2.2 Butterworth filters, Chebyshev filters. Design of IIR filters from analog filters (LPF, HPF, BPF, BRF)

1. Butterworth Filter

Definition

A **Butterworth filter** is an analog IIR filter with a **maximally flat magnitude response** in the passband (no ripple).

Magnitude Response

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where

- Ω_c = cutoff frequency
- N = order of the filter

Characteristics

- Flat passband (no ripple)
- Smooth monotonic response
- Moderate roll-off
- Phase is non-linear
- Widely used when smooth response is required

Pole Location

- Poles lie on a circle of radius Ωc
- Evenly spaced in the left half of the s-plane

2. Chebyshev Filters

Chebyshev filters provide **sharper transition** than Butterworth at the cost of **ripples**.

Magnitude Response

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

where

- ϵ = ripple factor
- T_N = Chebyshev polynomial of order N

Characteristics

- Ripple in **passband**
- Monotonic stopband
- Sharper roll-off than Butterworth
- Non-linear phase

(b) Chebyshev Type-II Filter (Inverse Chebyshev)

Characteristics

- Flat passband
- Ripple in **stopband**
- Better stopband attenuation
- More complex design

Comparison: Butterworth vs Chebyshev-I

Feature	Butterworth	Chebyshev-I
Passband ripple	No	Yes
Transition width	Wide	Narrow
Roll-off	Moderate	Sharp
Complexity	Simple	Higher

3. Design of Digital IIR Filters from Analog Filters

Digital IIR filters are designed by **transforming an analog prototype**.

General Steps

1. Specify digital filter requirements
2. Pre-warp frequencies
3. Design analog LPF prototype
4. Frequency transformation (LPF → HPF/BPF/BRF)
5. Convert analog filter to digital filter
6. Obtain digital transfer function $H(z)$

4. Analog to Digital Transformation Methods

(a) Bilinear Transformation (Most Common)

$$s = \frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$

Advantages

- No aliasing
- Stable analog → stable digital
- One-to-one frequency mapping

(b) Impulse Invariance Method

- Preserves impulse response
- Causes **aliasing**
- Used mainly for LPF

5. Frequency Transformations (Analog Domain)

(a) LPF → HPF

$$s \to \frac{\Omega_c}{s}$$

(b) LPF → BPF

$$s \to \frac{s^2 + \Omega_0^2}{Bs}$$

where

- $\Omega_0 = \sqrt{\Omega_1 \Omega_2}$
- $B = \Omega_2 - \Omega_1$

(c) LPF → BRF

$$s \to \frac{Bs}{s^2 + \Omega_0^2}$$

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6. IIR Filter Design for Different Types

(a) Digital Low-Pass Filter (LPF)

- Design analog LPF
- Apply bilinear transformation
- Obtain $H(z)$

(b) Digital High-Pass Filter (HPF)

- Convert analog LPF → HPF
- Apply bilinear transformation

(c) Digital Band-Pass Filter (BPF)

- Convert LPF → BPF
- Requires two cutoff frequencies

(d) Digital Band-Reject Filter (BRF)

- Convert LPF → BRF
- Rejects a specific frequency band

7. Advantages of IIR Filters

- Lower order than FIR
- Efficient computation
- Suitable for real-time applications

Disadvantages

- Non-linear phase
- Potential stability issues

Butterworth filters are preferred for **smooth passband response**, while Chebyshev filters are used when **sharp transition** is required. Digital IIR filters are efficiently designed by transforming analog prototypes using **bilinear transformation**, enabling LPF, HPF, BPF, and BRF realizations.

Design an analog Butterworth filter that has a -2dB pass band attenuation at a frequency of 20 rad/sec and atleast -10 dB stop band attenuation at 30 rad/sec.

Solution:

Given data:

Pass band attenuation α_P= 2 dB;

Stop band attenuation α_S= 10 dB;

Pass band frequency Ω_P= 20 rad/sec.

Stop band frequency Ω_S=30 rad/sec.

The order of the filter

$$N \geq \left[\frac{\log_{10} \sqrt{\left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right)}}{\log_{10} \left(\frac{\Omega_s}{\Omega_P} \right)} \right]$$

$$N \geq \left[\frac{\log_{10} \sqrt{\left(\frac{10^{0.1*10} - 1}{10^{0.1*2} - 1} \right)}}{\log_{10} \left(\frac{30}{20} \right)} \right]$$

$$\geq \frac{\log \sqrt{\left(\frac{10 - 1}{10^{0.2} - 1} \right)}}{\log \frac{30}{20}}$$

$$\geq 3.37$$

Rounding off 'N' to the next higher integer, we get

N=4

The normalized transfer function for *N=4*.

$$H_a(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

To find cut off frequency

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha p} - 1)^{1/2N}}$$

$$\Omega_c = \frac{20}{(10^{0.1*20} - 1)^{\frac{1}{2*4}}} = 21.3868$$

The transfer function for Ωc=21.3868,

$$H(s) = H_a(s)\Big|_{S->\frac{s}{21.3868}}$$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}\Bigg|s->\frac{s}{21.3868}$$

$$H(s) = \frac{1}{\left(\frac{s}{21.868}\right)^2 + 0.76535\left(\frac{s}{21.3868}\right)+1} * \frac{1}{\left(\frac{s}{21.868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right)+1}$$

$$H(s) = \frac{0.20921\times 10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

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H.W: Challenge 1: For the given specification design an analog Butterworth filter

$$0.9 \le |H(j\Omega)| \le 1 \quad \text{for } 0 \le \Omega \le 0.2\pi$$

$$|H(j\Omega)| \le 0.2 \text{ for } 0.4\pi \le \Omega \le \pi.$$

Ans: $$H(s) = \frac{0.323}{(s^2+0.577s+0.0576\pi^2)(s^2+1.393s+0.0576\pi^2)}$$

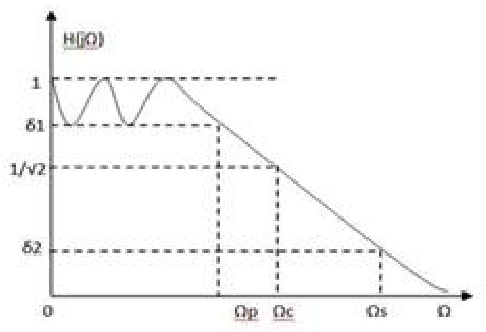
Challenge 2: Determine the order and the poles of low pass Butterworth filter that has 3 dB attenuation at 500 Hz and an attenuation of 40dB at 1000Hz.

Ans: $H(s) = (s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.445s+1)$

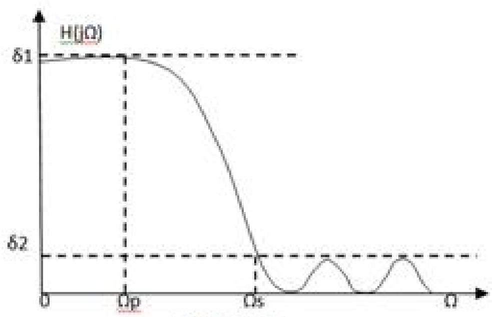
Given the specification $\alpha_p = 1dB; \alpha_s = 30dB; \Omega_p = 200\,rad/\sec; \Omega_s = 600\,rad/sec.$ **determine the order of the filter.** **Ans:** ***N=4***

Analog Low pass Chebyshev Filter:

There are two types of Chebyshev filters.



