3.4 SOLUTION OF DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM TECHNIQUE

There are so many methods to solve a linear differential equation. If the initial conditions are known, then Laplace transform technique is easier to solve the differential equation. The Laplace transform transforms the differential equation into an algebraic equation.

$$L[y'(t)] = sL[y(t)] - y(0)$$

$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Problems using Partial Fraction

Example: 1.Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

Given $x'' - 3x' + 2x = 2; x(0) = 0; x'(0) = 5$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0; x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$

$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s} + 5$$

$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

Consider $\frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)+Bs(s-2)+Cs(s-1)}{s(s-1)(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

Put $s = 0$ in (1)	Put $s = 1$ in (1)	Put $s = 2$ in (1)
$A(-1)(-2) = 2$	$B(1)(-1) = 7$	$C(2)(1) = 2 + 10$
$A = 1$	$B = -7$	$C = 6$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$\therefore \bar{x} = \frac{1}{s} - 7\frac{1}{s-1} + 6\frac{1}{s-2}$

$x(t) = L^{-1}\left[\frac{1}{s}\right] - 7L^{-1}\left[\frac{1}{s-1}\right] + 6L^{-1}\left[\frac{1}{s-2}\right]$

$x(t) = 1 - 7e^{t} + 6e^{2t}$

Example: 2. Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$, with $y(0) = 1 = y'(0)$.

Solution:

Given $y'' - 3y' - 4y = 2e^{-t}$; with $y(0) = 1 = y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2\frac{1}{s+1}$

Substituting $y(0) = 1 = y'(0)$.

$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$

$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$

$[s^2 - 3s - 4]\bar{y} = \frac{2+s(s+1)-2(s+1)}{s+1}$

$= \frac{2+s^2+s-2s-2}{s+1}$

$(s+1)(s-4)\bar{y} = \frac{s^2-s}{s+1}$

$\bar{y} = \frac{s^2-s}{(s+1)(s+1)(s-4)}$

$\bar{y} = \frac{s^2-s}{(s+1)^2(s-4)}$

Consider $\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4)+B(s-4)+C(s+1)^2}{(s+1)^2(s-4)}$

$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \cdots (1)$

Put $s = -1$ in (1)	Put $s = 4$ in (1)	equating the coefficients of s^2, we get
$-5B = 1 + 1$	$25C = 16 - 4$	$A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$
$B = \frac{-2}{5}$	$C = \frac{12}{25}$	$A = \frac{13}{25}$

$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$

$y(t) = \frac{13}{25} L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{2}{5} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25} L^{-1}\left[\frac{1}{s-4}\right]$

$y(t) = \frac{13}{25} e^{-t} - \frac{2}{5} te^{-t} + \frac{12}{25} e^{4t}$

Example: 3. Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$, with $y(0) = 1$ *and* $y'(0) = 0$ using Laplace transform.

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ *and* $y'(0) = 1$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$

Substituting $y(0) = 1$ *and* $y'(0) = 0$.

$[s^2L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$

$s^2L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$

$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$

$[s^2 - 3s + 2]\bar{y} = \frac{1+s(s+1)-3(s+1)}{s+1}$

$= \frac{1+s^2+s-3s-3}{s+1}$

$(s-1)(s-2)\bar{y} = \frac{s^2-2s-2}{s+1}$

$\bar{y} = \frac{s^2-2s-2}{(s+1)(s-1)(s-2)}$

Consider $\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$

$\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2)+B(s+1)(s-2)+C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$

$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \cdots (1)$

Put $s = -1$ in (1)	put $s = 1$ *in* (1)	put $s = 2$ *in* (1)
$6A = 1 + 2 - 2$	$-2B = 1 - 4$	$3C = 4 - 4 - 2$
$A = \frac{1}{6}$	$B = \frac{3}{2}$	$C = \frac{-2}{3}$

$\therefore \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$

$y(t)=\frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right]+\frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right]-\frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$

$y(t)=\frac{1}{6}e^{-t}+\frac{3}{2}e^{t}-\frac{2}{3}e^{2t}$

Example: 4. Using Laplace transform solve the differential equation $y''+2y'-3y=sint$, with $y(0)=y'(0)=0$.

