1.5.BILINEAR TRANSFORMATION

- **1.5.a. Introduction**

The transformation $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

This transformation was first introduced by A.F. Mobius, So it is also called Mobius transformation.

A bilinear transformation is also called a linear fractional transformation because $\frac{az+b}{cz+d}$ is a fraction formed by the linear fuctions $az - b$ and $cz + d$.

Theorem:1 Under a bilinear transformation no two points in z plane go to the same point in wplane.

Proof:

Suppose $z_1$andz_2 go to the same point in the wplane under the transformation$w = \frac{az+b}{cz+d}$.

Then $\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$

$$\Rightarrow (az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d)$$

$$i.e., \quad (az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d) = 0$$

$$\Rightarrow acz_1 z_2 + adz_1 + bcz_2 + bd - acz_1 z_2 - adz_2 - bcz_1 - bd = 0$$

$$\Rightarrow (ad - bc)(z_1 - z_2) = 0$$

$$\text{or} z_1 = z_2 \qquad [\because ad - bc \neq 0]$$

This implies that no two distinct points in the z plane go to the same point in wplane. So, each point in the z plane go to a unique point in the wplane.

Theorem: 2 The bilinear transformation which transforms z_1, z_2, z_3,into w_1, w_2, w_3 is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Proof:

If the required transformation $w = \frac{az+b}{cz+d}$.

$$\Rightarrow w - w_1 = \frac{az + b}{cz + d} - \frac{az_1 + b}{cz_1 + d} = \frac{(ad - bc)(z - z_1)}{(cz + d)(cz_1 + d)}$$

$$\Rightarrow (cz + d)(cz_1 + d)(w - w_1) = (ad - bc)(z - z_1)$$

$$\Rightarrow (cz_2 + d)(cz_3 + d) (w_2 - w_3) = (ad - bc)(z_2 - z_3)$$

$$\Rightarrow (cz + d) (cz_3 + d)(w - w_3) = (ad - bc)(z - z_3)$$

$$\Rightarrow (cz_2 + d)(cz_1 + d)(w_2 - w_1) = (ad - bc)(z_2 - z_1)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{\left[\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\right]\left[\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}\right]}{\left[\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}\right]\left[\frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}\right]}$$

$$= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\text{Now,} \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad \dots(1)$$

$$\text{Let}: A = \frac{w_2-w_3}{w_2-w_1}, B = \frac{z_2-z_3}{z_2-z_1}$$

$$(1) \Rightarrow \frac{w-w_1}{w-w_3}A = \frac{z-z_1}{z-z_3}B$$

$$\frac{wA-w_1A}{w-w_3} = \frac{zB-z_1B}{z-z_3}$$

$$\Rightarrow wAz - wAz_3 - w_1Az + w_1Az_3 = wBz - wz_1B - w_3zB + w_3z_1B$$

$$\Rightarrow w[(A-B)z + (Bz_1 - Az_3)] = (Aw_1 - Bw_3)z + (Bw_3z_1 - Aw_1z_3)$$

$$\Rightarrow w = \frac{(Aw_1-Bw_3)z + (Bw_3z_1 - Aw_1z_3)}{(A-B)z + (Bz_1 - Az_3)}$$

$\frac{az+b}{cz+d}$, Hence $a = Aw_1 - Bw_3, b = Bw_3z_1 - Aw_1z_3, c = A - B, d = Bz_1 - Az_3$

Cross ratio

Definition:

Given four point z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z-z_1)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

Note: (1) $w = \frac{az+b}{cz+d}$ can be expressed as $cwz + dw - (az+b) = 0$

It is linear both in w *and* z that is why, it is called bilinear.

Note: (2) This transformation is conformal only when $\frac{dw}{dz} \neq 0$

$$i.e., \frac{ad-bc}{(cz+d)^2} \neq 0$$

$$i.e., ad - bc \neq 0$$

If $ad - bc \neq 0$, every point in the z plane is a critical point.

Note: (3) Now, the inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

Note: (4) Each point in the plane except $z = \frac{-d}{c}$ corresponds to a unique point in the w plane.

The point $z = \frac{-d}{c}$ corresponds to the point at infinity in the w plane.

Note: (5) The cross ratio of four points

$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ is invariant under bilinear transformation.

Note: (6) If one of the points is the point at infinity the quotient of those difference which involve this points is replaced by 1.

