

4.4 CONVOLUTION OF FOURIER TRANSFORM

Evaluate (a) $\int_0^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$ **(b)** $\int_0^{\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx$ **using Fourier transforms.**

Solution:

(a) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$\begin{aligned} F_s[f(x)] &= F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx \end{aligned}$$

$$\boxed{F_s(s) = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]} \quad \because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$\boxed{G_s(s) = F_s[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right]}$$

We know that

$$\begin{aligned} \int_0^{\infty} F_s(s) G_s(s) ds &= \int_0^{\infty} f(x) g(x) dx \\ \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right] ds &= \int_0^{\infty} e^{-ax} e^{-bx} dx \\ \frac{2}{\pi} \int_0^{\infty} \left[\frac{s^2}{(a^2 + s^2)(b^2 + s^2)} \right] ds &= \int_0^{\infty} e^{-ax-bx} dx \\ \int_0^{\infty} \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds &= \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty} \\ &= \frac{-\pi}{2(a+b)} [e^{-\infty} - e^{-0}]_0^{\infty} \\ &= \frac{-\pi}{2(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1 \\ \int_0^{\infty} \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds &= \frac{\pi}{2(a+b)} \end{aligned}$$

Put s=x we get

$$\int_0^{\infty} \left[\frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \right] dx = \frac{\pi}{2(a+b)}$$

(b) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$\begin{aligned} F_c[f(x)] &= F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx \end{aligned}$$

$$F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \quad \because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$G_c(s) = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right]$$

We know that

$$\begin{aligned} \int_0^{\infty} F_c(s) G_c(s) ds &= \int_0^{\infty} f(x) g(x) dx \\ \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right] ds &= \int_0^{\infty} e^{-ax} e^{-bx} dx \\ \frac{2ab}{\pi} \int_0^{\infty} \left[\frac{1}{(a^2 + s^2)(b^2 + s^2)} \right] ds &= \int_0^{\infty} e^{-ax-bx} dx \\ \int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} [e^{-\infty} - e^{-0}]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1 \\ \int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds &= \frac{\pi}{2ab(a+b)} \end{aligned}$$

Put s=x we get

$$\int_0^{\infty} \left[\frac{1}{(x^2 + a^2)(x^2 + b^2)} \right] dx = \frac{\pi}{2ab(a+b)}$$

Evaluate (a) $\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+16)} dx$, (b) $\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx$ using Fourier transforms.

Solution:

(a) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$F_s(s) = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right]$$

$$\because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$G_s(s) = G_s[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right]$$

We know that

$$\int_0^{\infty} F_s(s) G_s(s) ds = \int_0^{\infty} f(x) g(x) dx$$

$$\int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{s}{b^2 + s^2} \right] ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\frac{2}{\pi} \int_0^{\infty} \left[\frac{s^2}{(a^2 + s^2)(b^2 + s^2)} \right] ds = \int_0^{\infty} e^{-ax-bx} dx$$

$$\int_0^{\infty} \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty}$$

$$= \frac{-\pi}{2(a+b)} [e^{-\infty} - e^{-0}]_0^{\infty}$$

$$= \frac{-\pi}{2(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \left[\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2(a+b)} \text{-----(1)}$$

Put a=3 & b=4 and s=x we get

$$(1) \Rightarrow \int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+16)} dx = \frac{\pi}{2(3+4)}$$

$$\boxed{\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+16)} dx = \frac{\pi}{14}}$$

(b) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$\begin{aligned} F_c[f(x)] &= F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx \end{aligned}$$

$$\boxed{F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right]} \quad \because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$\boxed{G_c(s) = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right]}$$

We know that

$$\begin{aligned} \int_0^{\infty} F_c(s) G_c(s) ds &= \int_0^{\infty} f(x) g(x) dx \\ \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \sqrt{\frac{2}{\pi}} \left[\frac{b}{b^2 + s^2} \right] ds &= \int_0^{\infty} e^{-ax} e^{-bx} dx \\ \frac{2ab}{\pi} \int_0^{\infty} \left[\frac{1}{(a^2 + s^2)(b^2 + s^2)} \right] ds &= \int_0^{\infty} e^{-ax-bx} dx \\ \int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx \\ &= \frac{\pi}{2ab} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} [e^{-\infty} - e^{-0}]_0^{\infty} \\ &= \frac{-\pi}{2ab(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1 \end{aligned}$$

$$\boxed{\int_0^{\infty} \left[\frac{1}{(s^2 + a^2)(s^2 + b^2)} \right] ds = \frac{\pi}{2ab(a+b)}}$$

Put a=1 & b=2 s=x we get

$$(1) \Rightarrow \int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx = \frac{\pi}{2(1)(2)(1+2)}$$

$$\boxed{\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx = \frac{\pi}{12}}$$