Odd - Type I



Even - Type II

Given specifications αp= 3dB,αs=16 dB, f_P=1KHz and f_S=2KHz. Determine the order of the filter using Chebyshev approximation. Find H(s).

Solution:

Given:

Step 1:

Pass band attenuation αp= 3dB,
Stop band attenuation αs=16 dB,
Pass band frequency f_P=1 KHz=2π*1000=2000π rad/sec
Stop band frequency f_S=2 KHz=2π*2*1000=4000π rad/sec

Step 2: Order of the filter

$$N \geq \frac{\cosh^{-1}\sqrt{\left(\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}\right)}}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)}$$

$$N \geq \frac{\cosh^{-1}\sqrt{\left(\frac{10^{0.1*16}-1}{10^{0.1*3}-1}\right)}}{\cosh^{-1}\left(\frac{4000\pi}{2000\pi}\right)}$$

$$\geq 1.91$$

Rounding the next higher integer value *N=2.*

Step 3: The value of minor axis and major axis can be found as below

$$\varepsilon = \sqrt{\left(10^{0.1*\alpha_P}-1\right)} = \sqrt{\left(10^{0.1*3}-1\right)} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 1^{-1} + \sqrt{1+1^{-2}} = 2.414$$

$$a = \Omega_P \frac{\left[\mu^{\frac{1}{N}} - \mu^{-\frac{1}{N}}\right]}{2} = 2000\pi \frac{\left[(2.414)^{\frac{1}{2}} - (2.414)^{-\frac{1}{2}}\right]}{2} = 910\pi$$

$$b = \Omega_S \frac{\left[\mu^{\frac{1}{N}} + \mu^{-\frac{1}{N}}\right]}{2} = 4000\pi \frac{\left[(2.414)^{\frac{1}{2}} + (2.414)^{-\frac{1}{2}}\right]}{2} = 2197\pi$$

Step 4: The poles are given by

$$S_K = a\cos\phi_k + jb\sin\phi_k; \quad k = 1,2$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}; \qquad k = 1,2$$

For k = 1

$$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{2*2} = \frac{\pi}{2} + \frac{\pi}{4} = 135^o$$

Fork = 2

$$\phi_2 = \frac{\pi}{2} + \frac{(2*2-1)\pi}{2*2} = \frac{\pi}{2} + \frac{3\pi}{4} = 225^o$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 = (910\pi * \cos 135) + j(2197 * \sin 135)$$

$$s_1 = -643.46\pi + j1553\pi$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2 = (910\pi * \cos 225) + j(2197 * \sin 225)$$

$$s_2 = -643.46\pi - j1553\pi$$

Step 5: The denominator of *H(s):*

$$H(s) = (s + 643.46\pi)^2 + (1554)^2$$

Step 6: The numerator of *H(s):*

$substitute\ , s = 0 \qquad H(s) = \frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}}$

$$= \frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+1^2}} = (1414.38)^2 \pi^2$$

The transfer function $H(s) = \dfrac{(1414.38)^2 \pi^2}{s^2 + 1287\pi s + (1682)^2 \pi^2}$

HW: Challenge 1: Obtain an analog Chebyshev filter transfer function that satisfies the constraints

$$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1; \ for\ 0 \le \Omega \le 2$$

$$|H(j\Omega)| \le 0.1 \ for\ \Omega \ge 4$$

Ans: $H(s) = \dfrac{2}{(s+0.596)(s^2 + 0.596s + 3.354)}$

2. Design a Chebyshev filter with a maximum pass band attenuation of 2.5dB at Ω_P=20rad/sec and stop band attenuation of 30 dB at Ω_S=50rad/sec.

Ans: ***N=3.*** $H(s) = \dfrac{2265.27}{(s+6.6)(s^2 + 6.6s + 343.2)}$

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3. For the given specifications find the order of the Chebyshev-I filter

$\alpha_P = 1.5dB; \alpha_S = 10dB; \Omega_P = 2rad/Sec; \ \Omega_S = 30rad/\sec.$

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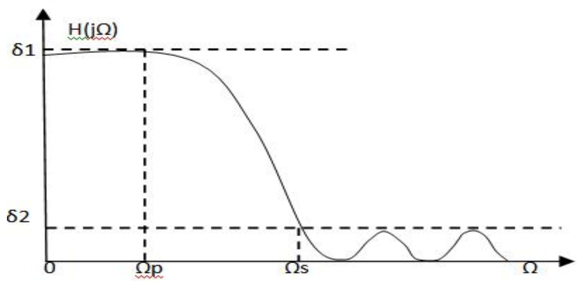
4. For the given specifications find the order of the Chebyshev-I filter

$\alpha_P = 1dB; \alpha_S = 25dB; \Omega_P = 1rad/Sec; \ \Omega_S = 20rad/\sec.$

<u>Discrete time IIR filter from analog filter:</u>

Magnitude Response of LPF:



Design of IIR filters from analog filters:

The different design techniques available for IIR filter are

1) Approximation of derivates
2) Impulse invariant method
3) Bilinear transformation
4) Matched z-transform techniques.

<u>Approximation of derivates:</u>

For analog to digital domain, we get

$$s = \frac{1 - z^{-1}}{T} \text{ -------------------------- (3)}$$

$$H(z) = H(s)\Big|_{s=\frac{1-z^{-1}}{T}} \text{ ---------------------- (4)}$$

Mapping of the s-plane to the z-plane using approximation of derivatives.

Convert the analog BPF with system IIR filter $H_a(s) = \frac{1}{(s+0.1)^2 + 9}$ into a digital IIR filter by use of the backward difference for the derivative. [Nov/Dec-2015]

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

$$H(z) = \frac{1}{(s+0.1)^2 + 9}\Bigg|_{s=\frac{1-z^{-1}}{T}}$$

$$= \frac{1}{\left(\frac{1-z^{-1}}{T} + 0.1\right)^2 + 9}$$

$$= \frac{T^2\left(1 + 0.2T + 9.01T^2\right)}{1 - \frac{2(1+0.1T)}{1+0.2T+9.01T^2} z^{-1} + \frac{1}{1+0.2T+9.01T^2} z^{-2}}$$

$T = 0.1\, sec,$

$$= 0.91 \pm j0.27$$

<u>**Design of IIR filter using Impulse Invariance Method:**</u>

<u>**Steps to design a digital filter using Impulse Invariance Method (IIM):**</u>

Step 1: For the given specifications, find Ha(s) the Transfer function of an analog filter.

Step 2: Select the sampling rate of the digital filter, T seconds per sample.

Step 3: Express the analog filter transfer function as the sum of single-pole filter.

