0$) and Put another condition $x = l$; $y = 0$ in whole part of same eqn to find C_1 value.

Step6: then substitute C_1 , C_2 value in slope & deflection Eqn to get real Slope and deflection eqn.

Step7: Now substitute required point distance in slope and deflection eqn in first part to find Near right support slope and deflection.

Step8: similarly include second part of eqn and substitute another x value to find slope and deflection of every point.

➤ $MI \frac{d^2y}{dx^2} =$ Moment equation

➤ $MI \frac{dy}{dx} =$ Slope equation

➤ $MI y =$

deflection equation

➤ To find maximum $\frac{dy}{dx} = 0$

deflection put to

get x value and substitute x value in deflection to find max. deflection

➤ To evaluate integration in whole part

➤ In UDL, unload distance also assume both side UDL load act.

DOUBLE INTEGRATION METHOD

Same procedure followed as mecaulay's method

But some difference are

1. Constant C_1 , C_2 are add at the end of equation
2. Equation are not separated only whole eqn used
3. Two condition are applied in whole eqn to find C_1 , C_2
4. Ordinary integration followed

THEORETICAL QUESTIONS

1. Derive an expression for the slope and deflection of a beam subjected to uniform bending moment.
2. Prove that the relation that $M = EI \frac{d^2y}{dx^2}$
where M = Bending moment, E = Young's modulus, I = M.O.I.
3. Find an expression for the slope at the supports of a simply supported beam, carrying a point load at the centre.
4. Prove that the deflection at the centre of a simply supported beam, carrying a point load at the centre, is given by $y_c = \frac{WL^3}{48EI}$ where W = Point load, L = Length of beam.
5. Find an expression for the slope and deflection of a simply supported beam, carrying a point load W at a distance 'a' from left support and at a distance 'b' from right support where $a > b$.
6. Prove that the slope and deflection of a simply supported beam of length L and carrying a uniformly distributed load of w per unit length over the entire length are given by
Slope at supports $= -\frac{WL^2}{24EI}$, and Deflection at centre $= \frac{5}{384} \frac{WL^3}{EI}$ Where W
 $=$ Total load $= w \times L$.
7. What is Macaulay's method ? Where is it used ? Find an expression for deflection at any section of a simply supported beam with an eccentric point load, using Macaulay's method.
8. What is moment- area method ? Where is it conveniently used ? Find the slope and deflection of a simply supported beam carrying a (i) point load at the centre and (ii) uniformly distributed load over the entire length using moment-area method.
9. What is a cantilever? What are the different methods of finding of slope and deflection of a cantilever ?

10. Derive an expression for the slope and deflection of a cantilever of length L , carrying a point load W at the free end by double integration method.
11. Solve questions 2, by moment area method.
12. Prove that the slope and deflection of a cantilever carrying uniformly distributed load over the whole length are given by,

$$\theta_B = \frac{wL^3}{6EI} \text{ and } y_B = \frac{wL^4}{8EI}$$

Where w = Uniformly distributed load and EI = Flexural rigidity.

13. Find the expression for the slope and deflection of a cantilever of length L which carries a uniformly distributed load over a length 'a' from the fixed end by (i) Double integration method and (ii) Moment area method.
14. Prove that the slope and deflection of a cantilever length L , which carries a gradually varying load from zero at the free end to w/m run at the fixed end are given by :

$$\theta_B = \frac{wL^3}{24EI} \text{ and } y_B = \frac{wL^4}{30EI}$$

Where EI = Flexural rigidity.

NUMERICAL PROBLEMS

1. A wooden beam 4 m long, simply supported at its ends, is carrying a point load of 7.25 kN at its centre. The cross-section of the beam is 140 mm wide and 240 mm deep. If E for the beam = $6 \times 10^3 \text{ N/mm}^2$, find the deflection at the centre.
2. A beam 5 m long, simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1° , find the deflection at the centre of the beam.
3. Determine : (i) slope at the left support, (ii) deflection under the load and (iii) maximum deflection of a simply supported beam of length 10 m, which is carrying a point load of 10 kN at a distance 6 m from the left end.
Take $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 1 \times 10^8 \text{ mm}^4$.
4. A beam of uniform rectangular section 100 mm wide and 240 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9.125 kN/m

run over the entire span of 4 m. Find the deflection at the centre if $E = 1.1 \times 10^4$ N/mm².

5. A beam of length 4.8 m and of uniform rectangular section is simply supported at its ends. It carries a uniformly distributed load of 9.375 kN/m run over the entire length. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm² and maximum deflection is not to exceed 0.95 cm.

Take E for beam material = 1.05×10^4 N/mm².

6. Solve problem 3, using Macaulay's method.

7. A beam of length 10 m is simply supported at its ends and carries two point loads of 100 kN and 60 kN at a distance of 2 m and 5 m respectively from the left support. Calculate the deflections under each load. Find also the maximum deflection.

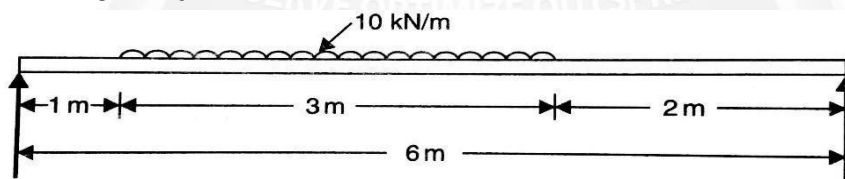
Take $I = 18 \times 10^8$ mm⁴ and $E = 2 \times 10^5$ N/mm².

