

UNIT V APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

5.2 ONE DIMENSIONAL HEAT EQUATION

One dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } \alpha^2 = \frac{k}{\rho c} = \frac{\text{Thermal conductivity}}{\text{density} \times \text{Specific heat capacity}}$$

$u(x, t) \rightarrow$ The temperature distribution at any point x from one end at time t .

The various Possible Solution of 1-D heat equation.

$$(i) u(x, t) = (Ae^{px} + Be^{-px})Ce^{\alpha^2 p^2 t}$$

$$(ii) u(x, t) = (A \cos px + B \sin px)Ce^{-\alpha^2 p^2 t}$$

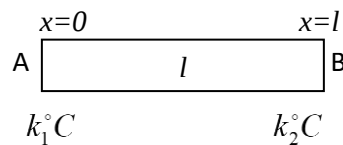
$$(iii) u(x, t) = Ax + B$$

The boundary and initial conditions.

$$i) u(0, t) = k_1^\circ C$$

$$ii) u(l, t) = k_2^\circ C$$

$$iii) u(x, 0) = f(x)$$



The correct solution is $u(x, t) = (A \cos px + B \sin px)Ce^{-\alpha^2 p^2 t}$

The steady state solution in 1-D heat equation:

Solution:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{-----(1)}$$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $u(x) = Ax + B$

A rod of length l has its ends A and B are kept at $0^\circ C$ and $100^\circ C$ until steady state condition prevail. If the temperature at B is reduced suddenly to $0^\circ C$ and kept so while that of A is maintained. Find the temperature $u(x, t)$ at a distance x from A and at time t .

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$

To find steady state solution $u(x, 0) = u(x)$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$ -----(1)

The boundary conditions are i) $u(0) = 0^\circ C$ ii) $u(l) = 100^\circ C$

Applying condn (i) in (1)

$$(1) \Rightarrow u(0) = 0 + B \Rightarrow \boxed{B = 0}$$

Sub B in (1)

$$u(x) = Ax \text{ -----(2)}$$

Applying condn (ii) in (2)

$$u(l) = Al \Rightarrow 100 = Al \Rightarrow \boxed{A = \frac{100}{l}}$$

Sub A in (2)

$$\boxed{u(x) = \frac{100x}{l}}$$

The boundary and initial conditions are

$$\text{i) } u(0, t) = 0^\circ C$$

$$\text{ii) } u(l, t) = 100^\circ \text{C}$$

$$\text{iii) } u(x, 0) = f(x) = \frac{100x}{l}, \quad 0 \leq x \leq l$$

The correct solution is

$$u(x, t) = (A \cos px + B \sin px) C e^{-\alpha^2 p^2 t} \text{ ----- (1)}$$

Apply condn. (i) in (1)

$$u(0, t) = (A \cos 0 + \cancel{B \sin 0}) C e^{-\alpha^2 p^2 t}$$

$$0 = A C e^{-\alpha^2 p^2 t}$$

$$\text{Here } C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \boxed{A = 0}$$

Sub A in (1)

$$u(x, t) = (B \sin px) C e^{-\alpha^2 p^2 t} \text{ ----- (2)}$$

Apply condn. (ii) in (2)

$$u(l, t) = (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$0 = (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$\text{Here } B \neq 0, C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \boxed{\sin pl = 0}$$

$$\sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub p in (2)

$$u(x, t) = \left(B \sin \frac{n\pi x}{l} \right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x, t) = b_1 \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad BC = b_1 (\text{say})$$

The most general solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}} \text{-----} (3)$$

Apply condn (iii) in (3)

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \because e^{-0} = 1$$

This is Fourier sine series of $f(x)$ in $(0,l)$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \int_0^l x \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \left[(x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{200}{l^2} \left(\frac{-l}{n\pi} \right) \left[x \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-200}{ln\pi} [l \cos n\pi - 0]$$

$$= \frac{-200(-1)^n}{n\pi}$$

$$\boxed{b_n = \frac{200}{n\pi} (-1)^{n+1}}$$

Sub b_n in (3)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

This is the required temperature.

The ends A and B of a rod l cm long have their temperatures kept at 30°C and 80°C , until steady state conditions prevail. The temperature of the end B is suddenly reduced to 60°C and that of A is increased to 40°C . Find the steady state temperature distribution in the rod after time t .