Suppose $z_1 = \infty$, then we replace $\frac{z - z_1}{z_2 - z_1}$ by 1 (or)Omit the factors involving ∞

Example: 1.59 Find the fixed points of $w = \frac{2zi+5}{z-4i}$.

Solution:

The fixed points are given by replacing $w\ by\ z$

$$z = \frac{2zi + 5}{z - 4i}$$

$$z^2 - 4iz = 2zi + 5\ ;\ z^2 - 6iz - 5 = 0$$

$$z = \frac{6i \pm \sqrt{-36 + 20}}{2} \qquad \therefore z = 5i, i$$

Example: 1.60 Find the invariant points of $w = \frac{1+z}{1-z}$

Solution:

The invariant points are given by replacing $w\ by\ z$

$$z = \frac{1 + z}{1 - z}$$

$$\Rightarrow z - z^2 = 1 + z$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

Example: 1.61 Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$. Solution:

The invariant points are given by

$$z = 2 - \frac{2}{z}\ ; \qquad z = \frac{2z - 2}{z}$$

$$z^2 = 2z - 2; \qquad z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Example: 1.62 Find the fixed point of the transformation $w = \frac{6z-9}{z}$.[A.U N/D 2005]

Solution:

The fixed points are given by replacing $w = z$

i.e., $w = \frac{6z-9}{z} \Rightarrow z = \frac{6z-9}{z}$

$$\Rightarrow z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)^2 = 0$$

$$\Rightarrow z = 3,3$$

The fixed points are 3, 3.

Example: 1.63 Find the invariant points of the transformation $w = \frac{2z+6}{z+7}$.[A.U M/J 2009]

Solution:

The invariant (fixed) points are given by

$$w = \frac{2z+6}{z+7}$$

$$\Rightarrow z^2 + 7z = 2z + 6$$

$$\Rightarrow z^2 + 5z - 6 = 0$$

$$\Rightarrow (z+6)(z-1) = 0$$

$$\Rightarrow z = -6, z = 1$$

Example: 1.64 Find the invariant points of $f(z) = z^2$. [A.U M/J 2014 R-13]

Solution:

The invariant points are given by $z = w = f(z)$

$$\Rightarrow z = z^2$$

$$\Rightarrow z^2 - z = 0$$

$$\Rightarrow z(z-1) = 0$$

$$\Rightarrow z = 0, \quad z = 1$$

Example 1.65 Find the invariant points of a function $f(z) = \frac{z^3+7z}{7-6zi}$. [A.U D15/J16 R-13]

Solution:

Given $w = f(z) = \frac{z^3+7z}{7-6zi}$

The invariant points are given by

$$\Rightarrow z = \frac{z^3+7z}{7-6zi}$$

$$\Rightarrow 7 - 6zi = z^2 + 7$$

$$\Rightarrow -6zi = z^2 \Rightarrow z^2 + 6zi = 0 \Rightarrow z(z+6i) = 0$$

$$\Rightarrow z = 0, \qquad z = -6i$$

PROBLEMS BASED ON BILINEAR TRANSFORMATION

Example: 3.66 Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively. Solution:

Given $z_1 = 0, z_2 = -1, z_3 = i,$

$$w_1 = i, w_2 = 0, w_3 = \infty,$$

Let the required transformation be

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\Rightarrow \frac{w - w_1}{w_2 - w_1} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\Rightarrow \frac{w - i}{0 - i} = \frac{(z - 0)(-1 - i)}{(z - i)(-1 - 0)}$$

$$\Rightarrow \frac{w - i}{-i} = \frac{z}{(z - i)}(1 + i)$$

$$\Rightarrow w - i = \frac{z}{(z - i)}(-i + 1)$$

$$\Rightarrow w = \frac{z}{(z - i)}(-i + 1) + i = \frac{-iz + z + iz + 1}{(z - i)} = \frac{z + 1}{z - i}$$