Solution:

Given $y''+2y'-3y=sint$ with $y(0)=0=y'(0)$.

Taking Laplace transform on both sides, we get,

$L[y''(t)]+2L[y'(t)]-3L[y(t)]=L(sint)$

$[s^2L[y(t)]-sy(0)-y'(0)]+2[sL[y(t)]-y(0)]-3L[y(t)]=\frac{1}{s^2+1}$

Substituting $y(0)=0=y'(0)$.

$[s^2L[y(t)]-0-0]+2[sL[y(t)]-0]-3L[y(t)]=\frac{1}{s^2+1}$

$s^2L[y(t)]+2sL[y(t)]-3L[y(t)]=\frac{1}{s^2+1}$

$s^2L[y(t)]+2sL[y(t)]-3L[y(t)]=\frac{1}{s^2+1}$

Put $L[y(t)]=\bar{y}$

$s^2\bar{y}+2s\bar{y}-3\bar{y}=\frac{1}{s^2+1}$

$[s^2+2s-3]\bar{y}=\frac{1}{s^2+1}$

$(s-1)(s+3)\bar{y}=\frac{1}{s^2+1}$

$\bar{y}=\frac{1}{(s-1)(s+3)(s^2+1)}$

Consider $\frac{1}{(s-1)(s+3)(s^2+1)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{s^2+1}$

$\frac{1}{(s-1)(s+3)(s^2+1)}=\frac{A(s^2+1)(s+3)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$

$A(s^2+1)(s+3)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)=1\cdots(1)$

Put $s=1$ in (1)	Put $s=-3$ in (1)	equating the coefficients of s^2, we get
$8A=0+1$	$B(-4)(10)=1$	$A+B+C=0\Rightarrow C=-A-B=\frac{-1}{8}+\frac{1}{40}$
$A=\frac{1}{8}$	$B=\frac{-1}{40}$	$C=\frac{-1}{10}$

Put $s=0$ in (1), we get

$3A-B-3D=1\Rightarrow\frac{3}{8}+\frac{1}{40}-3D=1$

$3D=\frac{3}{8}+\frac{1}{40}-1$

$3D=\frac{15+1-40}{40}\Rightarrow D=\frac{-24}{40\times3}\Rightarrow D=\frac{-1}{5}$

$\frac{1}{(s-1)(s+3)(s^2+1)}=\frac{1}{8(s-1)}-\frac{1}{40(s+3)}+\frac{\left(\frac{-1}{10}\right)s-\frac{1}{5}}{s^2+1}$

$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$

$y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$

$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(cost - 2sint)$

Example: 5. Using Laplace transform solve the differential equation $y'' - 3y' + 2y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ $and y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$

Substituting $y(0) = -3$ and $y'(0) = 5$.

$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$

$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$

$[s^2 - 3s + 2]\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$(s-1)(s-2)\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$\bar{y} = \frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2}$

Consider $\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$

$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \cdots (1)$

Put $s = 1$ in (1)	Put $s = 2\ in\ (1)$	equating the coefficients of s^2, we get
$A = 4 - 11$	$C = 4 + 0$	$A + B = -3 \Rightarrow -7 + B = -3$
$A = -7$	$C = 4$	$B = 4$

$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$y(t) = -7L^{-1}\left[\frac{1}{(s-1)}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$

$$= -7e^{t} + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = -7e^{t} + 4e^{2t} + 4e^{2t}t$$

Example: 6. Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with $y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2$ and $y'(0) = -2$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$$

Substituting $y(0) = 2$ and $y'(0) = -2$.