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

Step 4: Compute the z-transform of the digital filter by using formula

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k T} z^{-1}}$$

For high sampling rate,

$$\therefore H(z) = \sum_{k=1}^{N} \frac{TC_k}{1 - e^{p_k T} z^{-1}}$$

For the analog transfer function $H(s) = \frac{2}{s^2 + 3s + 2}$ Determine H (z) using impulse invariant transformation if (a) T=1 second and (b) T=0.1 second. [Nov/Dec-15]

Solution:

$$Given\,that\,, H(s) = \frac{2}{s^2+3s+2} = \frac{2}{(s+1)(s+2)}$$

By partial fraction exp*ansion tchnique we can writ e,*

$$\boxed{\begin{aligned}&The\,roots\,of\,quadratic,\\&S2+3S+2=0\,are,\\&s=\frac{-3\pm\sqrt{3^2-4X2}}{2}\\&=\frac{-3\pm1}{2}=-1,-2\end{aligned}}$$

$$H(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \frac{2}{\cancel{(s+1)}\,(s+2)}X\,\cancel{(s+1)}\Bigg|_{s=-1} = \frac{2}{-1+2} = 2$$

$$B = \frac{2}{(s+1)\,\cancel{(s+2)}}X\,\cancel{(s+2)}\Bigg|_{s=-2} = \frac{2}{-2+1} = -2$$

$$\therefore H(s) = \frac{2}{s+1} + \frac{-2}{s+2}$$

By impulse in var*iant transformation we know that,*

$$\frac{A_i}{s+p_i} \to \frac{A_i}{1-e^{-p_iT}z^{-1}}$$

$$H(z) = \frac{2}{1-e^{-p_iT}z^{-1}} + \frac{-2}{1-e^{-p_iT}z^{-1}}\ Where\ p_1 = 1, p_2 = -2$$

$$\boxed{H(z) = \frac{2}{1-e^{-T}z^{-1}} + \frac{-2}{1-e^{-2T}z^{-1}}}$$

(a) When T $=1$ second

$$H(\text{z})=\frac{2}{1-e^{-1}z^{-1}}+\frac{-2}{1-e^{-2}z^{-1}}$$

$$H(z)=\frac{2}{1-0.1353z^{-1}}+\frac{2}{1-0.1353z^{-1}}=\frac{2(1-0.1353\,z^{-1})-2(1-0.3679\,z^{-1})}{(1-0.367z^{-1})(1-0.1353z^{-1})}$$

$$=\frac{2-0.27606z^{-1}-2+0.7358z^{-1}}{1-0.1353z^{-1}-0.3679z^{-1}+0.0498z^{-2}}=\frac{0.4652z^{-1}}{1-0.5032\,z^{-1}+0.0498z^{-2}}$$

$$H(z)=\frac{0.4652z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}=\frac{0.4652z^{-1}}{\text{z}^{-2}(\text{z}^2-0.5032z+0.0498)}$$

$$\boxed{H(z)=\frac{0.4652z}{(\text{z}^2-0.5032z+0.0498)}}$$

(b) *When* $T=0.1$ sec*ond*

$$H(z)=\frac{2}{1-e^{-0.1}z^{-1}}+\frac{-2}{1-e^{-0.2}z^{-1}}$$

$$=\frac{2}{1-0.9048z^{-1}}+\frac{-2}{1-0.8187z^{-1}}=\frac{2(1-0\ .8187\,z^{-1})-2(1-0.9048z^{-1})}{(1-0.8187z^{-1})(1-0.9048z^{-1})}$$

$$=\frac{2-1.6374z^{-1}-2+1.8096z^{-1}}{1-0.8187z^{-1}-0.9048z^{-1}+0.7408z^{-2}}=\frac{0.1722z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}}$$

$$H(z)=\frac{0.1722z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}}=\frac{0.1722z^{-1}}{z^{-2}(z^2-1.7235z+0.7408)}$$

$$=\frac{0.1722z}{z^2-1.7235z+0.7408}$$

Since, $T<1$, *we can compute magnitude normalized tra nsfer function*, $H(z)$.

$$H_N(z)=T\ X\,H(z)=0.1X\ \frac{0.1722z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}}=\frac{0.1722z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}}$$

$$H_N(z)=T\ X\,H(z)=0.1X\ \frac{0.1722z}{z^2-1.7235z+0.7408}=\frac{0.172z}{z^2-1.7235z+0.7408}$$

Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period *T=1 sec.*

Given: For N=3, the transfer function of a normalized Butterworth filter is given by

$$H(s)=\frac{1}{(s+1)(s^2+s+1)}$$

Ans: $H(z) = \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$

Apply impulse invariant method and find *H(z) for* $H(s) = \frac{s+a}{(s+a)^2+b^2}$

Solution:

Given: The transfer function $H(s) = \frac{s+a}{(s+a)^2+b^2}$

Sampling the function produces

$$h(nT) = \begin{cases} e^{-anT}\cos(bnT) & for\ n \geq 0 \\ 0 & otherwise \end{cases}$$

$$H(z) = \sum_{n=0}^{\infty} e^{-anT}\cos(bnT)z^{-n}$$

$$= \sum_{n=0}^{\infty}\left[e^{-anT}z^{-n}\left(\frac{e^{jbnT}+e^{-jbnT}}{2}\right)\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(e^{-(a-jb)T}z^{-1}\right)^n + \left(e^{-(a+jb)T}z^{-1}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$H(z) = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

Convert analog filter $H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$ **into digital IIR filter using impulse invariant method**

[Nov/Dec-2015]

Solution:

Given: Analog filter $H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$

$$\frac{s+a}{(s+a)^2+b^2} = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{s+0.1}{(s+0.1)^2+9} = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}; T = 1\,sec$$

$$\frac{s+0.1}{(s+0.1)^2+9}=\frac{1-e^{-0.1}\cos(3)z^{-1}}{1-2e^{-0.1}\cos(3)z^{-1}+e^{-2*0.1}z^{-2}};$$

$$=\frac{1-0.9048*(-0.9899)z^{-1}}{1+1.791z^{-1}+0.818z^{-2}}$$

$$H(z)=\frac{1+0.89566z^{-1}}{1+1.7915z^{-1}+0.818z^{-2}}$$

Convert analog filter $H_a(s)=\frac{6}{(s+0.1)^2+36}$ into digital IIR filter whose system function is given above. The digital filter should have ($\omega r = 0.2\pi$). Use impulse invariant mapping T=1sec.

Solution:

Given: Analog filter $H_a(s)=\frac{6}{(s+0.1)^2+36}$

$$\frac{b}{(s+a)^2+b^2}=\frac{e^{-aT}\sin bTz^{-1}}{1-2e^{-aT}\cos bTz^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{6}{(s+0.1)^2+36}=\frac{e^{-0.1}\sin(6)z^{-1}}{1-2e^{-0.1}\cos(6)z^{-1}+e^{-2*0.1}z^{-2}}$$

Assume T = 1 sec.

$$\frac{6}{(s+0.1)^2+36}=\frac{-0.2528z^{-1}}{1-2*0.8687z^{-1}+0.818z^{-2}}$$

$$=\frac{-0.2528z^{-1}}{1-1.7374z^{-1}+0.818z^{-2}}$$

H.W: Challenge 1: An analog filter has a transfer function $H_a(s)=\frac{10}{s^2+7s+10}$. Design a digital filter equivalent to this using impulse invariant method for T=0.2 sec. [Nov/Dec-15]

Ans : $H(z)=\frac{0.2012z^{-1}}{1-1.0378z^{-1}+0.247z^{-2}}$

2. An analog filter has a transfer function $H(s)=\frac{5}{s^3+6s^2+11s+6}$. Design a digital equivalent to this using impulse invariant method for T=1 sec.