Aliter: Given $z_1 = 0, z_2 = -1, z_3 = i,$

$$w_1 = i, w_2 = 0, w_3 = \infty,$$

Let the required transformation be

$$w = \frac{az + b}{cz + d} \ldots (1), \qquad ad - bc \neq 0$$

$$i = \frac{b}{d}$$

$w_1 = \frac{az_1+b}{cz_1+d}$ $w_2 = \frac{az_2+b}{cz_2+d}$ $w_3 = \frac{az_3+b}{cz_3+d}$

$$i = \frac{b}{d} \quad 0 = \frac{-a + b}{-c + d} \quad \frac{1}{0} = \frac{ai + b}{ci + d}$$

$$b = di \Rightarrow -a + b = 0 \Rightarrow ci + d = 0$$

$$\Rightarrow a = b \Rightarrow d = -ci$$

$$\therefore a = b = di = c$$

$$\therefore (1) \Rightarrow w = \frac{az + a}{az + \frac{a}{i}} = \frac{z + 1}{z + \frac{1}{i}} = \frac{z + 1}{z - i}$$

Example:1.67 Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Solution:

Given $z_1 = \infty, z_2 = i, z_3 = 0, w_1 = 0, w_2 = i, w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving z_1, and w_3, since $z_1 = \infty, w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z_2-z_3)}{z-z_3}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow w = \frac{-1}{z}$$

Example:1.68Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$, show that the transformation maps the interior of the unit circle of the $z-$ plane onto the upper half of the $w-$ plane. [A.U. May 2001] [A.U M/J 2014] [A.U D15/J16 R-13]

Solution:

Given $z_1 = 1, z_2 = i, z_3 = -1$

$w_1 = 0, w_2 = 1, w_3 = \infty,$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Omit the factors involving w_3, since , $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \because \left[\left(\frac{i+1}{i-1}\right)\left(\frac{i+1}{i+1}\right)\right] = \left[\frac{i^2+i+i+1}{i^2-i^2}\right] = \left[\frac{2i}{-2}\right] = -i$$

$$\Rightarrow w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{z-1}{z+1}[-i]$$

$$\Rightarrow w = \frac{(-i)z+i}{(1)z+1}\left[\because w = \frac{az+b}{cz+d}, ad-bc \neq 0 \text{ Form}\right]$$

To find z:

$$\Rightarrow wz + w = -iz + i$$

$$\Rightarrow wz + iz = -w + i$$

$$\Rightarrow z[w + i] = -w + i$$

$$\Rightarrow z = \frac{(w - i)}{w + i}$$

To prove: $|z| < 1$ maps $v > 0$

$$\Rightarrow |z| < 1$$

$$\Rightarrow \left|\frac{-(w - i)}{w + i}\right| < 1$$

$$\Rightarrow \left|\frac{w - i}{w + i}\right| < 1$$

$$\Rightarrow |w - i| < |w + i|$$

$$\Rightarrow |u + iv - i| < |u + iv + i|$$

$$\Rightarrow |u + i(v - 1)| < |u + i(v + i)|$$

$$\Rightarrow u^2 + (v - 1)^2 < u^2 + (v + 1)^2$$

$$\Rightarrow (v - 1)^2 < (v + 1)^2$$

$$\Rightarrow \quad v^2 - 2v + 1 < v^2 + 2v + 1$$

$$\Rightarrow \; -4v < 0$$

$$\Rightarrow v > 0$$

Example: 1.69 Determine the bilinear transformation that maps the points $-1, 0, 1$, in the z plane onto the points $0, i, 3i$ in the w plane. [Anna, May 1999]

Solution:

Given $z_1 = -1, z_2 = 0, z_3 = 1,$

$$w_1 = 0, w_2 = i, w_3 = 3i,$$

Let the required transformation be

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\Rightarrow \quad \frac{(w - 0)(i - 3i)}{(w - 3i)(i - 0)} = \frac{[z - (-1)][0 - 1]}{(z - 1)[0 - (-1)]}$$

$$\Rightarrow \quad \frac{w(-2i)}{(w - 3i)(i)} = \frac{(z + 1)(-1)}{(z - 1)(1)}$$

$$\Rightarrow \quad \frac{-2w}{w - 3i} = \frac{z + 1}{z - 1}$$

$$\Rightarrow \frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$\Rightarrow 2wz - 2w = wz + w - 3zi - 3i$$

$$\Rightarrow 2wz - 2w - wz - w = -3i(z+1)$$

$$\Rightarrow w[2z - 2 - z - 1] = -3i(z+1)$$

$$\Rightarrow w[z-3] = -3i(z+1)$$

$$\Rightarrow w = -3i\frac{(z+1)}{(z-3)}$$

Note: Either image or object or both are infinity should not apply the following Aliter method.