$$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1+(2s-10)(s-2)}{s-2}$$

$$\bar{y} = \frac{1+(2s-10)(s-2)}{(s-2)(s^2-4s+8)}$$

$$= \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$$

Consider $\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$

$$= \frac{A[(s-2)^2+4]+B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$$

$$A[(s-2)^2 + 4] + B[(s-2) + C](s-2) = 1 + (2s-10)(s-2) \cdots (1)$$

Put $s = 2$ in (1)	Put $s = 0$ in (1)	equating the coefficients of s^2, we get
$4A = 1 + 0$	$8A + 4B - 2C = 21$	$A + B = 2 \Rightarrow \frac{1}{4} + B = 2$
$A = \frac{1}{4}$	$C = -6$	$B = \frac{7}{4}$

$$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4}\frac{(s-2)}{(s-2)^2+4} - 6\frac{1}{(s-2)^2+4}$$

$$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{(s-2)}\right] + \frac{7}{4}L^{-1}\left[\frac{(s-2)}{(s-2)^2+4}\right] - 6L^{-1}\left[\frac{1}{(s-2)^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}\left[\frac{s}{s^2+4}\right] - 6e^{2t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 6e^{2t}\frac{sin2t}{2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 3e^{2t}sin2t$$

Problems without using Partial Fraction

Example: 7. Solve using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$

Solution:

Given $x'' - 2x' + x = e^t; x(0) = 2; x'(0) = -1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$$

Substituting $x(0) = 2; x'(0) = -1$

$$[s^2L[x(t)] - 2s + 1] - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 2s\bar{x} + \bar{x} = \frac{1}{s-1} + 2s - 5$$

$$[s^2 - 2s + 1]\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$\bar{x} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^tL^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2!} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - 5e^tt$$

$$= e^t\frac{t^2}{2} + 2e^t + 2e^tt - 5e^tt$$

$$\therefore x = \frac{t^2e^t}{2} + 2e^t - 3e^tt$$

Example: 8. Solve the following differential equation using Laplace transform

$(D^2 - 2D + 1)y = t^2e^t$ Given $y(0) = 2$ and $Dy(0) = 3$

Solution:

Given $(D^2 - 2D + 1)y = t^2e^t$ with $y(0) = 2$ and $Dy(0) = 3$

$ie., D^2y - 2Dy + y = t^2e^t$

$y'' - 2y' + y = t^2e^t$ With $y(0) = 2$ and $y'(0) = 3$

Apply Laplace transform on both sides, we get

$$L[y''(t)] - 2L[y'(t)] + L[y(t)] = L(t^2e^t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 2[sL[y(t)] - y(0)] + L[y(t)] = L[t^2]_{s\to s-1}$$

Substituting $y(0) = 2\ and y'(0) = 3.$

$$[s^2L[y(t)] - 2s - 3] - 2[sL[y(t)] - 2] + L[y(t)] = \left[\frac{2!}{s^3}\right]_{s\to s-1}$$

$$s^2L[y(t)] - 2s - 3 - 2sL[y(t)] + 4 + L[y(t)] = \frac{2}{(s-1)^3}$$

$$s^2L[y(t)] - 2sL[y(t)] + L[y(t)] = \frac{2}{(s-1)^3} + 2s - 1$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} - 2s\bar{y} + \bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$[s^2 - 2s + 1]\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$(s-1)^2\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$\bar{y} = \frac{2}{(s-1)^5} + \frac{2s}{(s-1)^2} - \frac{1}{(s-1)^2}$$

$$y(t) = L^{-1}\left[\frac{2}{(s-1)^5}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= 2e^tL^{-1}\left[\frac{1}{s^5}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - e^tL^{-1}\left[\frac{1}{s^2}\right]$$

$$= 2e^t\frac{t^4}{4!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - e^tt$$

$$= 2e^t\frac{t^4}{24} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - e^tt$$

$$= e^t\frac{t^4}{12} + 2e^t + 2e^tL^{-1}\left[\frac{1}{s^2}\right] - e^tt$$

$$= e^t\frac{t^4}{12} + 2e^t + 2e^tt - e^tt$$

$$\therefore x = \frac{t^4e^t}{12} + 2e^t + e^tt$$

Example: 9. Solve using Laplace transform $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$, given that $y(0) = 0$ and $y'(0) = 0$