3. An analog filter has a transfer function $H(s)=\frac{s+3}{s^2+6s+25}$. Design a digital filter equivalent to this using impulse invariant method T=1 sec.

<u>Design of IIR filters using Bilinear Transformation:</u>

Steps to design digital filter using bilinear transform technique:

1. From the given specifications, find prewarping analog frequencies using formula $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$
2. Using the analog frequencies find *H(s)* of the analog filter.
3. Select the sampling rate of the digital filter, call it T seconds per sample.
4. Substitute $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ into the transfer function found in step2.

Apply bilinear transformation of $H(s) = \frac{2}{(s+1)(s+2)}$ with *T=1 sec* and find *H(z).[Nov/Dec-13]*

Solution:

Given: The system function $H(s) = \frac{2}{(s+1)(s+2)}$

$$\text{Substitute } s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \text{ in } H(s) \text{ to get } H(z)$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Given *T=1 sec.*

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2\left(1+z^{-1}\right)^2}{\left(3-z^{-1}\right)(4)}$$

$$= \frac{\left(1+z^{-1}\right)^2}{6-2z^{-1}}$$

$$H(z) = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

Using the bilinear transformation, design a high pass filter, monotonic in pass band with cut off frequency of 1000Hz and down 10dB at 350 Hz. The sampling frequency is 5000Hz. [May/June-16]

Solution:

Given: Pass band attenuation $\alpha_P = 3dB$; Stop band attenuation $\alpha_S = 10dB$

Pass band frequency $\omega_P = 2\pi * 1000 = 2000\pi$ *rad/sec.*

Stop band frequency $\omega_S = 2\pi * 350 = 700\pi$ *rad/sec.*

$$T = \frac{1}{f} = \frac{1}{5000} = 2*10^{-4}\ \text{sec}.$$

Prewarping the digital frequencies, we have

$$\Omega_P = \frac{2}{T}\tan\frac{\omega_P T}{2} = \frac{2}{2*10^{-4}}\tan\frac{\left(2000\pi * 2*10^{-4}\right)}{2} = 10^4\ \tan(0.2\pi) = 7265\ \textit{rad/sec.}$$

$$\Omega_S = \frac{2}{T}\tan\frac{\omega_S T}{2} = \frac{2}{2*10^{-4}}\tan\frac{\left(700\pi * 2*10^{-4}\right)}{2} = 10^4\ \tan(0.07\pi) = 2235\ \textit{rad/sec.}$$

The order of the filter

$$N \ge \frac{\log\sqrt{\dfrac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{\log\dfrac{\Omega_S}{\Omega_P}}$$

$$= \frac{\log\sqrt{\dfrac{10^{0.1\times 10}-1}{10^{0.1\times 3}-1}}}{\log\dfrac{7265}{2235}}$$

$$= \frac{\log(3)}{\log(3.25)} = \frac{0.4771}{0.5118} = 0.932$$

$$N = 1$$

The first order Butterworth filter for Ω_C=1 rad/sec is H(s) = 1/S+1

The high pass filter for Ω_C=Ω_P=7265 rad/sec can be obtained by using the transformation.

$$S \to \frac{\Omega_C}{s}$$

$$S \to \frac{7265}{s}$$

The transfer function of high pass filter

$$H(s) = \left.\frac{1}{s+1}\right|_{s=\frac{7265}{s}}$$

$$= \frac{s}{s+7265}$$

$U \sin g$ *bilinear transformation*

$$H(z) = H(s)\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{s}{s+7265}\bigg|_{s=\frac{2}{2*10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+7265}$$

$$= \frac{0.5792\left(1-z^{-1}\right)}{1-0.1584z^{-1}}$$

H.W: 1. Determine H(z) that results when the bilinear transformation is applied to

Ha(s)$=\dfrac{s^2+4.525}{s^2+0.692s+0.504}$ **[Nov/Dec-15]**

Ans: $H(z) = \dfrac{1.4479+0.1783z^{-1}+1.4479z^{-2}}{1-1.18752z^{-1}+0.5299z^{-2}}$

2. An analog filter has a transfer function $H(s) = \dfrac{1}{s^2+6s+9}$,design a digital filter using bilinear transformation method.

Additional Examples:

Design a digital Butterworth filter satisfying the constraints

$$0.707 \le \left|H(e^{j\omega})\right| \le 1 \qquad for\ 0 \le \omega \le \frac{\pi}{2}$$

$$\left|H(e^{jw})\right| \le 0.2 \quad for\ \frac{3\pi}{4} \le \omega \le \pi$$

With *T=1 sec using bilinear transformation. [April/May-2015][May/June-14]*

Solution:

Given data:

Pass band attenuation $\alpha_P = 0.707$ *; Pass band frequency* $\omega_P = \dfrac{\pi}{2}$*;*

Stop band attenuation $\alpha_S = 0.2$*; Stops band frequency* $\omega_S = \dfrac{3\pi}{4}$*;*

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_P = -20\log\delta 1 = -20\log(0.707) = 3.0116\,dB$

Stop band attenuation $\alpha_S = -20\log\delta 2 = -20\log(0.2) = 13.9794\,dB$

Step2. Choose T and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P = \frac{2}{T}\tan\left(\frac{\omega_P T}{2}\right) = \frac{2}{1}\tan\left(\frac{\frac{\pi}{2}}{2}\right) = 2\tan\left(\frac{\pi}{4}\right) = 2 Rad / Sec$$

$$\Omega_S = \frac{2}{T}\tan\left(\frac{\omega_S T}{2}\right) = \frac{2}{1}\tan\left(\frac{\frac{3\pi}{4}}{2}\right) = 2\tan\left(\frac{3\pi}{8}\right) = 4.828 Rad / Sec$$

Step3. To find order of the filter

$$N \geq \left[\frac{\log_{10}\sqrt{\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)}}{\log_{10}\left(\frac{\Omega_s}{\Omega_P}\right)}\right]$$

$$N \geq \frac{\log\sqrt{\left(\frac{10^{0.1*3.01}-1}{10^{0.1*13.97}-1}\right)}}{\log\left(\frac{4.828}{2}\right)}$$

$$\geq \frac{log\sqrt{\left(\frac{0.9998}{23.945}\right)}}{log(2.414)}$$

$$\geq \frac{\log(0.20433)}{\log(2.414)}$$

$$\geq \frac{-0.6896}{0.382}$$

$$\geq 1.8017$$

Rounding the next higher value N=2

Step 4: The normalized transfer function

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Step 5: Cut off frequency

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha p}-1)^{1/2N}}$$

$$\Omega_c = \frac{2}{\left(10^{0.1*3.01}-1\right)^{\frac{1}{2*2}}} = \frac{2}{(0.9998)^{\frac{1}{4}}} = 2 Rad / Sec$$

