1.5 NATURE OF QUADRATIC FORM DETERMINED BY PRINCIPAL MINORS

Let A be a square matrix of order n say $A = \left[\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ddots & a_{2n} \\ & \ldots & & \cdots & \ldots. \end{pmatrix}\right]$

The principal sub determinants of A are defined as below.

$$s_1 = a_{11}$$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$s_n = |A|$$

The quadratic form $Q = X^T AX$ is said to be

1. Positive definite: If $s_{1,}s_{2,}s_3 \ldots\ldots\ldots s_n > 0$
2. Positive semidefinite: If $s_{1,}s_{2,}s_3 \ldots\ldots\ldots s_n \geq 0$ and atleast one $s_i = 0$
3. Negative definite: If $s_{1,}s_{3,}s_5 \ldots\ldots\ldots < 0$ and$s_{2,}s_{4,}s_6 \ldots\ldots\ldots > 0$
4. Negative semidefinite: If $s_{1,}s_{3,}s_5 \ldots\ldots\ldots < 0$ and$s_{2,}s_{4,}s_6 \ldots\ldots\ldots > 0$ and

   atleast one $s_i = 0$
5. Indefinite: In all other cases

Example: Determine the nature of the Quadratic form $\mathbf{12x_1^2 + 3x_2^2 + 12x_3^2 + 2x_1x_2}$

Solution:

$$A = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

$s_1 = a_{11}$=12 > 0

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 12 & 1 \\ 1 & 3 \end{vmatrix} = 35 > 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{vmatrix} = 430 > 0 \text{ , Positive definite}$$

Example: Determine the nature of the Quadratic form $\mathbf{x_1^2 + 2x_2^2}$

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$s_1 = a_{11} = 1 > 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 2 > 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \text{ , Positive semidefinite}$$

Example: Determine the nature of the Quadratic form

$$\mathbf{x^2 - y^2 + 4z^2 + 4xy + 2yz + 6zx}$$

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

$s_1 = a_{11} = 1 > 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -5 < 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 0 \text{ , Indefinite}$$

Example: Determine the nature of the Quadratic form $\mathbf{xy + yz + zx}$

Solution:

$$\text{Let } A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$s_1 = a_{11} = 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -1/4 < 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{vmatrix} = \frac{1}{4} > 0 \text{ , Indefinite}$$

RANK, INDEX AND SIGNATURE OF A REAL QUADRATIC FORMS

Let $Q = X^T AX$ be quadratic form and the corresponding canonical form is $d_1{y_1}^2 + d_2{y_2}^2 + \cdots + d_n{y_n}^2$.

The **rank** of the matrix A is number of non –zero Eigen values of A. If the rank of A is ‘r’, the canonical form of Q will contain only “r” terms .Some terms in the canonical form may be positive or zero or negative.

The number of positive terms in the canonical form is called the **index**(p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form .i.e,$p - (r - p) = 2p - r$ is called the signature of the quadratic form and usually denoted by s. Thus $s = 2p - r$.

Example: Reduce the Quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form through an orthogonal transformation .Find the nature rank, index, signature

Solution:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 2 + 1 + 1 = 4$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3 + 1 + 1 = -1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -4$$

Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\lambda = -1, 1, 4$$

To find the Eigen vectors:

Case (i) When $\lambda = -1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \ldots (1)$$

$$x_1 + 2x_2 - 2x_3 = 0 \ldots (2)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 0 \ldots (4)$$

$$x_1 + 0x_2 - 2x_3 = 0 \ldots (5)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-2+0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (7)$$

$$x_1 - 3x_2 - 2x_3 = 0 \ldots (8)$$

$$-x_1 - 2x_2 - 3x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$\mathrm{N} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$\mathrm{N^T} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $\mathrm{D} = \mathrm{N^T A N}$

$$= \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$\mathrm{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Canonical form $= Y^T\mathrm{DY}$ where $\mathrm{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T\mathrm{DY} = (y_1, y_2, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -{y_1}^2 + {y_2}^2 + 4{y_3}^2$$

Rank = 3

Index = 2

Signature = 2 – 1 = 1

Nature is indefinite.