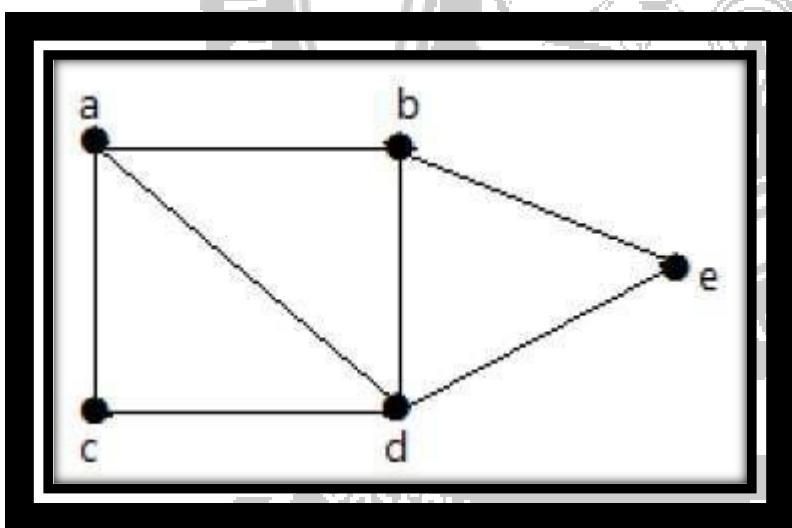


Connectivity:

A graph is said to be connected if there is a path between every pair of vertex. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

Example 1

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Theorem: 1

Show that graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2 .

Proof:

Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G such that $a \in V_1$ and $b \in V_2$.

No path can exist between vertices a and b .

Otherwise, there would be atleast one edge whose one end vertex be in V_1 and the other in V_2 .

Hence if partition exists, G is not connected.

Conversely, let G be a disconnected graph.

Consider a vertex a in G .

Let V_1 be the set of all vertices that are joined by paths to a .

Since G is disconnected, V_1 does not include all vertices of G .

The remaining vertices will form a set V_2 .

No vertex in V_1 is joined to any in V_2 by an edge.

Hence the partition.

Hence the proof.

Components of a graph:

The connected subgraphs of a graph G are called components of the graph G .

Theorem: 1

A simple graph with n vertices and k components can have atmost

$$\frac{(n-k)(n-k+1)}{2} \text{ edges.}$$

Proof:

Let n_1, n_2, \dots, n_k be the number of vertices in each of k components of the graph G .

$$\text{Then } n_1 + n_2 + \dots + n_k = n = |V(G)|$$

$$\sum_{i=1}^k n_i = n \quad \dots (1)$$

$$\text{Now, } \sum_{i=1}^k (n_i - 1) = (n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1)$$

$$= \sum_{i=1}^k n_i - k$$

$$\Rightarrow \sum_{i=1}^k (n_i - 1) = n - k$$

Squaring on both sides

$$\Rightarrow \left[\sum_{i=1}^k (n_i - 1) \right]^2 = (n - k)^2$$

$$\Rightarrow (n_1 - 1)^2 + (n_2 - 1)^2 + \dots + (n_k - 1)^2 \leq n^2 + k^2 - 2nk$$

$$\Rightarrow n_1^2 + 1 - 2n_1 + n_2^2 + 1 - 2n_2 + \dots + n_k^2 + 1 - 2n_k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^k n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^k n_i^2 \leq n^2 + k^2 - 2nk + 2n - k$$

$$\Rightarrow \sum_{i=1}^k n_i^2 = n^2 + k^2 - k - 2nk + 2n$$

$$= n^2 + k(k-1) - 2n(k-1)$$

$$= n^2 + (k-1)(k-2n) \dots (2)$$

Since, G is simple, the maximum number of edges of G in its components is

$$\frac{n_i(n_i-1)}{2}.$$

$$\text{Maximum number of edges of } G = \sum_{i=1}^k \frac{n_i(n_i-1)}{2}$$

$$= \sum_{i=1}^k \left[\frac{n_i^2 - n_i}{2} \right]$$

$$= \frac{1}{2} \sum_{i=1}^k n_i^2 - \frac{1}{2} \sum_{i=1}^k n_i$$

$$\leq \frac{1}{2} [n^2 + (k-1)(k-2n)] - \frac{n}{2} \quad (\text{Using (1) and (2)})$$

$$= \frac{1}{2} [n^2 - 2nk + k^2 + 2n - k - n]$$

$$= \frac{1}{2} [n^2 - 2nk + k^2 + n - k]$$

$$= \frac{1}{2} [(n-k)^2 + (n-k)]$$

$$= \frac{1}{2} [(n - k)(n - k + 1)]$$

Maximum number of edges of $G \leq \frac{(n-k)(n-k+1)}{2}$

Hence the proof.