Step 6: To find Transfer function of $H(s)$:

$$H(s) = H_a(s)\Big|_{S \to \frac{s}{2}}$$

$$H(s)=\left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{S\to\frac{s}{2}}$$

$$=\frac{1}{\left(\frac{s}{2}\right)^2+\sqrt{2}\left(\frac{s}{2}\right)+1}$$

$$H(s)=\frac{4}{s^2+2.828s+4}$$

Step 7. Apply Bilinear Transformation with to obtain the digital filter

$$H(z)=H(s)\Big|_{S=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$=\left.\frac{4}{s^2+2.828s+4}\right|_{S=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$=\frac{4\left(1-z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2+5.656\left(1-z^{-2}\right)+4\left(1+z^{-1}\right)^2}$$

$$H(z)=\frac{0.2929\left(1+z^{-1}\right)^2}{1+0.1716z^{-2}}$$

Design a digital Butterworth filter satisfying the constraints

$$0.707\le\left|H(e^{j\omega})\right|\le 1\qquad for\ 0\le\omega\le\frac{\pi}{2}$$

$$\left|H(e^{jw})\right|\le 0.2\ \ for\ \frac{3\pi}{4}\le\omega\le\pi$$

With *T=1 sec using Impulse invariant method. [Nov/Dec-13]*

Solution:

Given data:

Pass band attenuation $\alpha_P=0.707$ *; Pass band frequency* $\omega_P=\frac{\pi}{2}$ *;*

Stop band attenuation $\alpha_S=0.2$ *; Stops band frequency* $\omega_S=\frac{3\pi}{4}$ *;*

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_P=-20\log\delta 1=-20\log(0.707)=3.0116dB$

Stop band attenuation $\alpha_S=-20\log\delta 2=-20\log(0.2)=13.9794dB$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\omega_P=\Omega_P T=\frac{\pi}{2}Rad/Sec$$

$$\omega_S=\Omega_S T=\frac{3\pi}{4}Rad/Sec$$

Step3. To find order of the filter

$$N \geq \left[\frac{\log_{10} \sqrt{\left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right)}}{\log_{10} \left(\frac{\Omega_s}{\Omega_P} \right)} \right]$$

$$N \geq \frac{\log \sqrt{\left(\frac{10^{0.1*3.01} - 1}{10^{0.1*13.97} - 1} \right)}}{\log \left(\frac{\frac{3\pi}{4}}{\frac{\pi}{2}} \right)}$$

$$\geq \frac{log \sqrt{\left(\frac{0.9998}{23.945} \right)}}{log(1.5)}$$

$$\geq \frac{\log(0.20433)}{\log(1.5)}$$

$$\geq \frac{-0.6896}{0.17609}$$

$$\geq 3.924$$

Rounding the next higher value N=4

Step 4: The normalized transfer function

$$H_a(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

Step 5: Cut off frequency

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha p} - 1)^{1/2N}}$$

$$\Omega_c = \frac{\frac{\pi}{2}}{(10^{0.1*3.01} - 1)^{\frac{1}{2*4}}} = \frac{\frac{\pi}{2}}{(0.9998)^{\frac{1}{8}}} = 1.57 Rad/Sec$$

Step 6: To find Transfer function of *H(s)*:

$$H(s) = H_a(s)\Big|_{S \to \frac{s}{1.57}}$$

$$H(s) = \left. \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)} \right|_{S \to \frac{s}{1.57}}$$

$$= \frac{1}{\left(\left(\frac{s}{1.57}\right)^2 + 0.76537\left(\frac{s}{1.57}\right) + 1\right)\left(\left(\frac{s}{1.57}\right)^2 + 1.8477\left(\frac{s}{1.57}\right) + 1\right)}\Bigg|$$

$$H(s) = \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)}$$

Step 7: Using partial fraction expansion, expand H(s) into

$$H(s) = \frac{A}{(s+1.45+j0.6)} + \frac{A^*}{(s+1.45-j0.6)} + \frac{B}{(s+0.6+1.45j)} + \frac{B^*}{(s+0.6-1.45j)}$$

To find A and A:*

$$A = H(s)\big|_{S=-1.45-j0.6}$$

$$= \cancel{(s+1.45-j0.6)}\frac{(1.57)^4}{\cancel{(s+1.45+j0.6)}(s+1.45-j0.6)(s^2+2.902s+2.465)}\Bigg|_{S=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(s+1.45-j0.6)(s^2+1.202s+2.465)}\Bigg|_{S=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(-1.45-j0.6+1.45-j0.6)\left((-1.45-j0.6)^2+1.202(-1.45-j0.6)+2.465\right)}$$

$$= \frac{(1.57)^4}{-j(1.2)[1.7425+1.74j-1.7429-j0.7212+2.465]}$$

$$= \frac{(1.57)^4}{-j(1.2)[2.465+j1.0188]} = \frac{5.063}{1.0188-j2.465}$$

$$= \frac{5.063(1.0188-j2.465)}{7.114}$$

$$A = 0.7253 + j1.754;\ A^* = 0.7253 - j1.754$$

To find B and B:*

$$A = H(s)\big|_{S=-0.6-j1.45}$$

$$= \cancel{(s+0.6+j1.45)}\frac{(1.57)^4}{(s+1.45+j0.6)(s+1.45-j0.6)\cancel{(s+0.6+1.45j)}(s+0.6-1.45j)}\Bigg|_{S=-0.6-j1.45}$$

$$=\frac{(1.57)^4}{(-0.6-j1.45+1.45+j0.6)(-0.6-j1.45+1.45-j0.6)(-0.6-j1.45+0.6-1.45j)}\Bigg|_{S=-0.6-j1.45}$$

$$=\frac{(1.57)^4}{(0.85-j0.85)(0.85-j0.85)(-2.9j)}$$

$$=\frac{(1.57)^4}{-j[-1.0187-j2.468]}$$

$$=\frac{2.095}{-2.468+j1.0187}$$

$$B=-0.7253-j0.3;\quad B^*=-0.7253+j0.3$$

$$H(s)=\frac{0.7253+j1.754}{s-(-1.45-j0.6)}+\frac{0.7253-j1.754}{s-(-1.45+j0.6)}+\frac{-0.7253-0.3j}{s-(-0.6-j1.45)}+\frac{-0.7253+0.3j}{s-(-0.6+j1.45)}$$

we know for T = 1 sec

$$H(z)=\sum_{k=1}^{N}\frac{C_k}{1-e^{p_k}z^{-1}}$$

$$\therefore\quad =\frac{0.7253+j1.754}{1-e^{-1.45}e^{-j0.6}z^{-1}}+\frac{0.7253-j1.754}{1-e^{-1.45}e^{j0.6}z^{-1}}+\frac{-0.7253-0.3j}{1-e^{-0.6}e^{-j1.45}}+\frac{-0.7253+0.3j}{1-e^{-0.6}e^{j1.45}}$$

$$H(z)=\frac{1.454+0.1839z^{-1}}{1-0.387z^{-1}+0.055z^{-2}}+\frac{-1.454+0.2307z^{-1}}{1-0.1322z^{-1}+0.301z^{-2}}$$

H.W: Challenge 1: Design a digital Butterworth filter satisfying the constraints

$$0.8\le\left|H(e^{j\omega})\right|\le 1\qquad for\ 0\le\omega\le 0.2\pi$$

$$\left|H(e^{jw})\right|\le 0.2\quad for\ 0.6\pi\le\omega\le\pi$$

With *T=1 sec using Impulse invariant method.*

Ans: $$H(z)=\frac{0.30109z^{-1}}{1-1.048z^{-1}+0.36z^{-2}}$$

Challenge 2: Design a digital Butterworth filter satisfying the constraints

$$0.8\le\left|H(e^{j\omega})\right|\le 1\qquad for\ 0\le\omega\le 0.2\pi$$

$$\left|H(e^{jw})\right|\le 0.2\quad for\ 0.6\pi\le\omega\le\pi$$

With *T=1 sec using Bilinear Transformation.*

Ans: $$H(s)=\frac{0.084\left(1+z^{-1}\right)^2}{1-1.028z^{-1}+0.3651z^{-2}}$$

Challenge 3: Determine the system function H(z) of the lowest order Butterworth digital filter with the following specification.

