4.2 Substitution Rule

Substitution Rule:

Let us see the suitable substitution to convert the given integral into a standard form.

The integrand of the form

(i) $\int F\big(f(x)\big)\,f'(x)dx$

(ii) $\int \big(f(x)\big)^n\, f'(x)dx$

(iii) $\int \frac{f'(x)}{(f(x))^n}\,dx$

(iv) $\int \frac{f'(x)}{F(f(x))}\,dx$

(v) $\int \frac{e^{f(x)}}{f'(x)}\,dx$

(vi) $\int e^{f(x)} f'(x)dx$

Substitute $u = f(x) \therefore du = f'(x)$ and then proceed.

Algebraic functions:

Example:

(i) Evaluate $\int \sqrt{2x+1}\,dx$.

Solution:

Put $u = 2x+1 \quad \Rightarrow \quad du = 2dx \quad \Rightarrow \quad dx = \frac{du}{2}$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{2}{2\times 3}(u)^{3/2} + C = \frac{(2x+1)^{3/2}}{3} + C$$

(ii) Evaluate $\int \frac{1}{(ax+b)^4}\,dx$.

Solution:

Put $u = ax+b \quad \Rightarrow \quad du = adx \quad \Rightarrow \quad dx = \frac{du}{a}$

$$\int \frac{1}{(ax+b)^4}\,dx = \int \frac{1}{u^4}\frac{du}{a}$$

$$= \frac{1}{a}\int u^{-4}du$$

$$= \frac{1}{a}\left[\frac{u^{-3}}{-3}\right] + C$$

$$= \frac{-1}{3a}\left[\frac{1}{u^3}\right] + C = \frac{-1}{3a}\left[\frac{1}{(ax+b)^3}\right] + C$$

(iii) Evaluate $\int x^5\sqrt{x^2+1}\,dx$.

Solution:

Put $u = x^2 + 1 \quad \Rightarrow x^2 = u - 1; \quad du = 2xdx \Rightarrow xdx = \frac{du}{2}$

$$\int x^5\sqrt{x^2+1}dx = \int x^4 \sqrt{x^2+1}\, x\, dx$$

$$= \int \sqrt{u}\,(u-1)^2\,\frac{du}{2}$$

$$= \frac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\,du$$

$$= \frac{1}{2}\int\left(u^{5/2} - 2\,u^{3/2} + u^{1/2}\right)\,du$$

$$= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - \frac{2u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + C$$

$$= \left(\frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} - \frac{u^{3/2}}{3}\right) + C$$

$$= \left(\frac{(x^2+1)^{7/2}}{7} - \frac{2(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3}\right) + C$$

(iv)Evaluate $\int \frac{x^2}{\sqrt{x+5}}\,dx$

Solution:

Given $\int \frac{x^2}{\sqrt{x+5}}\,dx$

$$Put\ u = \sqrt{x+5} \quad \Rightarrow \quad du = \frac{1}{2\sqrt{x+5}}dx$$

$$\Rightarrow\ 2du = \frac{1}{\sqrt{x+5}}dx$$

$$u^2 = x+5 \quad \Rightarrow\ x = u^2 - 5 \quad \Rightarrow\ x^2 = (u^2-5)^2 = u^4 - 10u^2 + 25$$

$$\int \frac{x^2}{\sqrt{x+5}}\,dx = \int (u^4 - 10u^2 + 25)\,2\,du = 2\int (u^4 - 10u^2 + 25)du$$

$$= 2\left[\frac{u^5}{5} - 10\frac{u^3}{3} + 25u\right] + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50(x+5)^{1/2} + C$$

(v)Evaluate $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

Solution:

Given $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

$$Put\ u = 1+\sqrt{x} \quad \Rightarrow\ du = \frac{1}{2\sqrt{x}}dx \quad \Rightarrow\ 2du = \frac{1}{\sqrt{x}}dx$$

$$= \int \frac{1}{u^2}\,2\,du = 2\int u^{-2}\,du = 2\left(\frac{u^{-1}}{-1}\right) + C$$

$$= -\frac{2}{u} + C$$

$$= -\frac{2}{1+\sqrt{x}} + C$$

(vi)Evaluate $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\, dx$

Solution:

Given $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\, dx$

$Put\ \ u = 1 + \sqrt{x} \quad \Rightarrow\ du = \frac{1}{2\sqrt{x}} dx \qquad \Rightarrow\ 2du = \frac{1}{\sqrt{x}} dx$

$$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\, dx = \int u^{1/3}\, 2\, du = 2\int u^{1/3}\, du = 2\, \frac{u^{4/3}}{(4/3)} + C$$

$$= \frac{3}{2}(u)^{4/3} + C$$

$$= \frac{3}{2}(1 + \sqrt{x})^{4/3} + C$$

Logarithmic functions:

Example :

(i) Evaluate $\int \frac{logx}{x}\, dx$

Solution:

Given $\int \frac{logx}{x}\, dx$

Put $u = logx \ \Rightarrow\ du = \frac{1}{x}\, dx$

$$\int \frac{logx}{x}\, dx = \int udu = \frac{u^2}{2} + C \quad = \frac{(logx)^2}{2} + C$$

(ii)Evaluate: $\int \frac{(logx)^2}{x}\, dx$

Solution:

Given $\int \frac{(logx)^2}{x}\, dx$

Put $u = logx \qquad \Rightarrow du = \frac{1}{x}\, dx$

$$\int \frac{(logx)^2}{x}\, dx = \int u^2 du = \frac{u^3}{3} + C \quad = \frac{(logx)^3}{3} + C$$

(iii)Evaluate $\int \frac{sin(2+logx)}{x}\, dx$

Solution:

Given $\int \frac{sin(2+logx)}{x}\, dx$

Put $u = 2 + logx \quad \Rightarrow du = \frac{1}{x}\, dx$

$$\int \frac{sin(2+logx)}{x}\,dx = \int sin\,u\,du = -cos\,u + C$$

$$= -\cos(2 + logx) + C$$

(iv)Evaluate $\int \frac{dx}{x\sqrt{logx}}$

Solution:

Given $\int \frac{dx}{x\sqrt{logx}}$

Put $u = logx \quad \Rightarrow du = \frac{1}{x}\,dx$

$$\int \frac{dx}{x\sqrt{logx}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C \quad = 2\sqrt{logx} + C$$

(v)Evaluate $\int secx\log(secx + tanx)\,dx$

Solution:

Given $\int secx\log(secx + tanx)\,dx$

Put $u = \log(secx + tanx) \quad \Rightarrow du = \frac{1}{(secx+tanx)}(secxtanx + sec^2x)dx$

$$\Rightarrow du = \frac{sec(tanx+secx)}{(secx+tanx)}dx$$

$$= \int udu = \frac{u^2}{2} + C$$

$$= \frac{1}{2}[log(secx + tanx)]^2 + C$$

Exponential functions

Example:

(i)Evaluate $\int e^{cosx}sinx\,dx$

Solution:

Given $\int e^{cosx}sinx\,dx$

$$Put\ u = e^{cosx} \quad \Rightarrow \quad du = e^{cosx}(-sinx)dx$$

$$\int e^{cosx}sinx\,dx = \int(-du) \quad = -\int du \quad = -u + C \quad = -e^{cosx} + C$$

(ii)Evaluate $\int e^{x^3}x^2dx$

Solution:

Given $\int e^{x^3}x^2dx$

$$Put\ u = e^{x^3} \quad \Rightarrow \quad du = e^{x^3}3x^2dx \quad \Rightarrow \quad \frac{du}{3} = e^{x^3}x^2dx$$

$$\int e^{x^3}x^2dx = \int \frac{du}{3} \quad = \frac{1}{3}\int du \quad = \frac{1}{3}u + C \quad = \frac{1}{3}e^{x^3} + C$$

(iii)Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution:

Given $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$$Put\ u = e^{\sqrt{x}} \quad \Rightarrow \quad du = e^{\sqrt{x}}\frac{1}{2\sqrt{x}}dx \quad \Rightarrow \quad 2du = \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int 2du \quad = 2\int du \quad = 2u + C \quad = 2e^{\sqrt{x}} + C$$

(iv)Evaluate $\int \frac{e^{tan^{-1}x}}{1+x^2} dx$

Solution:

Given $\int \frac{e^{tan^{-1}x}}{1+x^2} dx$

Put $u = tan^{-1}x$ $\qquad du = \frac{1}{1+x^2} dx$

$$\int \frac{e^{tan^{-1}x}}{1+x^2} dx = \int e^u du = e^u + C$$

$$= e^{tan^{-1}x} + C$$

(v)Evaluate $\int \frac{1}{e^x+e^{-x}} dx$

Solution:

Given $\int \frac{1}{e^x+\frac{1}{e^x}} dx = \int \frac{e^x dx}{e^{2x}+1}$

Put $e^x = u \quad \Rightarrow \quad e^x dx = du$

$$= \int \frac{du}{u^2+1}$$

$$= tan^{-1}u + C \quad = tan^{-1}e^x + C$$