Sinusoidal steady state analysis

A linear electric circuit consist of two responses,

1) Natural response :- short-lived transient response of a circuit to any sudden changes in its condition.

2) Forced response :- It is the long term steady-state response of a circuit to any independent sources present.

Sinusoidal signal

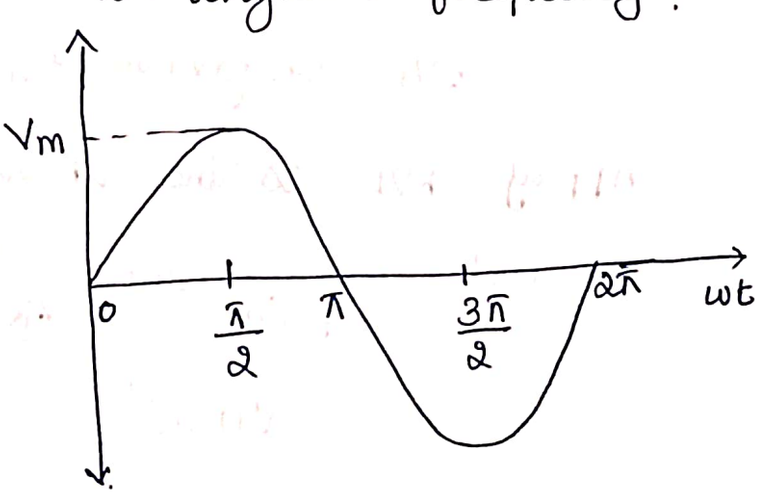
Let a sinusoidal signal be represented as $v(t) = V_m \sin \omega t$

$v(t)$ = Instantaneous voltage.

where V_m = Amplitude of sine wave. (Peak value)

ω = angular frequency.

The waveform repeats itself after every 2π radians, therefore the time period of the signal is 2π.



Timeperiod $T = \frac{1}{f}$

where f is frequency in Hz.

and $\omega t = 2\pi$

$$\omega = \frac{2\pi}{t} \Rightarrow \boxed{\omega = 2\pi f}$$

Frequency :- Number of cycles per second.

Complex Forcing Function

* This helps to relate the current and voltage associated with any element using a simple algebraic expression.

* Here the sinusoids and exponents are related through complex numbers.

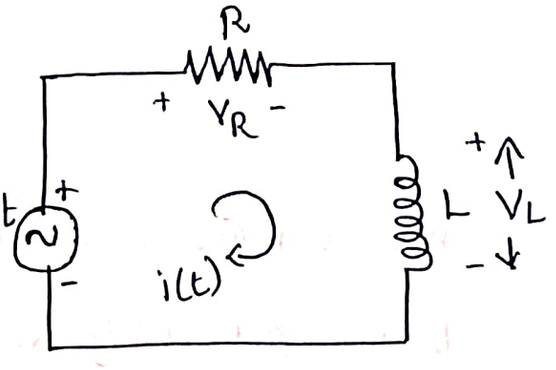
By Euler identity

$$e^{j\theta} = \cos\theta + j\sin\theta .$$

From the circuit,

$$V_m e^{j\omega t} = V_m[\cos\omega t + j\sin\omega t]$$

$$= V_m\cos\omega t + jV_m\sin\omega t \cdot V_m e^{j\omega t}$$



Apply KVL to the above circuit,

$$+V_m e^{j\omega t} - V_R - V_L = 0$$

$$V_m e^{j\omega t} = V_R + V_L$$

$$V_m e^{j\omega t} = R\, i(t) + L\frac{di(t)}{dt} \quad \text{—— (1)}$$

Let $i(t) = I_m e^{j(\omega t+\varphi)}$ —— (2)

$$\frac{di(t)}{dt} = I_m e^{j(\omega t+\varphi)} \cdot j\omega$$

$$= j\omega I_m e^{j(\omega t+\varphi)}$$

$\therefore$ (1) $\Rightarrow$ $V_m e^{j\omega t} = R\, i(t) + j\omega I_m e^{j(\omega t+\varphi)}$

$$V_m e^{j\omega t} = R\, I_m e^{j(\omega t+\varphi)} + j\omega I_m e^{j(\omega t+\varphi)}$$

$$V_m e^{j\omega t} = I_m\left[R \cdot e^{j\omega t} \cdot e^{j\varphi} + j\omega e^{j\omega t} \cdot e^{j\varphi}\right]$$

$$V_m e^{j\omega t} = I_m e^{j\omega t}\left[R + j\omega\right] e^{j\varphi}$$

$$V_m = I_m (R + j\omega L) e^{j\varphi}$$

$$I_m e^{j\varphi} = \frac{V_m}{R + j\omega L}$$

$$\left|I_m e^{j\varphi}\right| = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \quad \text{—— (3)}$$

$$\underline{I_m e^{j\varphi}} = -\tan^{-1}\frac{\omega L}{R} \quad \text{—— (4)}$$