8. A beam of length 20 m is simply supported at its ends and carries two point loads of 4 kN and 10 kN at a distance of 8 m and 12 m from left end respectively. Calculate : (i) deflection under each load (ii) maximum deflection.

Take $E = 2 \times 10^6$ N/mm² and $I = 1 \times 10^9$ mm⁴.

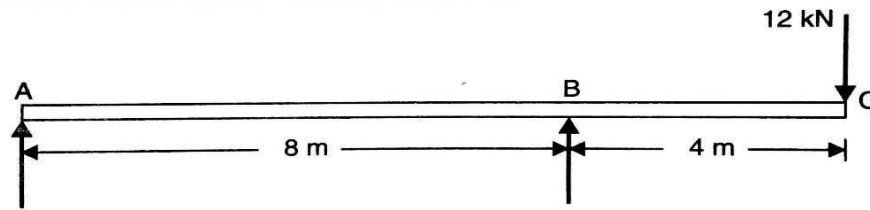
9. A beam of length 6 m is simply supported at its ends. It carries a uniformly distributed load of 10 kN/m as shown in Fig. Determine the deflection of the beam at its mid-point and also the position and the maximum deflection.

Take $EI = 4.5 \times 10^8$ N/mm².



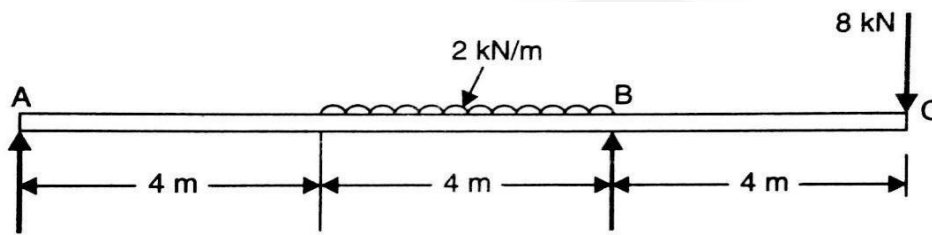
10. A beam ABC of length 12 metre has one support at the left end and other support at a distance of 8 m from the left end. The beam carries a point load of 12 kN at the right end as shown in Fig. Find the slopes over each supports and at the right end. Find also the deflection at the right end.

Take $E = 2 \times 10^5$ N/mm² and $I = 5 \times 10^8$ mm⁴.



11. An overhanging beam ABC is loaded as shown in Fig.. Determine the deflection of the beam at point C.

Take $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 5 \times 10^8 \text{ mm}^4$.



12. A beam of span 8 m and of uniform flexural rigidity $EI = 40 \text{ MN-m}^2$, is simply supported at its ends. It carries a uniformly distributed load of 15 kN/m run over the entire span. It is also subjected to a clockwise moment of 160 kNm at a distance of 3 m from the left support. Calculate the slope of the beam at the point of application of the moment.

13. A cantilever of length 2 m carries a point load of 30 kN at the free end. If moment of inertia $I = 10^8 \text{ mm}^4$ and value of $E = 2 \times 10^5 \text{ N/mm}^2$, then find :

- i) slope of the cantilever at the free end and
- ii) deflection at the free end.

14. A cantilever of length 3 m carries a point load of 60 kN at a distance of 2 m from the fixed end. If $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 10^8 \text{ mm}^4$, find :

- i) slope at the free end and
- ii) deflection at the free end.

15. A cantilever of length 30 m carries a uniformly distributed load of 24 kN/m length over the entire length. If moment of inertia of the beam $= 10^8 \text{ mm}^4$ and value of $E = 2 \times 10^5 \text{ N/mm}^2$, determine the slope and deflection at the free end.

16. A cantilever of length 3 m carries a uniformly distributed load over the entire length. If the slope at the free end is 0.01777 radians, find the deflection at the free end.
17. Determine the slope and deflection at the free end of a cantilever of length 4 m which is carrying a uniformly distributed load of 12 kN/m over a length of 3 m from the fixed end. Take $EI = 2 \times 10^{13} \text{ N/mm}^2$.
18. A cantilever of length 3 m carries a uniformly distributed load of 15 kN/m over a length of 2 m from the free end. If $I = 10^8 \text{ mm}^4$ and $E = 2 \times 10^5 \text{ N/mm}^2$, find
:
i) slope at the free end and ii) deflection at the free end.
19. A cantilever of length 2 m carries a load of 20 kN at the free end and 30 kN at a distance 1 m from the end. Find the slope and deflection at the free end. Take $E = 2.0 \times 10^5 \text{ N/mm}^2$ and $I = 1.5 \times 10^8 \text{ mm}^4$.
20. Determine the deflection at the free end of a cantilever which is 2 m long and carries a point load of 9 kN at the free end and a uniformly distributed load of 8 kN/m over a length of 1 m from the fixed end.
21. A cantilever of length 2 m carries a uniformly varying load of zero intensity at the free end, and 45 kN/m at the fixed end. If $E = 2 \times 10^5 \text{ N/mm}^2$ and $I = 10^8 \text{ mm}^4$, find the slope and deflection of the free end.
22. A cantilever of length 2 m carries a point load of 30 kN at the free end and another load of 30 kN at its centre. If $EI = 10^{13} \text{ N/mm}^2$ for the cantilever then determine by moment area method, the slope and deflection at the free end of cantilever.
23. A cantilever of length 'L' carries a U.D.L. of w per unit for a length of $\frac{L}{2}$ from the fixed end. Determine the slope and deflection at the free end using area moment method.