2.3 EULER'S EQUATION ALONG A STREAMLINE - BERNOULLI'S EQUATION – APPLICATIONS

EULER'S EQUATION OF MOTION

In this equation of motion the forces due to gravity and pressure are taken in to consideration. This is derived by considering the motion of the fluid element along a stream- line as:

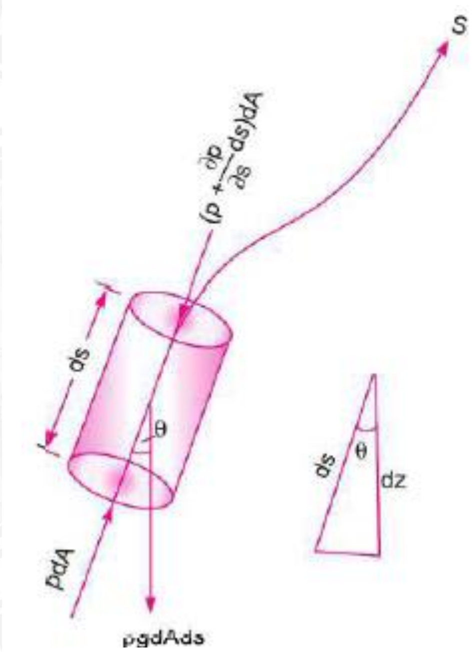
Consider a stream-line in which flow is taking place in s- direction. Consider a cylindrical element of cross-section dA and length ds.

The forces acting on the cylindrical element are:

1. Pressure force p dA in the direction of flow.
2. Pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow
3. Weight of element ρ g dA.ds

Let θ is the angle between the direction of flow and the line of action of the weight of the element.

The resultant force on the fluid element in the direction of S must be equal to the mass of fluid element× acceleration in the direction of s.



$$\therefore \quad pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos\theta$$

$$= \rho dA ds \times a_s \quad \text{———————}(1)$$

Whereas is the acceleration in the direction of s.

Now $\quad a_s = \frac{dv}{dt}$, where v is a function of s and t.

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{\because \frac{ds}{dt} = v\right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\therefore \quad a_s = \frac{v\partial v}{\partial s}$$

Substituting the value of as in equation (1) and simplifying, we get

$$-\frac{\partial p}{\partial s}\, dsdA - \rho g\, dAds \cos\theta = \rho dAds \times \frac{\partial v}{\partial s}$$

Dividing by $\rho dsdA$, $-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{v\partial v}{\partial s}$

or $$\frac{\partial p}{\rho \partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0$$

But from Fig., we have $\cos\theta = \frac{dz}{ds}$

$\therefore$ $$\frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + \frac{vdv}{ds} = 0 \quad \text{or} \quad \frac{dp}{\rho} + gdz + vdv = 0$$

or $$\frac{dp}{\rho} + gdz + vdv = 0$$

$\therefore$ This equation is known as **Euler"s equation of motion**.

BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli"s equation is obtained by integrating the Euler"s equation of motion as

$$\int \frac{dp}{\rho} + \int gdz + \int vdv = \text{constant}$$

$\therefore$ $$\frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

or $$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

or $$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}$$

The above equation is Bernoulli"s equation in which

$\frac{p}{\rho g}$ = Pressure energy per unit weight of fluid or pressure head.

$\frac{v^2}{2g}$ = Kinetic energy per unit weight of fluid or Kinetic head.

z = Potential energy per unit weight of fluid or Potential head.

The following are the assumptions made in the derivation of Bernoulli"s equation.

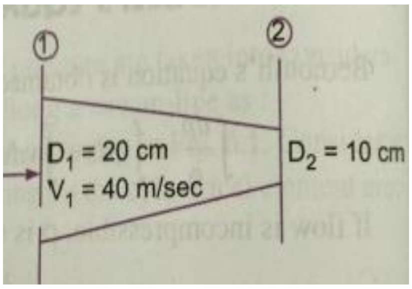
i. The fluid is ideal. i.e. Viscosity is zero.

ii. The flow is steady.

iii. The flow is incompressible.

iv. The flow is irrotational.

PROBLEM 1. Water is flowing through a pipe of 5cm dia. Under a pressure of 29.43N/cm2 and with mean velocity of 2 m/sec. find the total head or total energy per unit weight of water at a cross-section, which is 5m above datum line.