(2) $\Rightarrow$ $I_m e^{j(\omega t+\varphi)} = I_m\left[\cos(\omega t+\varphi) + j\sin(\omega t+\varphi)\right]$

$$= \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}\left[\cos\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right) + j\sin\left(\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right)\right]$$

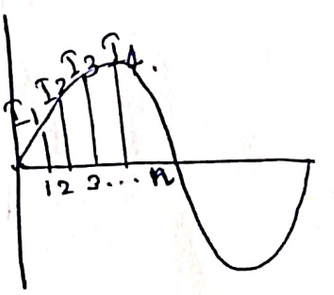
Real part of $i(t)$ is,

$$i(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \cdot \cos\left[\omega t - \tan^{-1}\left(\frac{\omega L}{R}\right)\right]$$

	Time domain	Frequency domain
Resistor	$V = iR$	$V = IR$
Inductor	$V = L \cdot \frac{di}{dt}$	$V = j\omega L \cdot I$
capacitor	$C = \frac{1}{C}\int i\,dt$	$V = \frac{1}{j\omega C} \cdot I$

Average value

It is the area under the curve waveform with respect to time.



For sinusoidal voltage (or) current the area under the waveform for a complete cycle is zero. so only positive half cycle is considered to take average.

$$I_{avg} = \frac{i_1 + i_2 + i_3 + \cdots + i_n}{n} = \frac{\text{area of alternation}}{\text{base.}}$$

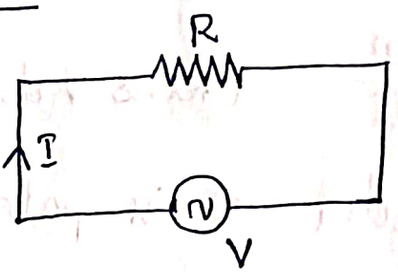
RMS value :-

RMS value of an alternating current is equal to the direct current flowing through a resistor for same period of time.

$$I_{rms} = 0.702\, I_m.$$

Analysis of AC circuit: Phasor relationship of R, L and C.

1. Purely reactive



$V = V_m \sin \omega t$ → Applied voltage.

R = Resistance of the circuit

i = instantaneous current flowing in the circuit

i) By ohm's law

$$i = \frac{V}{R}$$

$$= \frac{V_m \sin \omega t}{R}$$

$$= I_m \sin \omega t \quad \left[\because I_m = \frac{V_m}{R}\right]$$

∴ we can say the instantaneous voltage & current as,

$V = V_m \sin \omega t$ — ①

$I = I_m \sin \omega t$ — ② both V & I are in-phase

writing ① and ② in polar form.

$V = |V| \angle 0^\circ$

$I = |I| \angle 0^\circ$

(6)

To represent V and $\underline{I}$ in polar form.

$$V = \frac{V_m}{\sqrt{2}} \quad \text{(Rms value)}$$

$$\underline{I} = \frac{\underline{I}_m}{\sqrt{2}} \quad \text{(Rms value)}$$

writing ① and ② in polar form.

$$V = |V| \angle 0^\circ$$

$$\underline{I} = |\underline{I}| \angle 0^\circ$$

Impedance (Z)

$$Z = \frac{V}{\underline{I}} = \frac{|V| \angle 0^\circ}{|\underline{I}| \angle 0^\circ}$$

$$Z = \left|\frac{V}{\underline{I}}\right| \angle 0^\circ$$

$$Z = |Z| \angle 0^\circ$$

4) Instantaneous power (w)

$$w = V\underline{I}$$

$$= V_m \sin \omega t \times \underline{I}_m \sin \omega t$$

$$= V_m \underline{I}_m \sin^2 \omega t$$

$$\boxed{w = V_m \underline{I}_m \left[\frac{1 - \cos 2(\omega t)}{2}\right]} \quad \text{③}$$