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$

To find steady state solution $1 \ u(x, 0) = u(x)$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$ -----(1)

The boundary conditions are i) $u(0) = 30^\circ\text{C}$ ii) $u(l) = 80^\circ\text{C}$

Applying condn (i) in (1)

$$(1) \Rightarrow u(0) = 0 + B \Rightarrow \boxed{B = 30}$$

Sub B in (1)

$$u(x) = Ax + 30 \text{ -----(2)}$$

Applying condn (ii) in (2)

$$u(l) = Al + 30 \Rightarrow 80 = Al + 30 \Rightarrow \boxed{A = \frac{50}{l}}$$

Sub A in (2)

$$u(x) = \frac{50x}{l} + 30 = f(x)$$

This $u(x)$ will be treated as the initial conditions $u(x, 0) = f(x)$

To find steady state solution 2 $u(x, 0) = u(x)$

Integrating twice we get $\boxed{u_t(x) = Ax + B}$ -----(3)

The boundary conditions are i) $u_t(0) = 40^\circ C$ ii) $u_t(l) = 60^\circ C$

Applying condn (i) in (3)

$$(3) \Rightarrow u_t(0) = 0 + B \Rightarrow \boxed{B = 40}$$

Sub B in (1)

$$u_t(x) = Ax + 40 \text{ -----(4)}$$

Applying condn (ii) in (4)

$$u_t(l) = Al + 40 \Rightarrow 60 = Al + 40 \Rightarrow \boxed{A = \frac{20}{l}}$$

Sub A in (2)

$$u_t(x) = \frac{20x}{l} + 40$$

This $u_t(x)$ will be treated as the transient state temperature.

The required temperature is

$$u(x, t) = u_t(x, 0) + (A \cos px + B \sin px) Ce^{-\alpha^2 p^2 t}$$

$$u(x, t) = \frac{20x}{l} + 40 + (A \cos px + B \sin px) Ce^{-\alpha^2 p^2 t} \text{ -----(5)}$$

The boundary and initial conditions are

$$\text{i) } u(0, t) = 40^\circ C$$

$$\text{ii) } u(l, t) = 60^\circ C$$

$$\text{iii) } u(x, 0) = f(x) = \frac{50x}{l} + 30, \quad 0 \leq x \leq l$$

Apply condn (i) in (5)

$$u(0, t) = 0 + 40 + (A \cos 0 + B \sin 0) C e^{-\alpha^2 p^2 t}$$

$$40 = 0 + 40 + (A \cos 0 + \cancel{B \sin 0}) C e^{-\alpha^2 p^2 t}$$

$$0 = A C e^{-\alpha^2 p^2 t}$$

$$\text{This } C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \boxed{A = 0}$$

Sub A in (5)

$$u(x, t) = \frac{20x}{l} + 40 + (B \sin px) C e^{-\alpha^2 p^2 t} \text{ ---- (6)}$$

Apply condn (ii) in (6)

$$u(l, t) = 20 + 40 + (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$60 = 20 + 40 + (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$0 = (B \sin pl) C e^{-\alpha^2 p^2 t}$$

$$B \neq 0, C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \sin pl = 0$$

$$\sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub p in (6)

$$u(x, t) = \frac{20x}{l} + 40 + \left(B \sin \frac{n\pi x}{l} \right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$u(x, t) = \frac{20x}{l} + 40 + b_1 \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} \quad \because BC = b_1$$

The most general solution is

$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} \text{ ----- (7)}$$

Apply condn (iii) in (7)

$$u(x,0) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-0}$$

$$\frac{50x}{l} + 30 = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\frac{30x}{l} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

To find b_n :

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \left(\frac{30x}{l} - 10 \right) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[\left(\frac{30x}{l} - 10 \right) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - \left(\frac{30}{l} \right) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{2}{l} \left[\left(\frac{-l}{n\pi} \right) \left(\frac{30x}{l} - 10 \right) \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-2}{n\pi} \left[(20) \cos n\pi + 10 \right]$$

$$\boxed{b_n = \frac{-20}{n\pi} \left[2(-1)^n + 1 \right]}$$

Sub b_n in (7)

$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} \frac{-20}{n\pi} \left[2(-1)^n + 1 \right] \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$u(x,t) = \frac{20x}{l} + 40 - \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[2(-1)^n + 1 \right] \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$