Given: dia. Of pipe = 5cm = 0.05m

Pressure P $= 29.43 N/cm^2 = 29.43 \times 10^4 N/m^2$

Velocity V = 2 m/sec

Datum head Z = 5m

Total head = Pressure head + Kinetic head + Datum head

$$\text{Pressure head} = \frac{P}{\rho g} = \frac{29.43 \times 10^4}{1000 \times 9.81} = 30m$$

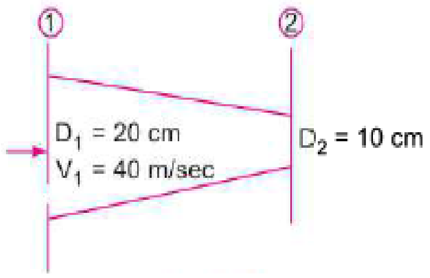
$$\text{Kinetic head} = \frac{V^2}{2g} = \frac{2 \times 2}{2 \times 9.81} = 0.204m$$

Datum head = Z = 5m

$$\frac{p}{\rho g} + \frac{V^2}{2g} + Z = 30 + 0.204 + 5 = 35.204m$$

Total head = 35.204m

PROBLEM 2. A pipe through which water is flowing is having diameters 20cms and 10cms at cross- sections 1 and 2 respectively. The velocity of water at section 1 is 4 m/sec. Find the velocity head at section 1 and 2 and also rate of discharge?



Given: D1 = 20cms = 0.2m

$\therefore$ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$

$V_1 = 4.0$ m/s

$D_2 = 0.1$ m

$\therefore$ $A_2 = \frac{\pi}{4} (.1)^2 = .00785 \text{ m}^2$

i) Velocity head at section 1

$$= \frac{V_1^2}{2g} = \frac{4.0 \times 4.0}{2 \times 9.81} = \mathbf{0.815\ m.}$$

ii) Velocity head at section 2

$$= V_2^2/2g$$

To find V_2, apply continuity equation

A1 V1 = A2 V2

$$V_2 = \frac{A_1V_1}{A_2} = \frac{.0314}{.00785} \times 4.0 = 16.0 \text{ m/s}$$

Velocity head at section 2

$$= \frac{V_2^2}{2g} = \frac{16.0 \times 16.0}{2 \times 9.81} = \mathbf{83.047\ m.}$$

iii) Rate of discharge

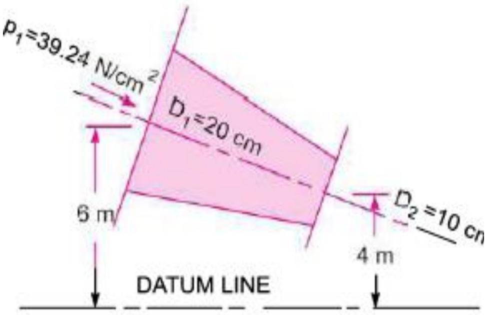
Q = A1 V1 = A2 V2

= 0.0314 ×4 = 0.1256 m3/sec

Q = 125.6 Liters/sec

PROBLEM 3. Water is flowing through a pipe having diameters 20cms and 10cms at sections 1 and 2 respectively. The rate of flow through pipe is 35 liters/sec. The section 1 is 6m above the datum and section 2 is 4m above the datum. If the pressure at section 1 is 39.24N/cm2. Find the intensity of pressure at section 2?

Given: At section 1 D1 = 20cm =0.2m



At section 1, $D_1 = 20 \text{ cm} = 0.2 \text{ m}$

$$A_1 = \frac{\pi}{4}(.2)^2 = .0314 \text{ m}^2$$

$$p_1 = 39.24 \text{ N/cm}^2 = 39.24 \times 10^4 \text{ N/m}^2$$

$$z_1 = 6.0 \text{ m}$$

At section 2, $D_2 = 0.10 \text{ m}$

$$A_2 = \frac{\pi}{4}(0.1)^2 = .00785 \text{ m}^2$$

$$z_2 = 4 \text{ m}$$

$$p_2 = ?$$

Rate of flow, $Q = 35 \text{ lit/s} = \frac{35}{1000} = .035 \text{ m}^3/\text{s}$

$$Q = A_1V_1 = A_2V_2$$

$$V_1 = \frac{Q}{A_1} = \frac{.035}{.0314} = 1.114 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{.035}{.00785} = 4.456 \text{ m/s}$$

Applying Bernoulli's equation at sections 1 and 2, we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\frac{39.24\times10^4}{1000\times9.81}+\frac{(1.114)^2}{2\times9.81}+6.0 = \frac{p_2}{1000\times9.81}+\frac{(4.456)^2}{2\times9.81}+4.0$$

$$40 + 0.063 + 6.0 = \frac{p_2}{9810} + 1.012 + 4.0$$

$$46.063 = \frac{p_2}{9810} + 5.012$$

$$\frac{p_2}{9810} = 46.063 - 5.012 = 41.051$$

$$p_2 = 41.051 \times 9810 \text{ N/m}^2$$

$$= \frac{41.051\times9810}{10^4} \text{ N/cm}^2 = \mathbf{40.27 \text{ N/cm}^2}.$$

PROBLEM 4. Water is flowing through a pipe having diameter 300mm and 200mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525N/cm^2 and the pressure at the upper end is 9.81N/cm^2. Determine the difference in datum head if the rate of flow through is 40lit/sec?