At $w = \pi/2$ we get max value,

$$w = \frac{V_m \underline{I}_m}{2}\left[1 - \cos \pi \times \frac{\pi}{2}\right] \qquad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right.$$

$$= \frac{V_m I_m}{2} [1 - (-1)]$$

$$\boxed{W = V_m I_m \text{ watts}}$$

Average power

$$P = \frac{1}{\pi} \int_0^{\pi} w \cdot d(wt) \quad \text{— (4)}$$

Sub (3) in (4) =>.

$$P = \frac{1}{\pi} \frac{V_m I_m}{2} \int_0^{\pi} [1 - \cos 2(wt)] \, d(wt)$$

$$= \frac{V_m I_m}{2\pi} \int_0^{\pi} 1 \, d(wt) - 1 \int_0^{\pi} \cos 2(wt)$$

$$= \frac{V_m I_m}{2\pi} [wt]_0^{\pi} - \left[\frac{\sin 2(wt)}{2w}\right]_0^{\pi}$$

$$= \frac{V_m I_m}{2\pi} [\pi - 0] - [\sin 2\pi - \sin 0]$$

$$= \frac{V_m I_m}{2\pi} \times \pi$$

$$\boxed{P = \frac{I_m V_m}{2}}$$

(8)

Power factor :- The angle between the voltage & current is called phase angle & is represented by ϕ

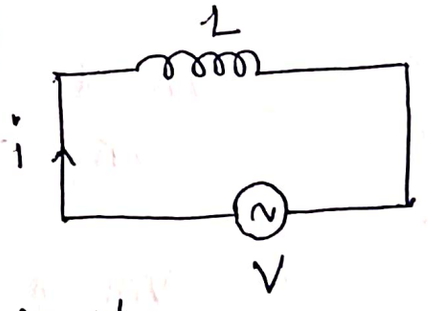
$\cos\phi$ = power factor.

$\cos\phi$ = power factor.

$\cos 0 = 1$

Power factor = 1

Purely Inductive circuit



Let $V = V_m \sin \omega t$

i → instantaneous current

L → self inductance of coil

$$V = L \cdot \frac{di}{dt} \quad \text{——} (1)$$

$$\frac{di}{dt} = \frac{V}{L}$$

$$di = \frac{V}{L}\, dt$$

Integrating on both sides

$$i = \int \frac{v}{L}\, dt$$

$$= \frac{1}{L} \int V_m \sin(\omega t)\, dt$$

$$= \frac{V_m}{L} \left[-\frac{\cos(\omega t)}{\omega} \right] = \frac{V_m}{\omega L} \left[-\sin\left(\frac{\pi}{2} - \omega t\right) \right]$$

$$= \frac{V_m}{\omega L} \left[\sin\left(\omega t - \frac{\pi}{2}\right) \right]$$

$\left[\because \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \right.$
$\left. \sin(-\theta) = -\sin\theta \right]$

$$i = \frac{V_m}{X_L} \sin\left(\omega t - \frac{\pi}{2}\right)$$

[$X_L = \omega L$ is the inductive reactance]

$$\boxed{i = I_m \sin\left(\omega t - \frac{\pi}{2}\right)}$$

$$i = I_m$$

[when $\sin\left(\omega t - \frac{\pi}{2}\right) = 1$ i.e. i is maximum]

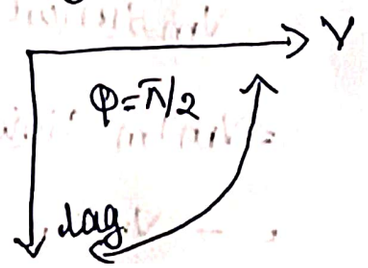
Now we know that,

$$V = V_m \sin(\omega t) \quad \text{——— (2)}$$

$$I = I_m \sin(\omega t - \pi/2) \quad \text{——— (3)}$$

From (2) and (3) we can say current lag voltage by $\pi/2$.