Aliter:

Given $z_1 = -1, z_2 = 0, z_3 = 1,$

$$w_1 = 0, w_2 = i, w_3 = 3i,$$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Let $A = \frac{w_2-w_3}{w_2-w_1} = \frac{i-3i}{i-0} = \frac{-2i}{i} = -2$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{0-1}{0+1} = -1$$

$$\Rightarrow a = Aw_1 - Bw_3 = 0 + 3i = 3i$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-1)(3i)(-1) - 0 = 3i$$

$$\Rightarrow c = A - B = (-2) - (-1) = -1$$

$$\Rightarrow d = Bz_1 - Az_3 = (-1)(-1) - (-2)(1) = 3$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(3i) + z(3i)}{(-1)z + 3}$$

Example: 1.70 Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, 1, -i$ respectively. [Anna, May 2002]

Solution:

Given $z_1 = -1, z_2 = 0, z_3 = 2,$

$$w_1 = 0, w_2 = i, w_3 = -i,$$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Let $A=\frac{w_2-w_3}{w_2-w_1}=\frac{i+i}{i-0}=\frac{2i}{i}=2$

$$B=\frac{z_2-z_3}{z_2-z_1}=\frac{0-2}{0+2}=-1$$

$$\Rightarrow a=Aw_1-Bw_3=(2)(0)-(-1)(-1)=-i$$

$$\Rightarrow b=Bw_3z_1-Aw_1z_3=(-1)(-i)(-2)-(2)(0)(2)=-2i$$

$$\Rightarrow c=A-B=2-(-1)=3$$

$$\Rightarrow d=Bz_1-Az_3=(-1)(-1)-(2)(2)=-2$$

We know that, $w=\frac{az+b}{cz+d}, ad-bc\neq 0$

$$\therefore w=\frac{(-i)z+(-2i)}{3z+(-2)}$$

Example: 1.71 Find the bilinear transformation which maps $z=1,i,-1$ respectively onto $w=i,0,-i$. Hence find the fixed points. [A.U, May 2001][A.U April 2016 R-15 U.D]

Solution:

Given$z_1=1, z_2=i, z_3=-1,$

$$w_1=i, w_2=0, w_3=-i,$$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let\quad A=\frac{w_2-w_3}{w_2-w_1}=\frac{0+i}{0-i}=-1$$

$$B=\frac{z_2-z_3}{z_2-z_1}=\frac{i+1}{i-1}=-i$$

$$\Rightarrow a=Aw_1-Bw_3=(-1)(i)-(-i)(-i)=-i+1$$

$$\Rightarrow b=Bw_3z_1-Aw_1z_3=(-i)(-i)(1)-(-1)(i)(-1)=-1-i$$

$$\Rightarrow c=A-B=(-1)-(-i)=-1+i$$

$$\Rightarrow d=Bz_1-Az_3=(-i)(1)-(-1)(-1)=-i-1$$

We know that, $w=\frac{az+b}{cz+d}, ad-bc\neq 0$

$$\therefore w=\frac{(-i+1)z+(-1-i)}{(-1+i)z+(-i-1)}=\frac{iz+1}{(-i)z+1}$$

Example: 1.72 Find the bilinear transformation which maps $z = 0$ onto $w = -i$ and has -1 and 1 as the invariant points. Also show that under this transformation the upper half of the z plane maps onto the interior of the unit circle in the w plane. Solution:

Given $z_1 = 0, z_2 = -1, z_3 = 1,$

$$w_1 = -i, w_2 = -1, w_3 = 1,$$

Let the required transformation be

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$Let \quad A = \frac{w_2 - w_3}{w_2 - w_1} = \frac{-1 - 1}{-1 + i} = \frac{-2}{-1 + i} = 1 + i$$

$$B = \frac{z_2 - z_3}{z_2 - z_1} = \frac{-1 - 1}{-1 - 0} = 2$$

$$\Rightarrow a = Aw_1 - Bw_3 = (1 + i)(-i) - 2(1) = -i + 1 - 2 = -i - 1$$

$$\Rightarrow b = Bw_3 z_1 - Aw_1 z_3 = (2)(1)(0) - (1 + i)(-i)(1) = i - 1$$

$$\Rightarrow c = A - B = (1 + i) - 2 = i - 1$$

$$\Rightarrow d = Bz_1 - Az_3 = (2)(0) - (1 + i)(1) = -(1 + i)$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i + 1)z + (i - 1)}{(i - 1)z + (-1 - i)} = \frac{z + (-i)}{(-i)z + 1}$$