Given :

Section 1, $D_1 = 300$ mm $= 0.3$ m

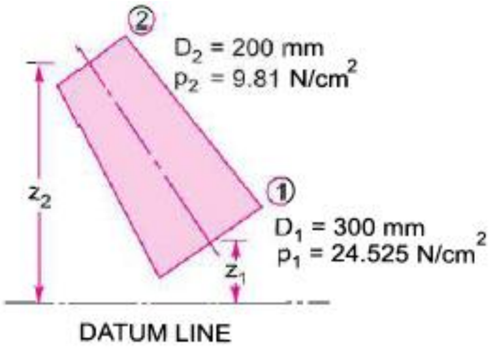
$p_1 = 24.525$ N/cm^2 $= 24.525 \times 10^4$ N/m^2

Section 2, $D_2 = 200$ mm $= 0.2$ m

$p_2 = 9.81$ N/cm^2 $= 9.81 \times 10^4$ N/m^2

Rate of flow $= 40$ lit/s

$$Q = \frac{40}{1000} = 0.04 \text{ m}^3/\text{s}$$



Now $A_1V_1 = A_2V_2 =$ rate of flow $= 0.04$

$$\therefore \quad V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4}D_1^2} = \frac{0.04}{\frac{\pi}{4}(0.3)^2} = 0.5658 \text{ m/s}$$

$$\simeq 0.566 \text{ m/s}$$

$$V_2 = \frac{.04}{A_2} = \frac{.04}{\frac{\pi}{4}(D_2)^2} = \frac{0.04}{\frac{\pi}{4}(0.2)^2} = 1.274 \text{ m/s}$$

Applying Bernoulli"s equation at sections 1 and 2

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1 = \frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2$$

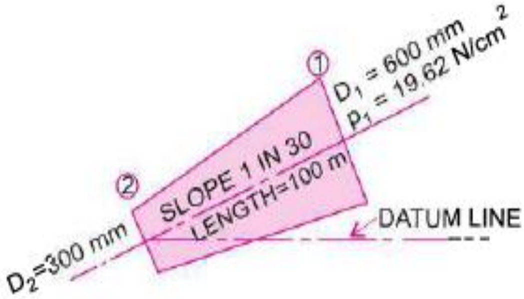
$$\frac{24.525\times10^4}{1000\times9.81}+\frac{.566\times.566}{2\times9.81}+z_1 = \frac{9.81\times10^4}{1000\times9.81}+\frac{(1.274)^2}{2\times9.81}+z_2$$

$$25 + .32 + z_1 = 10 + 1.623 + z_2$$
$$25.32 + z_1 = 11.623 + z_2$$
$$\therefore \quad z_2 - z_1 = 25.32 - 11.623 = 13.697 = 13.70 \text{ m}$$
$\therefore$ Difference in datum head $= z_2 - z_1 =$ **13.70 m. Ans.**

PROBLEM 5.The water is flowing through a taper pipe of length 100m having diameters 600mm at the upper end and 300mm at the lower end, at the rate of 50lts/sec. the pipe has a slope of 1 in 30. Find the pressure at the lower end, if the pressure at the higher level is 19.62N/cm^2?

Given: Length of pipe L = 100m

Dia. At the upper end D_1 = 600mm = 0.6m



$$\therefore \quad \text{Area,} \quad A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times (.6)^2$$
$$= 0.2827 \text{ m}^2$$
$$p_1 = \text{pressure at upper end}$$
$$= 19.62 \text{ N/cm}^2$$

Dia. at the lower end D2 = 300mm = 0.3m

$$\therefore \text{Area,} \quad A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} (.3)^2 = 0.07068 \text{ m}$$

$$Q = \text{rate of flow} = 50 \text{ litres/s} = \frac{50}{1000} = 0.05 \text{ m}^3\text{/s}$$

Let the datum line is passing through the centre of the lower end. Then Z_2 = o

As slope is 1 in 30 means $\quad z_1 = \frac{1}{30} \times 100 = \frac{10}{3}$ m

Also we know $\quad Q = A_1V_1 = A_2V_2$

$$V_1 = \frac{Q}{A} = \frac{0.05}{.2827} = 0.1768 \text{ m/sec} = 0.177 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.5}{.07068} = 0.7074 \text{ m/sec} = 0.707 \text{ m/s}$$

Applying Bernoulli"s equation at sections 1 and 2

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$\frac{19.62 \times 10^4}{1000 \times 9.81} + \frac{.177^2}{2 \times 9.81} + \frac{10}{3} = \frac{p_2}{\rho g} + \frac{.707^2}{2 \times 9.81} + 0$$

$$20 + 0.001596 + 3.334 = \frac{p_2}{\rho g} + 0.0254$$

$$23.335 - 0.0254 = \frac{p_2}{1000 \times 9.81}$$

$$p_2 = 23.3 \times 9810 \text{ N/m}^2 = 228573 \text{ N/m}^2 = \mathbf{22.857\ N/cm^2.}\ \textbf{Ans.}$$