Phasor diagram



(10) Polar Form

$$V = |V| \angle 0^\circ$$

$$I = |I| \angle -\pi/2$$

Impedance

$$Z = \frac{V}{I} = \frac{|V| \angle 0^\circ}{|I| \angle -\pi/2} = \left|\frac{V}{I}\right| \angle \pi/2$$

$$\boxed{Z = \left|\frac{V}{I}\right| \angle \pi/2}$$

$$\left[\because Z = X_L \right] \quad X_L = \left[\frac{V}{I}\right]$$

Phase Angle

$$\varphi = \pi/2 \text{ lag}$$

Power factor

$$P.F \quad \cos\varphi = \cos \pi/2$$

$$\therefore \cos\left(\frac{\pi}{2}\right) = 0$$

$$\boxed{PF = 0}$$

Instantaneous power

$$W = VI$$

$$= V_m \sin(\omega t)\, I_m \sin(\omega t - \pi/2)$$

$$= V_m I_m \sin(\omega t)\left[-\cos(\omega t)\right]$$

$$= -V_m I_m . \sin(\omega t)\cos(\omega t)$$

$$W = -V_m I_m \frac{\sin 2(\omega t)}{2}$$

$\left[\because \sin 2\theta = 2\sin\theta\cos\theta \right.$

$\left. \sin\theta\cos\theta = \frac{\sin 2\theta}{2}\right]$

Max value

$$W = \frac{V_m I_m}{2}.$$

Average power (P)

$$P = \frac{1}{\pi}\int_0^{\pi} W \, d(\omega t)$$

$$= \frac{1}{\pi}\int_0^{\pi} -\frac{V_m I_m}{2} \sin 2(\omega t)\, dt$$

$$= -\frac{V_m I_m}{2\pi}\left[-\frac{\cos 2(\omega t)}{2}\right]_0^{\pi}$$

$$= \frac{V_m I_m}{2\pi \times 2}\left[\cos 2\pi - \cos 0\right]$$

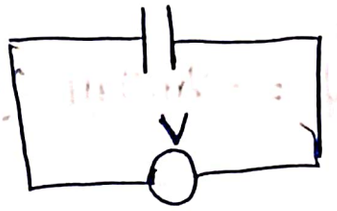
$$= \frac{V_m I_m}{4\pi}\left[+1 - 1\right]$$

$$P = 0 W$$

X_L Vs ω

(12)

Purely capacitive



$V = V_m \sin \omega t$

we know that,

$q = CV$

q = charge
C = capacitance
V = voltage.

current $i = \frac{dq}{dt} = \frac{d}{dt} CV$

$= C \cdot \frac{dv}{dt}$

$= C \frac{d}{dt} V_m \sin(\omega t)$

differentiating with respect to t.

$C \cdot V_m \cos(\omega t)(\omega)$

$= V_m C\omega \cos(\omega t)$

$= \frac{V_m}{\frac{1}{C\omega}} \cos(\omega t)$

$i = \frac{V_m}{X_c} \cos(\omega t)$ $\left[X_c = \frac{1}{C\omega} = \text{Capacitive Reactance} \right]$

$i = \frac{V_m}{X_c} \sin\left(\omega t + \frac{\pi}{2}\right)$ $\left[\because \cos\theta = \sin\left(\theta + \frac{\pi}{2}\right) \right]$

$i = I_m \sin\left(\omega t + \frac{\pi}{2}\right)$ ——— (1)

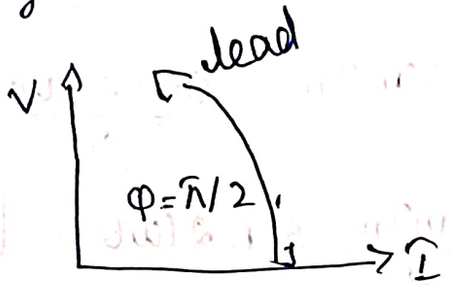
∴ current lead voltage by $\pi/2$.

Polar Form

$$V = |V| \angle 0^\circ$$

$$I = |I| \angle -\pi/2$$

Phasor diagram



Impedance

$$|Z| = \frac{|V| \angle 0^\circ}{|I| \angle -\pi/2^\circ}$$

$$= \left|\frac{V}{I}\right| \angle -\pi/2^\circ$$

$$|Z| = |X_C| \angle -\pi/2$$

$\therefore Z = -jX_C$ in Rectangular Form.

Power factor

$= \cos\varphi$ $\quad \varphi = \pi/2$ lead.

$\therefore \cos \pi/2 = 0$

Powerfactor = 0.