(a) 3db ripple in pass band $0\le\omega\le 0.2\pi$

(b) 25db attenuation in stop band $0.45\pi\le\omega\le\pi$

Ans: $$H(z)=\frac{0.0687\left(1+z^{-1}\right)^{3}}{\left(1-0.823z^{-1}\right)\left(1-1.6z^{-1}+0.915z^{-2}\right)}$$

Challenge 4: Enumerate the various steps involved in the design of low pass digital Butterworth IIR filter. (ii) The specification of the desired low pass filter is

$$0.8\le\left|H(e^{j\omega})\right|\le 1 \qquad for\ 0\le\omega\le 0.2\pi$$

$$\left|H(e^{jw})\right|\le 0.2 \quad for\ 0.32\pi\le\omega\le\pi$$

Design a Butterworth digital filter using impulse invariant transformation.

Ans:

$$H(z)=\frac{-0.6242+0.2747z^{-1}}{1-0.253z^{-1}+0.5963z^{-2}}+\frac{0.6242-0.1168z^{-1}}{1-1.0358z^{-1}+0.2869z^{-2}}$$

Design a chebyshev filter for the following specification using bilinear transformation.

$$0.8\le\left|H\left(e^{j\omega}\right)\right|\le 1 \qquad 0\le\omega\le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right|\le 0.2 \quad 0.6\pi\le\omega\le\pi.$$

Solution:

Given data:

Pass band attenuation $\alpha_P=0.8$; *Pass band frequency* $\omega_P=0.2\pi$;

Stop band attenuation $\alpha_S=0.2$; *Stops band frequency* $\omega_S=0.6\pi$;

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_P=-20\log\delta 1=-20\log(0.8)=1.938dB$

Stop band attenuation $\alpha_S=-20\log\delta 2=-20\log(0.2)=13.9794dB$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P=\frac{2}{T}\tan\left(\frac{\omega_P}{2}\right)=2\tan\left(\frac{0.2\pi}{2}\right)=0.649dB$$

$$\Omega_S=\frac{2}{T}\tan\left(\frac{\omega_S}{2}\right)=2\tan\left(\frac{0.6\pi}{2}\right)=2.75dB$$

Step3. To find order of the filter

$$N\ge\frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{Cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1*13.97}-1}{10^{0.1*1.938}-1}}}{Cosh^{-1}\left(\dfrac{2.75}{0.649}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\dfrac{23.945}{0.562}}}{Cosh^{-1}\left(\dfrac{2.75}{0.649}\right)}$$

$$\geq \frac{Cosh^{-1}(6.5273)}{Cosh^{-1}(4.2372)}$$

$$\geq \frac{2.5632}{2.1228}$$

$$\geq 1.207$$

Rounding the next higher integer value N=2

Step4. The poles of chebyshev filter can be determined by

$$S_k = a\cos\varphi_k + jb\sin\varphi_k, k = 0,1,\cdots,N$$

Where,

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right] \quad \text{And calculate a, b,} \varepsilon, \mu$$

$$\varepsilon = \sqrt{10^{0.1\alpha p}-1},$$

$$= \sqrt{10^{0.1*1.938}-1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1+\varepsilon^{-2}}\right]$$

$$= (0.75)^{-1} + \left[\sqrt{1+(0.75)^{-2}}\right]$$

$$\mu = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right]$$

$$= 0.649\left[\frac{(3)^{\frac{1}{2}}-(3)^{-\frac{1}{2}}}{2}\right]$$

$$a = 0.375$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right]$$

$$= 0.649\left[\frac{(3)^{\frac{1}{2}}-(3)^{-\frac{1}{2}}}{2}\right]$$

$$b = 0.75$$

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right] ; \quad k = 1,2$$

$$\phi_1 = \left[\frac{(2(1)+2-1)\pi}{2*2}\right] = \frac{3\pi}{4} = 135^0$$

$$\phi_2 = \left[\frac{(2(2)+2-1)\pi}{2*2}\right] = \frac{5\pi}{4} = 225^0$$

$$S_K = a\cos\varphi_k + jb\sin\varphi_k , \quad k = 1,2$$

$$for\ k = 1,$$

$$S_1 = 0.375\cos\varphi_1 + j(0.75)\sin\varphi_1$$

$$= 0.375cos135^0 + j(0.75)\sin 135^0$$

$$S_1 = -0.265 + j0.53$$

$$for\ k = 2,$$

$$S_1 = 0.375\cos\varphi_2 + j(0.75)\sin\varphi_2$$

$$= 0.375cos225^0 + j(0.75)\sin 225^0$$

$$S_1 = -0.265 - j0.53$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s) = \{S + 0.265 - j0.53\}\{S + 0.265 - j0.53\}$$

$$= \{(S + 0.265)^2 - (j0.53)^2\}$$

$$= (S + 0.265)^2 + (0.53)^2$$

$$= S^2 + 0.5306s + 0.3516$$

Step 6 : The numerator of the transfer function depends on the value of N.

- If N is Even substitute s=0 in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$ Find the value. This value is equal to numerator

$$= \frac{0.3516}{\sqrt{1+\varepsilon^2}} = \frac{0.3516}{\sqrt{1+(0.75)^2}}$$

$$H(s) = 0.28$$

Step 7: The Transfer function is

$$H(s) = \frac{NM}{DM}$$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Step 8: Apply bilinear transformation with to obtain the digital filter

$$H(z) = H(s)\big|\, s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \left.\frac{0.28}{s^2 + 0.5306s + 0.3516}\right| s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \left.\frac{0.28}{s^2 + 0.5306s + 0.3516}\right| s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{0.28}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + 0.5306\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 0.3516}$$

$$H(z) = \frac{0.28\left(1+z^{-1}\right)^2}{1 - 1.348z^{-1} + 0.608z^{-2}}$$

==

Design a chebyshev filter for the following specification using impulse invariance method.

$$0.8 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.2 \qquad 0.6\pi \le \omega \le \pi. \text{[May/June} - 2016\text{]}$$

Solution:

Given data:

Pass band attenuation $\alpha_P = 0.8$ *; Pass band frequency* $\omega_P = 0.2\pi$ *;*

Stop band attenuation $\alpha_S = 0.2$ *; Stops band frequency* $\omega_S = 0.6\pi$ *;*

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_P = -20\log\delta 1 = -20\log(0.8) = 1.938dB$