$$\text{We know that,} z = \frac{-dw + b}{cw - a} = \frac{-w - i}{-iw - 1} = \frac{w + i}{1 + wi}$$

$$z = \frac{u + iv + i}{1(u + iv)i}$$

$$= \frac{u + iv + i}{1 + iu - v} = \frac{u + iv + i}{(1 - v) + iu}$$

$$= \left[\frac{u + iv + i}{(1 - v) + iu}\right]\left[\frac{1 - v - iu}{(1 - v) - iu}\right]$$

$$= \frac{u - uv - iu^2 + iv - iv^2 + uv + i - iv + u}{(1 - v)^2 + u^2}$$

$$x + iy = \frac{2u + i[-u^2 - v^2 + 1]}{(1 - v)^2 + u^2}$$

$$\Rightarrow \quad y \quad = \frac{1 - u^2 - v^2}{(1 - v)^2 + u^2}$$

Upper half of the z $-$plane

$$\Rightarrow y \geq 0$$

$$\Rightarrow \frac{1-u^2-v^2}{(1-v)^2+u^2} \geq 0$$

$$\Rightarrow 1-u^2-v^2 \geq 0$$

$$\Rightarrow 1 \geq u^2+v^2$$

$$\Rightarrow u^2+v^2 \leq 1$$

Therefore the upper half of the z −plane maps onto the interior of the unit circles in the w-plane.

Example: 1.73 Find the Bilinear transformation that maps the points $1+i, -i, 2-i$ of the z −plane into the points $0, 1, i$ of the w-plane. [A.U M/J 2007, N/D 2007]

Solution:

Given $z_1 = 1+i \quad w_1 = 0$

$z_2 = -i \quad w_2 = 1$

$z_3 = 2-i \quad w_3 = i$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{1-i}{1-0} = 1-i = \frac{1-i}{1+2i}(1+2i) = \frac{3+i}{1+2i}$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{-i-2+i}{-i-1-i} = \frac{-2}{-1-2i} = \frac{2}{1+2i}$$

$$\Rightarrow a = Aw_1 - Bw_3 = \left(\frac{3+i}{1+2i}\right)(0) - \left(\frac{2}{1+2i}\right)(i) = \frac{-2i}{1+2i}$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = \left(\frac{2}{1+2i}\right)(i)(1+i) - 0 = \frac{-2+2i}{1+2i}$$

$$\Rightarrow c = A - B = \frac{3+i}{1+2i} - \frac{2}{1+2i} = \frac{1+i}{1+2i}$$

$$\Rightarrow d = Bz_1 - Az_3 = \left(\frac{2}{1+2i}\right)(1+i) - \left(\frac{3+i}{1+2i}\right)(2-i) = \frac{-5+3i}{1+2i}$$

We know that, $w = \frac{az+b}{cz+d}, ad-bc \neq 0$

$$\Rightarrow \; w = \frac{\left(\frac{-2i}{1+2i}\right)z + \left(\frac{2i-2}{1+2i}\right)}{\left(\frac{1+i}{1+2i}\right)z + \left(\frac{3i-5}{1+2i}\right)}$$

$$\Rightarrow w = \frac{(-2i)z + (2i-2)}{(1+i)z + (3i-5)}$$

Verification:

(i) If $z = 1 + i$,then

$$w = \frac{(-2i)(1+i) + (2i-2)}{(1+i)(1+i) + (3i-5)}$$

$$= \frac{-2i + 2 + 2i - 2}{(1+i)(1+i) + (3i-5)} \quad = 0$$

(ii) If $z = -i$, then

$$w = \frac{(-2i)(-i) + (2i-2)}{(1+i)(-i) + (3i-5)}$$

$$= \frac{-2 + 2i - 2}{-i + 1 + 3i - 5} = \frac{2i-4}{2i-4} = 1$$

(iii)If $z = -i$, then

$$w = \frac{(-2i)(2-i) + (2i-2)}{(1+i)(2-i) + (3i-5)} = \frac{-4i - 2 + 2i - 2}{2 - i + 2i + 1 + 3i - 5}$$

$$= \frac{-2i-4}{4i-2} = \frac{-i-2}{2i-1} \times \frac{2i+1}{2i+1}$$

$$= \frac{2 - i - 4i - 2}{-4-1} = \frac{-5i}{-5} = i$$