Instantaneous power (W)

$$W = VI$$

$$= V_m \sin(\omega t)\, I_m \sin(\omega t + \pi/2)$$

$$= V_m I_m \sin(\omega t)\cos(\omega t)$$

$$= V_m I_m \frac{\sin 2(\omega t)}{2}$$

$$\boxed{W = \frac{V_m I_m}{2} \sin 2(\omega t)} \quad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

Average power

$$P = \frac{1}{\pi}\int_0^{\pi} W \, d(\omega t)$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{V_m I_m}{2} \sin 2(\omega t)\, d(\omega t)$$

$$= \frac{V_m I_m}{2\pi}\left[-\frac{\cos 2(\omega t)}{2}\right]_0^{\pi}$$

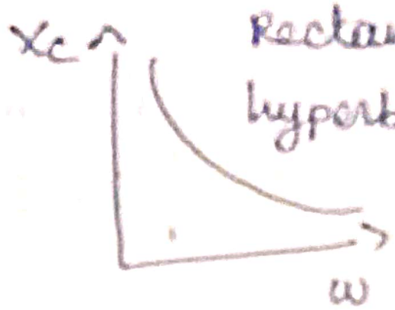
$$= -\frac{V_m I_m}{4\pi}\left[\cos 2\pi - \cos 0\right]$$

$$= \frac{-V_m I_m}{4\pi}\left[\cos' 1 - 1\right]$$

Average power = 0

Capacitive reactance

$$X_C = \frac{1}{\omega C}$$



Rectangular hyperbola

1) A voltage $V = 100 \sin(\pi t)$ is applied to a circuit consisting of a pure resistor of $20\,\Omega$. Find the following quantities.

i) maximum voltage b) Frequency of supply.

3) The expression for instantaneous current.

Soln

$$V = 100 \sin \pi t$$

$V_m = 100$

$\omega t = 100\pi t$

$\Rightarrow \omega = 100\pi$

i) max voltage $V_m = 100$

ii)

$$2\pi f = \omega = 100\pi$$

$$2\pi f = 100\pi$$

$$f = \frac{100\pi}{2\pi}$$

$$\boxed{f = 50\,Hz}$$

iii)

$$I_m = \frac{V_m}{R} = \frac{100}{20} \Rightarrow I_m = 5A$$

(16)

WKT for pure resistive network

$i = I_m \sin \omega t$

$\boxed{i = 5 \sin 100\pi t}$

2) An inductive coil has negligible resistance and inductance of 0.1 H. It is connected across 220V, 50Hz supply. Find current & power. Also write down expression for instantaneous applied voltage and current.

Soln

Gn = $L = 0.1$ H $\quad V_{rms} = 220$V $\quad f = 50$Hz.

$V = V_m \sin \omega t$

$= 220 \sin 2\times\pi\times 50 t$

$\boxed{V = 220 \sin 100\pi t}$

$V_{rms} = 220$

$V_{rms} = \frac{V_m}{\sqrt{2}} \Rightarrow V_m = \sqrt{2} V_{rm}$

$(\omega = 100\pi)$

$= 100 \times 3.14$

$\omega = 314$ rad/sec

~~Rms value~~

peak voltage $V_m = 220 \times \sqrt{2}$.

$\boxed{V_m = 311 \text{ volts}}$

$X_L = \omega L = 2\times\pi\times f\times L$

$= 2\times\pi\times 50\times 0.1$

$\boxed{X_L = 31.4}$

$I_m = \frac{V_m}{X_L} = \frac{311}{31.4} \simeq 9.9A$.