Stop band attenuation $\alpha_S = -20\log\delta 2 = -20\log(0.2) = 13.9794dB$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P = \frac{\omega_P}{T} = 0.2\pi\ Rad/Sec$$

$$\Omega_S = \frac{\omega_S}{T} = 0.6\pi\ Rad/Sec$$

Step3. To find order of the filter

$$N \ge \frac{Cosh^{-1}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{Cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\frac{10^{0.1*13.97}-1}{10^{0.1*1.938}-1}}}{Cosh^{-1}\left(\frac{0.6\pi}{0.2\pi}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\frac{23.945}{0.562}}}{Cosh^{-1}(3)}$$

$$\geq \frac{Cosh^{-1}(6.5273)}{Cosh^{-1}(3)}$$

$$\geq \frac{2.5632}{1.7627}$$

$$\geq 1.454$$

Rounding the next higher integer value N=2

Step4. The poles of chebyshev filter can be determined by

$$S_k = a\cos\varphi_k + jb\sin\varphi_k, k = 0,1,\cdots,N$$

Where,

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right]$$ And calculate a, b, ε, μ

$$\varepsilon = \sqrt{10^{0.1\alpha p}-1},$$

$$= \sqrt{10^{0.1*1.938}-1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1+\varepsilon^{-2}}\right]$$

$$= (0.75)^{-1} + \left[\sqrt{1+(0.75)^{-2}}\right]$$

$$\mu = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right]$$

$$= 0.2\pi\left[\frac{(3)^{\frac{1}{2}}-(3)^{-\frac{1}{2}}}{2}\right]$$

$$a = 0.362$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right]$$

$$= 0.2\pi\left[\frac{(3)^{\frac{1}{2}}-(3)^{-\frac{1}{2}}}{2}\right]$$

$$b = 0.7255$$

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right]; \quad k = 1,2$$

$$\phi_1 = \left[\frac{(2(1)+2-1)\pi}{2*2}\right] = \frac{3\pi}{4} = 135^0$$

$$\phi_2 = \left[\frac{(2(2)+2-1)\pi}{2*2}\right] = \frac{5\pi}{4} = 225^0$$

$$S_K = a\cos\varphi_k + jb\sin\varphi_k, \quad k = 1,2$$

$$for\ k = 1,$$

$$S_1 = 0.362\cos\varphi_1 + j(0.7255)\sin\varphi_1$$

$$= 0.362cos135^0 + j(0.7255)\sin 135^0$$

$$S_1 = -0.256 + j0.513$$

$$for\ k = 2,$$

$$S_1 = 0.362\cos\varphi_2 + j(0.7255)\sin\varphi_2$$

$$= 0.362cos225^0 + j(0.7255)\sin 225^0$$

$$S_1 = -0.256 - j0.513$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s) = \{S + 0.256 - j0.513\}\{S + 0.256 - j0.513\}$$

$$= \{(S + 0.256)^2 - (j0.513)^2\}$$

$$= (S + 0.256)^2 + (0.513)^2$$

$$= S^2 + 0.513s + 0.33$$

Step 6 : The numerator of the transfer function depends on the value of N.

- If N is Even substitute s=0 in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$ Find the value. This value is equal to numerator

$$= \frac{0.33}{\sqrt{1+\varepsilon^2}} = \frac{0.33}{\sqrt{1+(0.75)^2}}$$

$$H(s) = 0.264$$

Step 7: The Transfer function is

$$H(s) = \frac{NM}{DM}$$

$$H(s) = \frac{0.264}{s^2 + 0.513s + 0.33}$$

Step 8: Using partial fraction expansion, expand H(s) into

$$H(s) = \sum_{k=1}^{2} \frac{A_k}{s - p_k} = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2}$$

$$\frac{0.264}{s^2 + 0.513s + 0.33} = \frac{A_1}{s - (-0.256 + j0.514)} + \frac{A_2}{s - (-0.256 - j0.514)}$$

$$= \frac{0.257j}{s - (-0.256 + j0.514)} - \frac{0.257j}{s - (-0.256 - j0.514)}$$

Step 9: Now transform analog poles $\{P_k\}$ into digital poles $\{e^{p_kT}\}$ to obtain the digital filter

$$H(z)=\sum_{k=1}^{N}\frac{A_k}{1-e^{p_kT}z^{-1}}$$

$$=\sum_{k=1}^{2}\frac{A_k}{1-e^{p_kT}z^{-1}}$$

$$=\frac{0.257j}{s-e^{-0.256T}e^{j0.513T}z^{-1}}-\frac{0.257j}{s-e^{-0.256T}e^{-j0.513T}z^{-1}}$$

$$H(z)=\frac{0.1954z^{-1}}{1-1.3483z^{-1}+0.5987z^{-2}}$$

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H.W: Challenge 1: Design a chebyshev filter to meet the constraints by using bilinear transformation and assume sampling period T=1 sec.

$$\frac{1}{\sqrt{2}}\le\left|H\left(e^{j\omega}\right)\right|\le 1 \qquad 0\le\omega\le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right|\le 0.1 \qquad 0.5\pi\le\omega\le\pi.$$

Solution:

Ans: $H(z)=\dfrac{0.0413\left(1+z^{-1}\right)^2}{1-1.44z^{-1}+0.675z^{-2}}$

=== ***H.W: Convert the following analog filter with transfer function*** $H_a(s)=\dfrac{S+0.2}{(s+0.2)^2+16}$ ***using bilinear transformation. Ans:***

$$H(z)=\frac{0.105+0.0192z^{-1}-0.0864z^{-2}}{1+1.155z^{-1}+0.9232z^{-2}}$$

Find the order and poles of a low pass Butterworth filter that has 3dB bandwidth of 500Hz and attenuation of 40dB at 1kHz.

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<u>Filter design using frequency translation (HPF, BPF, BRF):</u>

A digital filter can be converted into a digital high pass, band stop or another digital filter. These transformations are given below.

Low pass to Low pass	**Low pass to high pass**
$z^{-1}\rightarrow\dfrac{z^{-1}-\alpha}{1-\alpha z^{-1}}$ $where\ \alpha=\dfrac{\sin\left[(\omega_p-\omega_p^{'})/2\right]}{\sin\left[(\omega_p+\omega_p^{'})/2\right]}$ $\omega_p = passband\ frequency\ of\ lowpass\ filter$ $\omega_p^{'} = passband\ frequency\ of\ new\ lowpass\ fi$	$z^{-1}\rightarrow-\left[\dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$ $where\ \alpha=-\dfrac{\cos\left[(\omega_p^{'}+\omega_p)/2\right]}{\cos\left[(\omega_p^{'}-\omega_p)/2\right]}$ $\omega_p = passband\ frequency\ of\ lowpass\ fil$ $\omega_p^{'} = passband\ frequency\ of\ highpass\ fi$

Low pass to Band pass	**Low pass to Band Stop**
$z^{-1} \to \dfrac{-\left(z^{-2} - \dfrac{2\alpha k}{1+k} z^{-1} + \dfrac{k-1}{k+1}\right)}{\dfrac{k-1}{k+1} z^{-2} - \dfrac{2\alpha k}{1+k} z^{-1} + 1}$ $where\ \alpha = \dfrac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$ $k = \cot\left[\dfrac{\omega_u - \omega_l}{2}\right] \tan\dfrac{\omega_p}{2}$ $\omega_u = upper\ cutoff\ \ frequency$ $\omega_l = lower\ cutoff\ \ frequency$	$z^{-1} \to \dfrac{z^{-2} - \dfrac{2\alpha}{1+k} z^{-1} + \dfrac{1-k}{1+k}}{\dfrac{1-k}{1+k} z^{-2} - \dfrac{2\alpha}{1+k} z^{-1} + 1}$ $where\ \alpha = \dfrac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$ $k = \tan\left[\dfrac{\omega_u - \omega_l}{2}\right] \tan\dfrac{\omega_p}{2}$ $\omega_u = upper\ cutoff\ \ frequency$ $\omega_l = lower\ cutoff\ \ frequency$

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Analog Domain:

The frequency transformation can be used to design on LPF with different pass band frequency HPF,BPF and BSF from a normalized Low pass filter Ω_C=1 rad/sec.