Solution:

Given $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$ with $y(0) = 0$ and $y'(0) = 0$

$y'' + 6y' + 9y = 6t^2e^{-3t}$ With $y(0) = 0$ and $y'(0) = 0$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + 6L[y'(t)] + 9L[y(t)] = 6L(t^2e^{-3t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 6[sL[y(t)] - y(0)] + 9L[y(t)] = 6L[t^2]_{s\to s+3}$$

Substituting $y(0) = 0\ and y'(0) = 0.$

$$[s^2L[y(t)] - 0 - 0] + 6[sL[y(t)] - 0] + 9L[y(t)] = 6\left[\frac{2!}{s^3}\right]_{s\to s+3}$$

$$s^2L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

$$s^2L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} + 6s\bar{y} + 9\bar{y} = \frac{12}{(s+3)^3}$$

$$[s^2 + 6s + 9]\bar{y} = \frac{12}{(s+3)^3}$$

$$(s+3)^2\bar{y} = \frac{12}{(s+3)^3}$$

$$\bar{y} = \frac{12}{(s+3)^5}$$

$$y(t) = L^{-1}\left[\frac{12}{(s+3)^5}\right] = 12e^{-3t}L^{-1}\left[\frac{1}{s^5}\right]$$

$$= 12e^{-3t}\frac{t^4}{4!}$$

$$\therefore y = \frac{t^4e^{-3t}}{2}$$

Example: 10. Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}sint;\ x(0) = 0$ and $x'(0) = 1$

Solution:

Given $x'' + 2x' + 5x = e^{-t}sint; x(0) = 0; x'(0) = 1$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] + 2L[x'(t)] + 5L[x(t)] = L(e^{-t}sint)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] + 2[sL[x(t)] - x(0)] + 5L[x(t)] = L[sint]_{s\to s+1}$$

Substituting $x(0) = 0; x'(0) = 1$

$$[s^2L[x(t)] - 0 - 1] + 2[sL[x(t)] - 0] + 5L[x(t)] = \left[\frac{1}{s^2+1}\right]_{s\to s+1}$$

$$s^2L[x(t)] + 2sL[x(t)] + 5L[x(t)] - 1 = \frac{1}{(s+1)^2+1}$$

$$s^2L[x(t)] + 2sL[x(t)] + 5L[x(t)] = \frac{1}{(s+1)^2+1} + 1$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} + 2s\bar{x} + 5\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{s^2+2s+2} + 1$$

$$\bar{x} = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{5-2}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right] + \frac{1}{s^2+2s+5}$$

$$\frac{1}{(s^2+ax+b)(s^2+ax+c)} = \frac{1}{c-b}\left[\frac{1}{s^2+ax+b} - \frac{1}{s^2+ax+c}\right]$$

$$= \frac{1}{3}\left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5}\right] + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{3(s^2+2s+2)} - \frac{1}{3(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$\bar{x} = \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)}$$

$$x(t) = \frac{1}{3}L^{-1}\left[\frac{1}{(s^2+2s+2)}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s^2+2s+5)}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{2}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{3}e^{-t}sint + \frac{2}{3}e^{-t}\frac{sin2t}{2}$$

$$\therefore x = \frac{1}{3}e^{-t}[sint + sin2t]$$

Example: 11. Solve using Laplace transform $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that $= 4$,$y' = -2$ when $t = 0$

Solution:

Given $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

$y'' + y' = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + L[y'(t)] = L(t^2) + L(2t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + [sL[y(t)] - y(0)] = \frac{2}{s^3} + 2\frac{1}{s^2}$$

Substituting $y(0) = 4\ and y'(0) = -2$.