For a purely Inductive N/w :-

$$i = I_m \sin(\omega t - \pi/2)$$

$$\boxed{i = 9.9 \sin(314t - \pi/2)}$$

$$V = V_m \sin \omega t$$

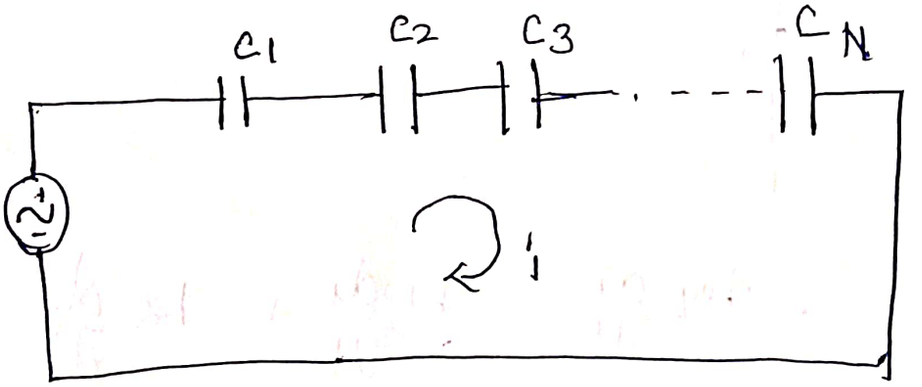
$$\boxed{V = 311 \sin(314t)}$$

$$\text{power} = |V||I| \cos\phi$$

$$= |V||I| \cos 90^\circ$$

$$\boxed{P = 0W}$$

Capacitor



$$i = \frac{d}{dt}(CV) \qquad V$$

$$d(CV) = i\,dt$$

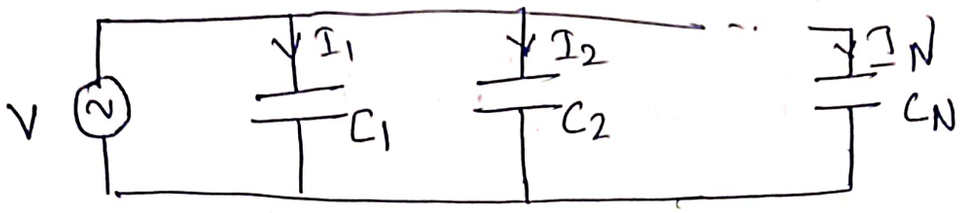
$$V = \frac{1}{C}\int i\,dt$$

$$V = \frac{1}{C_1}\int i(x)dt + \frac{1}{C_2}\int i(x)dt + \frac{1}{C_2}\int i(x)\,dt.$$

$$+ \cdots \frac{1}{C_N}\int i(t)\,dt.$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \frac{1}{C_N}$$

(18).

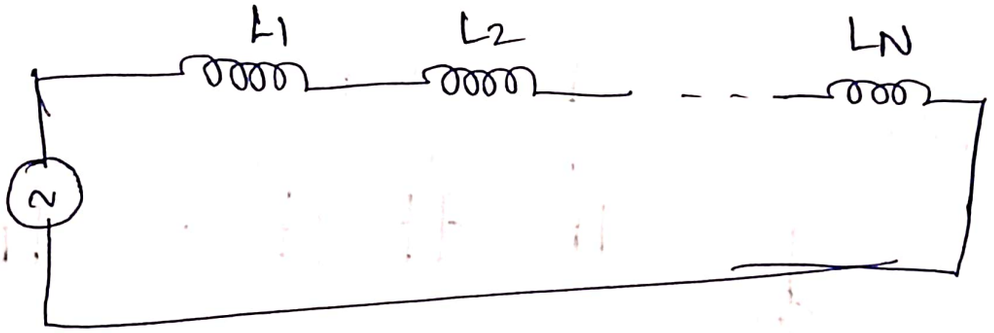


$$I = I_1 + I_2 + \cdots I_N$$

$$C\frac{dv}{dt} = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \cdots C_N\frac{dv}{dt}$$

$$C\frac{dv}{dt} = \left[C_1 + C_2 + C_3 \cdots C_N\right]\frac{dv}{dt}$$

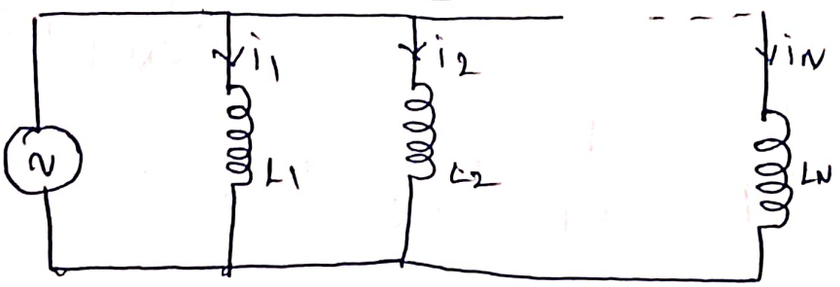
Inductor.



$$L_{eq}\frac{di}{dt} = L_1\frac{di}{dt} + L_2\frac{di}{dt} + \cdots L_N\frac{di}{dt}$$

$$= \left(L_1 + L_2 + \cdots + L_N\right)\frac{di}{dt}$$

$$L_{eq} = \left[L_1 + L_2 + \cdots L_N\right]$$



$$i_1 = \frac{1}{L_1}\int v\,dt$$

$$i_2 = \frac{1}{L_2}\int v_2\,dt$$

$$i_N = \frac{1}{L_N}\int V_N\,dt$$

$$= \frac{1}{L_1}\int v\,dt + \frac{1}{L_2}\int v_2\,dt \cdots \int V_N\,dt$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots \frac{1}{L_N}$$