Low pass to Low pass	**Low pass to high pass**
$S \to \dfrac{S}{\Omega_C}$	$S \to \dfrac{\Omega_C}{S}$
Low pass to band pass	**Low pass to band stop**
$s \to \dfrac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$ $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \dfrac{-\Omega_1^{\ 2} + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$ $B = \dfrac{\Omega_2^{\ 2} - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$	$s \to \dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$ $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \dfrac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^{\ 2} + \Omega_l \Omega_u}$ $B = \dfrac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^{\ 2} + \Omega_l \Omega_u}$

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H:W: 1. Design a digital chebyshev filter

$$\frac{1}{\sqrt{2}} \le H\left(e^{j\omega}\right) \le 1 \qquad for\ 0 \le \omega \le 0.2\pi$$

$$0 \le \left|H\left(e^{j\omega}\right)\right| \le 0.1 \qquad for\ 0.5\pi \le \omega \le \pi\,|$$

by using bilinear transformation and assume period T=1 sec.

Ans :

$$N=2; \quad H(s)=\frac{0.212}{s^2+0.4172s+0.3} \quad ; \; H(z)=\frac{0.0413\left(1+z^{-1}\right)^2}{1-1.44z^{-1}+0.675z^{-2}}$$

2. Enumerate the various steps involved in the design of low pass digital Butterworth IIR filter.

$$0.8 \le H\left(e^{j\omega}\right) \le 1 \qquad for\; 0 \le \omega \le 0.2\pi$$

$$\left\| H\left(e^{j\omega}\right) \right| \le 0.2 \quad for\; 0.32\pi \le \omega \le \pi \Big|$$

Design Butterworth digital filter using impulse invariant transformation.

Ans:

$$N=4; \quad H(s)=\frac{0.2084}{\left(s^2+0.5171s+0.4565\right)\left(s^2+1.2485s+0.4565\right)} \quad ;$$

$$H(z)=\frac{-0.6242+0.2747z^{-1}}{1-0.253z^{-1}+0.5963z^{-2}}+\frac{0.6242-0.1168z^{-1}}{1-1.03582z^{-1}+0.2869z^{-2}}$$

3. Design a chebyshev low pass filter with the specifications $\alpha s = 1dB$ ripple in the pass band $0 \le \omega \le 0.2\pi, \alpha s = 15dB$ ripple in the stop band $0.3\pi \le \omega \le \pi$, using (a) Bilinear transformation (b) Impulse invariance.

(a) Bilinear transformation:

Ans:

$$N=4; \quad H(s)=\frac{0.04381}{\left(s^2+0.1814s+0.4165\right)\left(s^2+0.4378s+0.1180\right)} \quad ;$$

$$H(z)=\frac{0.001836\left(1+z^{-1}\right)^4}{\left(1-1.499z^{-1}+0.8482z^{-2}\right)\left(1-1.5548z^{-1}+0.6493z^{-2}\right)}$$

(b) Impulse invariance:

Ans:

$$N=4; \quad H(s)=\frac{0.03834}{\left(s^2+0.175s+0.391\right)\left(s^2+0.423s+0.11\right)} \quad ;$$

$$H(z)=\frac{-0.083-0.0245z^{-1}}{1-1.49z^{-1}+0.839z^{-2}}+\frac{0.083+0.0238z^{-1}}{1-1.56z^{-1}+0.655z^{-2}}$$

4. Use the backward difference for the derivative to convert the analog low pass filter with system function.

$$H(s)=\frac{1}{s+2}$$

Ans: $$H(z)=\frac{1}{3-z^{-1}}$$

5. For the analog transfer function determine H(z) using impulse invariant technique. Assume T=1sec.

$$H(s)=\frac{1}{(s+1)(s+2)}$$ **Ans:** $$H(z)=\frac{0.2326z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$ **[May/Jue-2016]**

6. Determine H(z) using the impulse invariant technique for the analog transfer function.

$$H(s)=\frac{1}{(s+0.5)\left(s^2+0.5s+2\right)}$$

Ans:

$$H(z)=\frac{0.5}{1-0.6065z^{-1}}-0.5\left(\frac{1-0.1385z^{-1}}{1+0.277z^{-1}+0.606z^{-2}}\right)+0.0898\left(\frac{0.7663z^{-1}}{1-0.277z^{-1}+0.606z^{-2}}\right)$$

7. Using bilinear transformation obtain H (z) if $H(s)=\dfrac{1}{(s+1)^2}$ **and T=0.1s.**

Ans: $H(z)=\dfrac{0.0476(1+z^{-1})^2}{(1-0.9048z^{-1})^2}$

8. **Convert the analog filter with system function** $H(s)=\dfrac{s+0.1}{(s+0.1)^2+9}$ **into a digital IIR filter using bilinear transformation. The digital filter should have a resonant frequency of** $\omega r=\dfrac{\pi}{4}$. **[Nov/Dec-2015]**

$$H(z)=\frac{1+0.027z^{-1}-0.973z^{-2}}{8.572-11.84z^{-1}+8.177z^{-2}}$$

9. **A digital filter with a 3 dB bandwidth of 0.25π is to be designed from the analog filter whose system response is** $H(s)=\dfrac{\Omega c}{s+\Omega c}$ **. Use bilinear transformation and obtain H(z). [Nov/Dec-15]**

$$H(z)=\frac{1+z^{-1}}{3.414-1.414z^{-1}}$$

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Prove that $\Omega c=\dfrac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}=\dfrac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$

Solution:

$$|H(j\Omega)|^2=\frac{1}{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}$$

$$we\,know\,|H(j\Omega)|^2=\frac{1}{1+\varepsilon^2\left(\dfrac{\Omega}{\Omega_p}\right)^{2N}}$$

by comparing

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}=1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\left(\frac{\Omega}{\Omega_c}\right)^{2N}=\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$= \Omega_c (10^{0.1\alpha_p} - 1)^{1/2N} \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2N}$$

$$\Omega_c = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$$

$$thus\ \Omega c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$$

$$where \left(\frac{\Omega_p}{\Omega_c} \right)^{2N} = 10^{0.1\alpha_p} - 1$$

$$there\ fore\ \Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_p}{\varepsilon^{1/N}}$$

$$and \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

$$\Omega_s = \Omega_p \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2N}$$