$$[s^2L[y(t)] - 4s + 2] + [sL[y(t)] - 4] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2L[y(t)] + sL[y(t)] - 4s + 2 - 4 = \frac{2+2s}{s^3}$$

$$s^2L[y(t)] + sL[y(t)] = \frac{2(1+s)}{s^3} + 4s + 2$$

Put$L[y(t)] = \bar{y}$

$$s^2\bar{y} + s\bar{y} = \frac{2(1+s)}{s^3} + 2(2s+1)$$

$$s(s^2+s)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s+1)$$

$$s(s+1)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s+1)$$

$$\bar{y} = \frac{2(s+1)}{s^4(s+1)} + \frac{2(2s+1)}{s(s+1)}$$

$$= \frac{2}{s^4} + 2\left[\frac{s+(s+1)}{s(s+1)}\right]$$

$$= \frac{2}{s^4} + 2\left[\frac{s}{s(s+1)} + \frac{s+1}{s(s+1)}\right]$$

$$= \frac{2}{s^4} + 2\left[\frac{1}{s+1} + \frac{1}{s}\right]$$

$$\bar{y} = \frac{2}{s^4} + \frac{2}{s+1} + \frac{2}{s}$$

$$y(t) = 2L^{-1}\left[\frac{2}{s^4}\right] + 2L^{-1}\left[\frac{1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s}\right]$$

$$= 2\frac{t^3}{3!} + 2e^{-t} + 2(1)$$

$$\therefore y = \frac{t^3}{3} + 2e^{-t} + 2$$

Example: 12. Solve using Laplace transform $\frac{d^2x}{dt^2} + 9x = cos2t$**, if** $(0) = 1\ ; x\left(\frac{\pi}{2}\right) = -1$

Solution:

Given $x'' + 9x = cos2t; x(0) = 1; x\left(\frac{\pi}{2}\right) = -1$

Since $x'(0)$ is not given assume $x'(0) = k$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] + L[x(t)] = L(cos2t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] + 9L[x(t)] = L(cos2t)$$

Substituting $x(0) = 1; x\left(\frac{\pi}{2}\right) = -1$

$$[s^2L[x(t)] - s - k] + 9L[x(t)] = \frac{s}{s^2+4}$$

$$s^2L[x(t)] + 9L[x(t)] = \frac{s}{s^2+4} + s + k$$

$$[s^2 + 9]L[x(t)] = \frac{s}{s^2+4} + s + k$$

Put $L[x(t)] = \bar{x}$

$$[s^2 + 9]\bar{x} = \frac{s}{s^2+4} + s + k$$

$$\bar{x} = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{9-4}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{5}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$= \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$\bar{x} = \frac{s}{5(s^2+4)} + \frac{(5s-s)}{5(s^2+9)} + \frac{k}{s^2+9}$$

$$= \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{s}{s^2+9} + \frac{k}{s^2+9}$$

$$x(t) = \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{4}{5}L^{-1}\left[\frac{s}{s^2+9}\right] + kL^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{1}{5}cos2t + \frac{4}{5}cos3t + k\frac{sin3t}{3} \cdots (1)$$

Given $x\left(\frac{\pi}{2}\right) = -1$

Put $t = \frac{\pi}{2}\ in\ (1)$

$$(1) \Rightarrow x\left(\frac{\pi}{2}\right) = \frac{1}{5}cos\frac{2\pi}{2} + \frac{4}{5}cos\frac{3\pi}{2} + k\frac{sin\frac{3\pi}{2}}{3}$$

$$-1 = \frac{1}{5}(-1) + 0 + \frac{k}{3}(-1)$$

$-\frac{k}{3} = \frac{1}{5} - 1 \Rightarrow -\frac{k}{3} = \frac{-4}{5} \Rightarrow k = \frac{12}{5}$

$$= \frac{1}{5}cos2t + \frac{4}{5}cos3t + \frac{12}{5}\frac{sin3t}{3}$$

$\therefore x(t) = \frac{1}{5}[cos2t + 4cos3t + 4sin3t]$